An AI Analysis of Networked Trade
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Abstract
The study of modeling economic exchange on networks has generally involved studying two separate phenomena - networks in which the participants are allowed to construct the network themselves through purchasing edges, and networks in which the participants have no input and are placed on a network created from some generative model. In this paper, I introduce a model combining elements of these two approaches, and examine various properties of the networks that arise in this model.

Introduction
In recent years there has been an explosion in research into modeling economic exchange on networks, as Kakade, Kearns, and Ortiz’s result (2004a) has allowed for the application on networks of the Arrow-Debreu framework for equilibrium analysis. The appeal is obvious, as networks allow modeling of exchange where parties are constrained as to who they can trade with - a situation arising often in the natural world whether it involves individuals or nations. Two strains of research within this field quickly appeared. One was concerned with examining networks which the participants constructed themselves through purchasing connections to other participants. Here, the model is based on the assumption that participants are active in the formation of their economic networks. Even-Dar, Kearns, and Suri (2007) present an in-depth analysis of the equilibrium conditions that occur on such networks, establishing sharp bounds on the amount of wealth variation that can be present in such graphs. Intuitively, participant-created graphs have limits on the amount of wealth variation because a poor node may, in essence, purchase its way out of poverty by establishing more links to the network, giving it more avenues for trade. The other strain of research was concerned with examining the economic exchange properties of networks that occur naturally in the environment (so called “social networks”); here, the model is based on the assumption that participants simply “find” themselves in economic networks and must operate within their bounds. Kakade, Kearns, Ortiz, Pemantle, and Suri (2004b) present an analysis of such a model, demonstrating that equilibrium wealth variation can vary greatly depending on the generative model used to create the exchange network. The model I propose here is a hybrid of the two models above. Essentially, the hypothesis that I attempt to model is that humans find themselves in social networks that they must engage in economic exchange upon, and while humans can modify the network slightly by purchasing connections beyond the ones that they get for free, the effect they can have on the overall network structure
is negligible (even if they can change their individual situations through their edge purchasing actions).

**Definition of Model**

The model I propose is defined as follows. It is an extension of the Fisher model, in which there is a set of consumers and a set of goods. We assume linear utility functions for all participants, and note that in such a setting, an equilibrium always exists, where supply equals demand and no participant has incentive to change its purchasing pattern (Gale, 1960). Extending this model to exchange networks, we start off with a set of consumers placed on a bipartite graph, with the vertices being divided into two sets; edges are only allowed to connect vertices in one set to vertices in the other. Each consumer is given a vertex and is allotted an endowment of one unit of milk (for the first set) or one unit of wheat (for the second set); additionally the consumer is assigned a preference of zero for the good that it is allotted, and a non-zero preference for the other good. Therefore, the consumer’s objective is to obtain as much of the other good as possible.

Next, we define the generative model we will use to create the basic graph structures. We will use the modified preferential attachment model presented by Kakade et al. (2004b). In this model, the network starts off with one “wheat” vertex connected to one “milk” vertex. At each time step, one milk vertex is attached to a chosen wheat vertex with probability proportional to the wheat vertex’s degree in the graph. The same thing is done to attach a wheat vertex to one of the milk vertices. Note that this model will always form a tree since only one edge is added to the graph for each vertex that is added. It is possible to modify this model by introducing two parameters, $\alpha$ and $\nu$. If $\alpha$ is introduced, then the edges will be chosen according to the preferential attachment model with probability $1 - \alpha$, and will be chosen randomly (each vertex equally likely) with probability $\alpha$. If $\nu$ is introduced, then it represents the number of edges that a new vertex will establish in the network; values of $\nu > 1$ produce structures other than trees. For now, we will consider the simple case where $\alpha = 0, \nu = 1$.

Finally, we define the extension of the model that will allow players to purchase edges. This is based on the ideas of Even-Dar et al. (2007), with some modifications. Once the consumers have been positioned on a graph according to the preferential attachment model, they engage in trade for a single round, attempting to maximize wealth in the manner laid out above. Thus, at the end of this round, they have some wealth value corresponding to the amount of the opposite good that the player was able to trade for. The second round of the game consists of buying edges in a one-shot, simultaneous move game. A global price for edges, $c$, is set; players purchase their edges in wealth (not in initial endowment), but are allowed to go into debt if necessary to pay for their purchased edges. Only “milk” vertices are allowed to
purchase edges - this will introduce some asymmetry in the relative performances of milks and wheats vertices, but since we are not concerned about relative performance, only maximizing wealth, this is not too big of an issue. There is one additional constraint on purchasing of edges, that edges can only be purchased to consumers who are already in your “neighborhood” (i.e. less than three network hops away). This is in order to help maintain some “natural” order to the network, as Kakade et al. (2004b) noted that real-world networks display a significant degree of clustering. We introduce a parameter \( m \) to denote how many edges a player may purchase in this round; for this analysis, we will assume that \( m = 2 \). Finally, after players are finished purchasing edges, a third trading round begins that is identical to the first. Thus, the payoff for each player for the entire, three-round game is as follows: (wealth accumulated in first round) + (wealth accumulated in third round) - (wealth spent purchasing edges in second round).

Questions

Some questions that we may ask, and that I answer in this paper, about this model:

- What is the expected value of buying an edge?
- How do edge-buying preferences change with your initial condition?
- How does buying an edge affect the distribution of wealth for the entire network?
- Under which cases might a “subsidy” be required to achieve optimal social welfare (i.e., the classic game theoretic situation where no player wants to pay for an edge, but if they all pay for an edge they will be better off)?
- How does the distribution of wealth at the end of the game compare to the distribution at the beginning?

Formal Analysis

Before we begin our analysis, we note that it presumes some prior knowledge about equilibria on economic networks. Please refer to Kakade et al. (2004b) for a discussion of such equilibria. Additionally, in this section we define a few terms to simplify the discussion. “Target” means a consumer to whom you have purchased an edge. A rich consumer is a vertex with equilibrium wealth greater than the mean; a normal consumer is a vertex with equilibrium wealth equal to the mean, and a poor consumer is a vertex with equilibrium wealth less than the mean.

We begin by analyzing some fundamental properties of edge purchasing, in the intention of eventually developing a strategy of edge purchasing within our model.
Theoretically, there are possibly a number of things that can happen when you purchase an edge - your equilibrium wealth can increase, decrease, or remain the same, your target’s equilibrium wealth can increase, decrease, or remain the same, and the wealths of all other vertices may increase, decrease, or remain the same. We want to rule out some of these possibilities and establish the magnitudes and likelihoods of the others happening, given a specific type of edge purchase.

First we examine what may happen to your own equilibrium wealth when purchasing an edge. It is clear that your equilibrium wealth cannot go down, because you still have the option to use your old exchange subgraph (refer to Kakade et al., 2004b for definition) when engaging in trade. It is possible that your equilibrium wealth will remain the same, either from using the same exchange subgraph or switching to a new one. Obviously, it is possible that your equilibrium wealth will increase if you buy an edge - this is your motivation for buying. However, a subtle characteristic of wealth enhancement through buying edges exists here. Consider the result of Even-Dar et al. (2007), that in a network where edges are bought, there are sharp limits on wealth variation. Thus, one might naively think that if you are already rich, you would not be able to increase it. This is not necessarily the case, as figure 1 shows. Of course, in order to increase your wealth in this manner, you have to “poach” the connections of a consumer even richer than you, so overall the axiom that the graph is moving toward equality holds. Finally, by argument from symmetry, this analysis applies to the target’s wealth as well - it matters not who the consumer is who is actually doing the buying.

Analyzing the consequences of an edge purchase from the perspective of a consumer that was not involved in the transaction is a little more interesting. Your wealth can go down, obviously; if you were rich, you might expect this to happen. However, your wealth can go down even if you are poor (also figure 1). In many cases, you will be unaffected by the transaction and your wealth will remain the same. Finally, your wealth may increase, but only if you are poor - normal and rich edges cannot benefit from edge transactions that do not involve them.

Figure 1: The graph indicates how a rich consumer may increase his wealth through “poaching”. Note that, simultaneously, the poor consumers connected to the buying rich consumer become even poorer after the rich consumer makes his purchase.
Basic Simulation of the Network

In order to get a better idea of the distribution of events that occur during edge purchasing, I first ran some trials on a simulated network, written in Python. The charts below depict the results of 500 test runs of the simulation, where a network is set up according to the preferential attachment model. A random vertex is chosen to be an edge buyer, and then it chooses a random edge out of the possible edges it is allowed to purchase. The simulation then collects statistics about how valuable the purchase turned out to be for the buyer, which I then transformed into the charts below (please see the caption for more information).

Figure 2: The left graph demonstrates the expected value a player with degree \( x \) that is buying a random edge will enjoy. As is apparent, it is more beneficial for players with low degree to purchase edges; in fact, in this run of the simulation it was never beneficial for a player with more than 4 edges to purchase another one. The right graph demonstrates the expected value a player that is buying a random edge will enjoy, given that the edge is to a vertex with degree \( x \) (note the difference from the left graph). The results, however, are similar to the left graph, and taken together give a rough indication of the expected value a player can enjoy by buying an edge, given these circumstances.

Building a Model to Approximate the Equilibrium

Obviously, the simulation described above cannot give us a detailed enough picture of the network to truly understand its properties, because it doesn’t model the game very accurately; only one vertex is allowed to purchase edges, and it purchases edges randomly. In order to understand the game better, I needed to develop a model that would allow us to understand the equilibrium conditions of the game. Solving for the exact equilibrium would obviously be the most desirable solution, but due to the complexity of the game this would have been extremely difficult, and perhaps unnecessary given that we could approximate the equilibrium reasonably well using other means, saving time both computationally and in the real world. The approximation model I developed uses artificial intelligence to determine each player’s move within a particular game, attempting to make the optimal decision for the player given what it believes other players will do. In particular, I attempted to model the game using both adversarial search and reinforcement learning; it turned out that reinforcement
learning was the better choice. The process of building the agent was quite complex, so I will not detail it here; for a complete discussion of this aspect of the project, please refer to my project write-up, “An Analysis of Trade on Complex Networks.”

After I had programmed the reinforcement learning agents to play this game, I ran 10,000 trials of the game, varying parameters such as the network size and the cost to purchase an edge, and collecting data from the trials for analysis (the results of which are detailed below). I kept the network formation model the same in all the trials, specifically preferential attachment with $\alpha = 0, \nu = 1$ as described above; I found this to be the the most interesting network to study because the starting inequality was large but the opportunity for wealth equalization was also large. It would not have been interesting, of course, to study this game on networks where everyone started off with the same wealth, nor would it have been interesting to examine networks where there was a great deal of inequality but it was too difficult for players to improve their equilibrium wealths by purchasing edges.

### Analysis of Player Behavior

An analysis of the game can intuitively be divided into two categories; an analysis of player behavior and an analysis of the outcomes of that behavior, on both a network-wide and individual level. Thus, it seemed logical to first discuss the player behavior that I observed and then look at the effects this behavior had. As far as player behavior, the most pressing question to ask seemed to me to be, how many edges did a player decide to purchase, and how was this decision affected by the player’s status in the network? To analyze this question, I derived a histogram for the purchasing decisions of each buyer in all 10,000 trials, with the bins corresponding to the number of edges the buyer purchased. The results can be seen in figure 3.

![Figure 3](image)

**Figure 3:** These graphs indicate the relative frequency of the purchasing decisions of players, with the x-axis being the number of edges purchased. The left graph divides all buyers into either high wealth (defined to be having equilibrium wealth greater or equal to 1 at the beginning of the game) or low wealth. The right graph divides all buyers into either high cost (defined to be participating in a game where the cost of an edge is greater than 0.1) and low cost. Thus, the left graph buckets players based on a personal characteristic, whereas the right one buckets them based on a characteristic of the trial they are participating in.
A number of interesting conclusions can be derived from the data displayed in figure 3. First of all, no matter what the edge cost or their beginning equilibrium wealth, players barely ever purchase more than one edge, even though they were allowed to purchase up to two edges. This is not because they did not have a second edge available to purchase; in the majority of the trials, there were at least three choices of edges a player had available to choose from. Thus, the data seem to indicate that just buying one edge, combined with the effects of the rest of the network making their purchases, is enough to achieve the desired increase in equilibrium wealth (or, if you are rich, to mitigate the decrease you will experience from others’ purchasing decisions).

Another interesting observation is that the majority of players, no matter what their wealth or edge cost, choose to buy an edge. This seems to reinforce our discussion from earlier, that although edge buying naturally helps the poorest players the most, even richer players may derive some benefit from purchasing an edge. The breakdown of purchasing decisions between high-income and low-income is not surprising; low-income players purchase edges more often because they tend to benefit more from the purchases. Likewise, players naturally buy more edges when the edge cost is lower, as the expected benefit outweighs the small cost that one must pay for the edge. I must say I was surprised to see that virtually all players declined to purchase a second edge, even with an extremely low price. This seems to reinforce the notion that players do not derive any additional benefit from a second purchased edge, or if they do it is extremely small and not enough to outweigh even a small expenditure for the second edge.

As I was intrigued by the relationship between edge cost and player welfare, I decided to investigate it further. Specifically, I wanted to find out if there was an edge cost at which the marginal benefit to the entire network of buying an edge equalled the marginal cost. From one perspective, any edge buying at all reduces the amount of wealth present in the network, because all edge buying does is redistribute wealth, not create it. However, as we saw in the “Milks and Wheats” behavioral experiments which this game is based off of, social welfare actually increases as network equality increases because money is left on the table when the network is highly unequal in terms of the distribution of equilibrium prices. Thus, as players purchase edges which, as we saw earlier in the paper, can only reduce the network’s inequality, they increase the social welfare. Thus, I wanted to find the point at which the two effects were equivalent. My first thought was to do a binary search on edge costs between 0 and 1, so if I ran a series of trials and the combined money spent on edges was greater than the combined increase in social wealth, I would decrease the edge cost, and it it was the other way around I would increase the edge cost and run another series of trials. This turned out not to work because players cut back on their purchasing as the edge cost grew, so the function was not monotonic and thus a binary search would not have worked. Thus, I simply increased the edge cost from zero until the marginal
cost equaled the marginal benefit, incrementing in steps of 0.005 and running 500 trials for each edge cost. The results are depicted in figure 4.

Figure 4: On this graph, the y-axis represents the aggregate utility gained and the aggregate money spent on edges, both measured in units of endowment; each figure is averaged over a total of 500 trials. The x-axis represents the cost of a single edge. The money spent appears to roughly balance the utility gained at a cost of around 0.2.

As the chart demonstrates, the utility gained stays roughly constant as the edge cost increases, and the money spent increases proportionally with the edge cost. The reason the utility gained is constant is because it is not taking into account the money spent, just the increase in social welfare given the reduced network inequality (if it had taken money spent into account, we just would have looked to see where the function crossed zero). That the money spent increases proportionally with the edge cost seems to indicate that players are willing to buy a certain amount of edges at any cost, up to a point (although this graph does not detail it, the amount of money spent actually levels off as the cost approaches 0.3). Thus, from this figure we can conclude that, to encourage maximum social welfare, the price of an edge should be no more than a fifth of the initial endowments of the players in the network. One could imagine applying this to a real world situation, such as in a developing country where an aid agency might need to make a decision to subsidize building roads to establish new trade routes if the cost is too high to promote the social welfare.

Analysis of the Effects of Player Behavior

After analyzing the behaviors that players in this game undertake in our approximate equilibrium, I turned my attention to the effects those behaviors had. Specifically, I looked at two effects: the effects of actions on individual players, and the effects on
the network structure as a whole. In doing this, I was greatly aided by the add-on module Hippodraw, which attached seamlessly to my Python code and allowed me to efficiently build the figures which I present below.

Before we begin the analysis, a note about the figures. For most of them, the histograms appear to have discontinuities where one might expect a smooth distribution. This is not due to having run too few trials (10,000 were run) but due to the nature of the network game, where equilibrium wealths are often whole numbers and inverses of whole numbers with none or few equilibrium wealths in between.

Figure 5: The above figures give varying perspectives on the social wealth of networks before and after the game. The figures are histograms of the data, with the y-axis representing the number of networks observed to fall into each bin and the x-axis representing the bins, each denoting either social wealth or change in social wealth depending on the graphs. As a reminder, social wealth is calculated as a function of the standard deviation of the network’s equilibrium wealths, with the function being interpolated from the data collected from the “Milks and Wheats” behavioral experiments (Kearns and Judd, 2008).

I first examined the effect of edge-purchasing behavior on the networks as a whole, the results of which can be seen in figure 5. The top left graph represents the distribution of social wealth at the beginning of games. Because the preferential attachment model naturally tends toward creating unequal distributions, it is no surprise that the majority of these games started off with a level of social wealth that was significantly below the optimum level of 1. Of course, since the networks are randomly generated, a small number of networks started out at full equality as can be seen by the short bar on the far right of the graph. At the end of the game, as can be seen by the top right graph, wealth has predictably shifted toward equality, with a plurality of games ending up at full wealth equality; this effect is not unexpected considering the incentive of poor players to purchase edges in order to alleviate their own poverty.
An important distinction is that the majority of games played do not end up with full network equality, where the cost of an edge outweighed the benefit gained from its purchase. This reinforces the conclusion above that the edge cost must be carefully set if network equality is to be cultivated.

The bottom three graphs in Figure 5 give us an idea about how social wealth changed during the games. The left graph represents the change in social wealth for all games, the middle graph represents the change for the 50% of games that started off most equal, and the right graph represents the change for the 50% of games that started off least equal. As can be seen from the left graph, almost all games experienced some improvement, except for the games that started off as equal (this can be deduced from the fact that the leftmost bar in this graph matches up with the rightmost bar in the upper left graph). This means that even when edge costs were high, some players found it beneficial to purchase edges. The lower middle and lower right graphs serve to disprove a null hypothesis, that social wealth increases at about the same rate no matter how the network starts off. It is clear that unequal networks benefit more, in terms of social wealth increase, through edge-purchasing activity.

![Figure 6:](image)

Figure 6: The above figures demonstrate how equilibrium wealth changed from the start of the game to the end as a function of one’s starting equilibrium wealth. The figures are histograms of the data, so the y-axis represents the number of players observed to fall into each bin and the x-axis represents the bins, each denoting a level of change in equilibrium wealth.

I next examined how a player’s equilibrium wealth was affected as a result of edge-purchasing actions during the game. To do this, I divided all players into five categories based on their starting equilibrium wealth and ranging from richest (with wealths above 2 units) to poorest (with wealths less than 0.5 units). As Figure
6 demonstrates, edge-purchasing activity had radically different effects on players depending on their wealth. The graphs are ordered by starting wealth, ranging from richest at the top left going across, and then down to poorest at the bottom right. For players with starting wealths greater than 2, the vast majority lost 1 unit of wealth (which means they didn’t regress all the way to 1, but still are richer than the average). This does not disagree with our analysis above, because as we saw the majority of the games do not go to full equality; this is especially the case for networks that start off the most unequal, which tends to be the case for networks on which these players exist. Likewise for the second richest class of players, a significant portion of them do not get dragged down to equality. Players with average wealth tend to stay average, although as we have seen it is possible for them to gain wealth through buying or lose wealth through others’ buying, and this does in fact happen occasionally. Poor players tend to fare well, with only a small fraction of them not gaining any wealth and none of them at all losing wealth; this seems to indicate that the “poaching” strategy described at the beginning of the paper is at best an attempt by rich players to ward off losing wealth and at worst completely ineffective (future investigators of this game may find it interesting to examine which is the case).

Next, I decided to examine change in wealth from a different perspective, which was how many edges a player purchased. Figure 6 details the results, with the top left graph representing change in wealth given zero edges purchased, the middle graph one edge, and the left graph two edges. Because of the extreme rarity of a player purchasing two edges (looking at the y-axis values for that histogram indicates that the sample size is small), we will not try to draw any conclusions from that graph but simply look at the other two. Players purchasing zero edges tended to lose wealth, although a significant minority stayed the same and some gained wealth through others’ purchases. The direction of causality is not apparent from this graph; did players lose wealth because they didn’t buy, or did they not buy because it was not worth it and losing wealth was a foregone conclusion? Other graphs seem to indicate that it is the latter; richer players both tended to buy fewer edges and lose wealth, and since our simulation approximates the equilibrium we can conclude that this was

![Figure 7](image-url): This figure examines change in wealth from a player’s perspective in the same way that Figure 6 does. The only difference is that in this one, players are divided up into the three graphs based on how many edges they purchased: 0, 1, or 2.
the optimal thing to do and buying an edge would not have helped them out. As far as players who bought one edge (middle graph), they generally benefitted moderately or did not benefit at all; and for the same reasons we conclude that even when they did not benefit directly from purchasing the edge, it was the case that they would have lost wealth had they not purchased it.

Figure 8: This figure again examines change in wealth from a player’s perspective, like Figures 6 and 7. The only difference is that this time, it divides up players into groups based on starting wealth as well as the edge cost of the game in which they were participating.

This last set of graphs in Figure 8 was formulated specifically to see if a null hypothesis would be disproven, that players with high initial wealth do not benefit from having high edge costs, the alternative hypothesis is that they do benefit because it inhibits poorer players from buying and thus evening out the wealth. As the upper and lower left graphs demonstrate, there is a some degree of truth to the alternative hypothesis, because the number of rich players maintaining their wealth increases by about 35% when edge costs are high. As might be expected, players with average wealth do not seem to benefit from increased edge costs, nor are they hurt by it. Finally, it is difficult to determine the effect of higher edge costs on the poor class of players. This is not inconsistent with the analysis for rich players, however, because poor players are much more numerous than rich players just by the mathematics of the game (which by the way seems to mirror real-world wealth distribution in the world at large). Thus, if a cause has an effect on one class of players and an opposite effect on another class of players, we would expect to see the smaller class of players (the rich players) exhibit a larger response per player.
Conclusion

As we have demonstrated in this paper, the network game described here can be an effective catalyst for reducing network inequality while not fundamentally altering most of the underlying characteristics of the network. Through examining the incentive structure of the players to purchase edges, we have outlined a general strategy of play for players of different starting wealths and different network characteristics, and seen how these strategies end up affecting other players and the wealth distribution of the network as a whole. In my project write-up, “An Analysis of Trade on Complex Networks,” I put these results into action by building a heuristic agent meant to model how a real person might play the game given the knowledge contained in this paper. My goal was to develop an agent that was even faster than the reinforcement learning agent which ran the trials and generated the results described here, while achieving performance that was almost as good. Of course, the potential applications of this analysis do not end there. As we have demonstrated, with this model’s applicability to real-world situations such as the decision to subsidize new trade routes, it is clear that the study of network games such as this one will continue to be a rewarding and promising area of research at the intersection of computer science and economics.
If it were the case, this is encouraging for someone who is concerned about the problem of relieving network-based inequality, because it means that

[Kakade et al. 2004a]

[Gale 1960]

[Even-Dar et al. 2007]

[Kakade et al. 2004b]

[Kearns and Judd 2008]
References


