CSE410/CIS510 – Curves and Surfaces: Theory and Applications
Fall 2005, 303 Towne, TR 10:30AM-12PM

Homework Assignment 1

Due: Wednesday, September 28, 2005, by 10.30 AM (IN CLASS)

1. [**20 points**]

   Let V be a vector space over the reals. What does it take for V to be called an *Euclidean* vector space?

2. [**20 points**]

   Let V be the vector space whose vectors are all pairs $(u_1, u_2) \in \mathbb{R}^2$ and the addition and scalar multiplication operations are the standard addition and scalar multiplication on such pairs: $\forall (u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$ and $\forall \alpha \in \mathbb{R}$, we define $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$ and $\alpha \cdot (u_1, u_2) = (\alpha u_1, \alpha u_2)$. Now, suppose we define a function $\varphi : V \times V \to \mathbb{R}$ such that for any two vectors (u_1, u_2) and (v_1, v_2) in V, we have that $\varphi((u_1, u_2), (v_1, v_2)) = u_1 + v_1 + u_2 + v_2$. Is φ an inner product? Justify your answer.

3. [**20 points**]

   Let $\mathbb{A}^2$ be the affine space $\langle \mathbb{R}^2, \mathbb{R}^2, + \rangle$, where $+$ is the standard addition on tuples of $\mathbb{R}^2$. Suppose that we are given four points p_1, p_2, p_3, and p_4 of $\mathbb{A}^2$ such that p_1, p_2, p_3, and p_4 are not all collinear. Since these points are not all collinear, any point $p \in \mathbb{A}^2$ can be expressed as an affine combination, $p = \sum_{i=1}^{4} \alpha_i p_i$, of $p_1, \ldots, p_4$ and some scalars $\alpha_1, \ldots, \alpha_4 \in \mathbb{R}$. If we are given the coordinates (x_i, y_i) of p_i, for each $i = 1, 2, 3, 4$, as well as the coordinates (x, y) of p, with respect to some affine frame of $\mathbb{A}^2$, how can we compute the scalars $\alpha_1, \ldots, \alpha_4$ such that $p = \sum_{i=1}^{4} \alpha_i p_i$? Are these scalars unique? Justify your answer.

4. [**40 points**]

   For this problem, you will develop a short procedure to compute the barycentric coordinates of a point in $\mathbb{A}^2$, which is expressed as an affine combination of the vertices of a *star polygon*. Your procedure will take as input the coordinates of a set of n ($n \geq 3$) points $p_1, \ldots, p_n$ in $\mathbb{A}^2$ with respect to the standard affine frame $(O, (\vec{v_1}, \vec{v_2})) = ((0,0), \{(1,0), (0,1)\})$ of $\mathbb{A}^2$. These points are supposed to form the vertices of a polygon in $\mathbb{A}^2$ that contains the origin $O = (0,0)$ of $((0,0), ((1,0), (0,1)))$ in its *interior*. Furthermore, any line segment from O to a vertex p_i ($i = 1, \ldots, n$) of the polygon must be entirely contained in the polygon. Your code need not verify any of these two conditions. You can assume that the input points will always satisfy both conditions. The output of your procedure is *a* set $\{\alpha_1, \ldots, \alpha_n\}$ of (barycentric) coordinates of the origin $O = (0,0)$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $O = \sum_{i=1}^{n} \alpha_i p_i$. To compute $\alpha_1, \ldots, \alpha_n$, you will use a very nice technique, called *mean value coordinates*, as illustrated by Figure 1.

To compute α_i, we need to compute the angles θ_{i-1} and θ_i determined by the vectors $\vec{Op_{i-1}}$ and $\vec{Op_i}$, and $\vec{Op_i}$ and $\vec{Op_{i+1}}$, respectively, for each $i = 2, \ldots, n$. Next, we compute the following scalars:

$$w_i = \frac{\tan(\theta_{i-1}/2) + \tan(\theta_i/2)}{\|\vec{Op_i}\|},$$

where $\tan(\theta_{i-1}/2)$ and $\tan(\theta_i/2)$ are the tangents of the angles $\dfrac{\theta_{i-1}}{2}$ and $\dfrac{\theta_i}{2}$, respectively, and $\|\vec{Op_i}\|$ is the length of the vector $\vec{Op_i}$. After computing w_i for each $i = 1, \ldots, n$, we compute α_i as follows:

$$\alpha_i = \frac{w_i}{\sum_{j=1}^{n} w_j}.$$

Note that you have to figure out how to compute the angle between two vectors and the norm of a vector. By looking at your slides, you will see that both computations can be done by using the dot product. Recall that the output of your procedure is just the barycentric coordinates $\alpha_1, \ldots, \alpha_n$. You can use the following points as test case: $(3, 0)$, $(\frac{3}{2}, 1)$, $(2, 0)$, $(-4, \frac{1}{2})$, $(-2, 0)$, $(-3, -2)$, and $(2, -2)$. The polygon defined by these points satisfies the two conditions mentioned before.

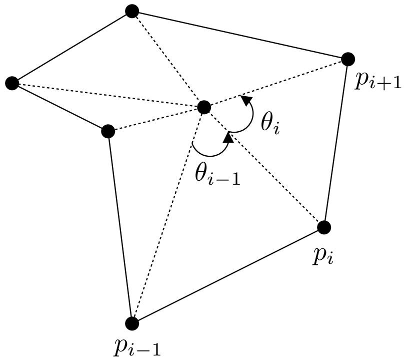

Figure 1: Mean value coordinates.

Some remarks. There are two very interesting issues related to mean value coordinates. First, it is the fact that $\tan(\theta_{i-1}/2)$ and $\tan(\theta_i/2)$ are *well-defined*; that is, since O is in the interior of the polygon defined by $p_1, \ldots, p_n$, and since each line segment whose endpoints are O and p_i, for $i = 1, \ldots, n$, is entirely contained in the polygon, the angle θ_j is less than 180^o, for every $j = 1, \ldots, n$. So, $\tan(\theta_j/2) \neq \infty$, and we can compute w_j and α_j, for each $j = 1, \ldots, n$. Second, since O is inside the polygon formed by $p_0, \ldots, p_n$, the point O is also inside the convex hull of $p_0, \ldots, p_n$. This means that we can always find some $\alpha_1, \ldots, \alpha_n$ that are nonnegative, i.e., $\sum_{i=1}^{n} \alpha_i p_i$ can be a convex combination of $p_1, \ldots, p_n$. It turns out that mean value coordinates always provides us with nonnegative $\alpha_1, \ldots, \alpha_n$, and this is the main feature of this technique. We will use mean values coordinates in the end of the course when we study polygonal surface parametrizations. So, keep your procedure, as you will need it later.