5

Hilbert Spaces

Our previous discussions have been concerned with algebra. The representation of systems (quantities and their interrelations) by abstract symbols has forced us to distill out the most significant and fundamental properties of these systems. We have been able to carry our exploration much deeper for linear systems, in most cases decomposing the system models into sets of uncoupled scalar equations.

Our attention now turns to the geometric notions of length and angle. These concepts, which are fundamental to measurement and comparison of vectors, complete the analogy between general vector spaces and the physical three-dimensional space with which we are familiar. Then our intuition concerning the size and shape of objects provides us with valuable insight. The definition of length gives rigorous meaning to our previous heuristic discussions of an infinite sequence of vectors as a basis for an infinite-dimensional space. Length is also one of the most widely used optimization criteria. We explore this application of the concept of length in Chapter 6. The definition of orthogonality (or angle) allows us to carry even further our discussion of system decomposition. To this point, determination of the coordinates of a vector relative to a particular basis has required solution of a set of simultaneous equations. With orthogonal bases, each coordinate can be obtained independently, a much simpler process conceptually and, in some instances, computationally.

5.1 Inner Products

The dot product concept is familiar from analytic geometry. If $\mathbf{x} = (\xi_1, \xi_2)$ and $\mathbf{y} = (\eta_1, \eta_2)$ are two vectors from $\mathcal{R}^2$, the dot product $\mathbf{x} \cdot \mathbf{y}$ between $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathbf{x} \cdot \mathbf{y} \triangleq \xi_1\eta_1 + \xi_2\eta_2 \tag{5.1}$$

The length $\|\mathbf{x}\|$ of the vector $\mathbf{x}$ is defined by

$$\|\mathbf{x}\| \triangleq \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\xi_1^2 + \xi_2^2} \tag{5.2}$$

The angle between the vectors $\mathbf{x}$ and $\mathbf{y}$ is defined in terms of the dot product between the normalized vectors:

$$\cos\phi = \frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot \frac{\mathbf{y}}{\|\mathbf{y}\|} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|} \tag{5.3}$$

Example 1. The Dot Product in $\mathcal{R}^2$. Let $\mathbf{x} = (1,1)$ and $\mathbf{y} = (2,0)$. Then $\mathbf{x} \cdot \mathbf{y} = 2$, $\|\mathbf{x}\| = \sqrt{2}$, $\|\mathbf{y}\| = 2$, and $\cos\phi = 1/\sqrt{2}$ (or $\phi = 45°$)). Figure 5.1 is an arrow space equivalent of this example.

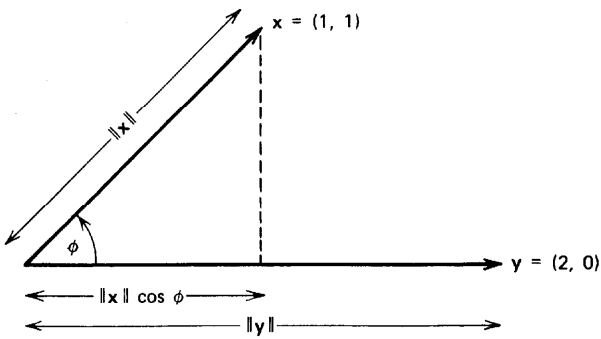


Figure 5.1. Arrow vectors corresponding to Example 1.

It is apparent from Example 1 that (5.3) can be interpreted, in terms of the natural correspondence to arrow space, as a definition of the dot product (as a function of the angle between the vectors):

$$\mathbf{x} \cdot \mathbf{y} \stackrel{\Delta}{=} \|\mathbf{y}\|(\|\mathbf{x}\| \cos\phi) \tag{5.4}$$

where $\|\mathbf{x}\| \cos\phi$ is the length of the projection of $\mathbf{x}$ on $\mathbf{y}$ along the perpendicular to $\mathbf{y}$. The following properties of the dot product seem fundamental:

1. Length is non-negative; that is,

$$\mathbf{x} \cdot \mathbf{x} \geqslant 0, \quad \text{with equality if and only if } \mathbf{x} = \boldsymbol{\theta}$$

2. The magnitude of ϕ (or $\cos\phi$) is independent of the order of $\mathbf{x}$ and $\mathbf{y}$; that is,

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$$

3. The length of $c\mathbf{x}$ equals $|c|$ times the length of $\mathbf{x}$, for any scalar c; that is,

$$c\mathbf{x} \cdot c\mathbf{x} = c^2(\mathbf{x} \cdot \mathbf{x})$$

4. In order that (5.4) be consistent with the rules for addition of vectors, the dot product must be distributive over addition (see Figure 5.2); that is,

$$(\mathbf{x}_1 + \mathbf{x}_2) \cdot \mathbf{y} = \mathbf{x}_1 \cdot \mathbf{y} + \mathbf{x}_2 \cdot \mathbf{y}$$

We now extend the dot product to arbitrary vector spaces with real or complex scalars in a manner which preserves these four properties.

Definition. An **inner product** (or **scalar product**) on a real or complex vector space $\mathcal{V}$ is a scalar-valued function $\langle \mathbf{x}, \mathbf{y} \rangle$ of the ordered pair of vectors $\mathbf{x}$ and $\mathbf{y}$ such that:

1. $\langle \mathbf{x}, \mathbf{x} \rangle \geqslant 0$, with equality if and only if $\mathbf{x} = \boldsymbol{\theta}$
2. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ (the bar denotes complex conjugation).
3. $\langle c_1\mathbf{x}_1 + c_2\mathbf{x}_2, \mathbf{y} \rangle = c_1\langle \mathbf{x}_1, \mathbf{y} \rangle + c_2\langle \mathbf{x}_2, \mathbf{y} \rangle$

It follows that $\langle \mathbf{y}, c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \rangle = \bar{c}_1\langle \mathbf{y}, \mathbf{x}_1 \rangle + \bar{c}_2\langle \mathbf{y}, \mathbf{x}_2 \rangle$. We describe these properties by saying that an inner product must be (1) **positive definite,** (2) **hermitian symmetric,** and (3) **conjugate bilinear.** Note that because of (2), $\langle \mathbf{x}, \mathbf{x} \rangle$ is necessarily real, and the inequality (1) makes sense. If the scalars are real, the complex conjugation bar is superfluous.

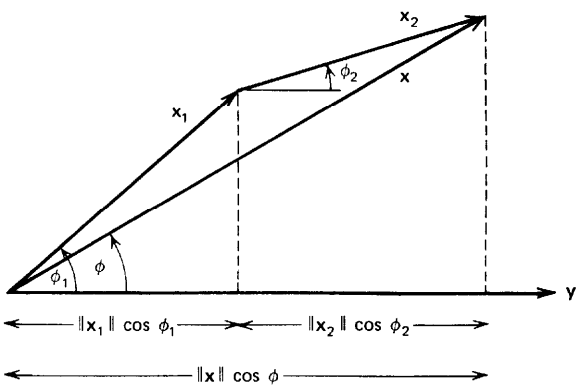


Figure 5.2. Dot products are distributive over addition.

We define the **norm** (or length) of $\mathbf{x}$ by

$$\|\mathbf{x}\| \triangleq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \tag{5.5}$$

When $\langle \mathbf{x}, \mathbf{y} \rangle$ is real, we can define the angle ϕ between $\mathbf{x}$ and $\mathbf{y}$ by

$$\cos\phi \triangleq \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|}$$

Practically speaking, we are interested in the angle ϕ only in the following two cases:

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ ($\mathbf{x}$ and $\mathbf{y}$ are said to be \textbf{orthogonal})} \tag{5.6}$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \pm \|\mathbf{x}\| \, \|\mathbf{y}\| \text{ ($\mathbf{x}$ and $\mathbf{y}$ are said to be \textbf{collinear})} \tag{5.7}$$

Example 2. The Standard Inner Product for $\mathcal{C}^n$ and $\mathcal{R}^n$. The standard inner product for $\mathcal{C}^n$ (and $\mathcal{R}^n$) is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \sum_{i=1}^{n} \xi_i \bar{\eta}_i \tag{5.8}$$

where ξ_i and η_i are the elements of $\mathbf{x}$ and $\mathbf{y}$, respectively. Of course, the complex conjugate bar is superfluous for $\mathcal{R}^n$. This inner product is simply the extension of the dot product to complex spaces and n dimensions. Consider the vector (i) in $\mathcal{C}^1$;

$$\|(i)\| = \sqrt{(i)(\bar{i})} = 1$$

The complex conjugation in (5.8) is needed in order to keep lengths non-negative for complex scalars.

Example 3. The Standard Inner Product for $\mathcal{M}_c^{n \times 1}$ and $\mathcal{M}^{n \times 1}$. The standard inner product for $\mathcal{M}_c^{n \times 1}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \bar{\mathbf{y}}^T \mathbf{x} \tag{5.9}$$

Again, if only real scalars are involved, the conjugate is unnecessary. For instance, if $\mathbf{x} = (1 \ 2 \ 4)^T$ and $\mathbf{y} = (-1 \ 3 \ 2)^T$ in $\mathcal{M}^{3 \times 1}$, then, by (5.9),

$$\langle \mathbf{x}, \mathbf{y} \rangle = (-1 \ 3 \ 2) \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = 13$$

Example 4. The Standard Inner Product for Function Spaces. The standard inner

product for a function space such as $\mathcal{P}(a, b)$ or $\mathcal{C}(a, b)$ is defined by

$$\langle \mathbf{f}, \mathbf{g} \rangle \triangleq \int_a^b \mathbf{f}(t)\, \mathbf{g}(t)\, dt \tag{5.10}$$

for each $\mathbf{f}$ and $\mathbf{g}$ in the space. We usually deal only with real functions and ignore the complex conjugation. Consider the function $\mathbf{f}(t) = 1$ in $\mathcal{C}(0, 1)$:

$$\|\mathbf{f}\| = \sqrt{\int_0^1 (1)^2\, dt} = 1$$

Any vector whose average value over the interval [0, l] is zero is orthogonal to $\mathbf{f}$; for then $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 (1)\mathbf{g}(t)\, dt = 0$. We easily verify, for the case of continuous functions and real scalars, that (5.10) possesses the properties of an inner product; by the properties of integrals:

(*a*) $\langle \mathbf{f}, \mathbf{f} \rangle = \int_a^b \mathbf{f}^2(t)\, dt \geqslant 0$, with equality if and only if $\mathbf{f}(t) = 0$ for all t in [a,b];

(*b*) $\int_a^b \mathbf{f}(t)\mathbf{g}(t)\, dt = \int_a^b \mathbf{g}(t)\mathbf{f}(t)\, dt$

(*c*) $\int_a^b [c_1\mathbf{f}_1(t) + c_2\mathbf{f}_2(t)]\mathbf{g}(t)\, dt = c_1 \int_a^b \mathbf{f}_1(t)\mathbf{g}(t)\, dt + c_2 \int_a^b \mathbf{f}_2(t)\mathbf{g}(t)\, dt$

Example 5. The Standard Inner Product for a Space of Two-Dimensional Functions. Let $\mathcal{C}^2(\Omega)$ denote the space of functions which are twice continuously differentiable over a two-dimensional region Ω. We define an inner product for $\mathcal{C}^2(\Omega)$ by

$$\langle \mathbf{f}, \mathbf{g} \rangle \triangleq \int_\Omega \mathbf{f}(\mathbf{p})\mathbf{g}(\mathbf{p})\, d\mathbf{p} \tag{5.11}$$

where $\mathbf{p} = (s, t)$, an arbitrary point in Ω.

An inner product assigns a real number (or norm) to each vector in the space. The norm provides a simple means for comparing vectors in applications. Example 1 of Section 3.4 is concerned with the state (or position and velocity) of a motor shaft in the state space $\mathfrak{M}^{2\times 1}$. In a particular application we might require both the position and velocity to approach given values, say, zero. As a simple measure of the nearness of the state to the desired position (θ), we use the norm corresponding to (5.9):

$$\|\mathbf{x}(t)\| = \sqrt{\xi_1^2 + \xi_2^2}$$

where ξ_1 and ξ_2 are the angular position and velocity of the motor shaft at instant *t*. However, there is no inherent reason why position and velocity should be equally important. We might be satisfied if the velocity stayed large as long as the position of the shaft approached the target position

$\xi_1 = 0$. In this case, some other measure of the performance of the system would be more appropriate. The following measure weights ξ_1 more heavily than ξ_2.

$$\|\mathbf{x}(t)\| = \sqrt{100\xi_1^2 + \xi_2^2}$$

This new measure is just the norm associated with the following weighted inner product for $\mathcal{R}^{2\times 1}$:

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq 100\xi_1\eta_1 + \xi_2\eta_2$$
$$= \mathbf{y}^T \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}$$

where $\mathbf{x} = (\xi_1\ \xi_2)^T$ and $\mathbf{y} = (\eta_1\ \eta_2)^T$. We generally select that inner product which is most appropriate to the purpose for which it is to be used.

Example 6. A Weighted Inner Product for Function Spaces. An inner product of the following form is often appropriate for such spaces as $\mathcal{P}(a, b)$ and $\mathcal{C}(a, b)$:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b \omega(t)\mathbf{f}(t)\,\overline{\mathbf{g}(t)}\,dt, \qquad 0 < \omega(t) < \infty \tag{5.12}$$

If the weight function is $\omega(t) = 1$, (5.12) reduces to the standard inner product (5.10). The weight $\omega(t) = e^t$ might be used to emphasize the values of functions for large t and deemphasize the values for t small or negative.

Example 7, A Weighted Inner Product for $\mathcal{R}^2$. Let $\mathbf{x} = (\xi_1, \xi_2)$ and $\mathbf{y} = (\eta_1, \eta_2)$ be arbitrary vectors in $\mathcal{R}^2$. Define the inner product on $\mathcal{R}^2$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \tfrac{1}{4}\xi_1\eta_1 - \tfrac{1}{4}\xi_1\eta_2 - \tfrac{1}{4}\xi_2\eta_1 + \tfrac{5}{4}\xi_2\eta_2 \tag{5.13}$$

We apply this inner product to the vectors $\mathbf{x} = (1,1)$ and $\mathbf{y} = (2,0)$, the same vectors to which we previously applied the standard (or dot) inner product: $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, $\|\mathbf{x}\| = 1$, and $\|\mathbf{y}\| = 1$. The same vectors which previously were displaced by 45° (Figure 5.1) are, by definition (5.13), orthogonal and of unit length. We see that (5.13) satisfies the properties required of an inner product:

1. By completing the square, we find

$$\langle \mathbf{x}, \mathbf{x} \rangle = \tfrac{1}{4}(\xi_1 - \xi_2)^2 + \xi_2^2 \geqslant 0$$

with equality if and only if $\xi_1 = \xi_2 = 0$;

2. Since the coefficients for the cross-product terms are equal,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$$

3. We rewrite (5.13) as

$$\langle \mathbf{x}, \mathbf{y} \rangle = (\eta_1 \ \eta_2) \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \triangleq \mathbf{y}^T \mathbf{Q} \mathbf{x}$$

Then, by the linearity of matrix multiplication,

$$\begin{aligned} \langle c_1\mathbf{x}_1 + c_2\mathbf{x}_2, \mathbf{y} \rangle &= \mathbf{y}^T\mathbf{Q}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) \\ &= c_1\mathbf{y}^T\mathbf{Q}\mathbf{x}_1 + c_2\mathbf{y}^T\mathbf{Q}\mathbf{x}_2 \\ &= c_1\langle \mathbf{x}_1, \mathbf{y} \rangle + c_2\langle \mathbf{x}_2, \mathbf{y} \rangle \end{aligned}$$

The last two examples suggest that we have considerable freedom in picking inner products. Length and orthogonality are, to a great extent, what we define them to be. Only if we use standard inner products in $\mathcal{R}^3$ do length and orthogonality correspond to physical length and 90° angles. Surprisingly, the concept suggested by (5.4) still holds in Example 7: $|\langle \mathbf{x}, \mathbf{y} \rangle|$ is the product of $\|\mathbf{y}\|$ and the norm of the projection of $\mathbf{x}$ on $\mathbf{y}$ along the direction orthogonal [in the sense of (5.13)] to $\mathbf{y}$. The sign of $\langle \mathbf{x}, \mathbf{y} \rangle$ is positive if the projection of $\mathbf{x}$ on $\mathbf{y}$ is in the same direction as $\mathbf{y}$; if the projection is in the opposite direction, the sign is negative.

Exercise 1. Let $\mathbf{x} = (0,1)$ and $\mathbf{y} = (1,0)$ in $\mathcal{R}^2$. Define the inner product in $\mathcal{R}^2$ by (5.13). Show that the projection of $\mathbf{x}$ on $\mathbf{y}$ along the direction orthogonal to $\mathbf{y}$ is the vector $(-1,0)$. Verify that $\langle \mathbf{x}, \mathbf{y} \rangle$ is correctly determined by the above rule which uses the projection of $\mathbf{x}$ on $\mathbf{y}$.

An **inner product space** (or **pre-Hilbert space)** is a vector space on which a particular inner product is defined. A *real* inner product space is called a **Euclidean space.** A **unitary space** is an inner product space for which the scalars are the complex numbers. We will often employ the symbols $\mathcal{R}^n$ and $\mathcal{M}^{n\times 1}$ to represent the Euclidean spaces consisting of the real vector spaces $\mathcal{R}^n$ and $\mathcal{M}^{n\times 1}$ together with the standard inner products (5.8) and (5.9), respectively. Similarly, we use $\mathcal{P}(a, b)$, $\mathcal{C}(a, b)$, etc. to represent real Euclidean function spaces which make use of the standard inner product (5.10). Whereever we use a different (nonstandard) inner product, we mention it explicitly.

Matrices of Inner Products

To this point, we have not used the concept of a basis in our discussion of inner products. There is no particular basis inherent in any inner product space, although we will find some bases more convenient than others. We found in Chapter 2 that by picking a basis $\mathcal{X}$ for an n-dimensional space

$\mathcal{V}$ we can represent vectors $\mathbf{x}$ in $\mathcal{V}$ by their coordinates $[\mathbf{x}]_{\mathcal{X}}$ in the "standard" space $\mathcal{M}^{n \times 1}$; moreover, we can represent a linear operator $\mathbf{T}$ on $\mathcal{V}$ by a matrix manipulation of $[\mathbf{x}]_{\mathcal{X}}$, multiplication by $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$. It seems only natural that by means of the same basis we should be able to convert the inner product operation to a matrix manipulation. We proceed by means of an example.

Let $\mathbf{x} = (\xi_1, \xi_2)$ and $\mathbf{y} = (\eta_1, \eta_2)$ be general vectors in the vector space $\mathcal{R}^2$. Let $\langle \mathbf{x}, \mathbf{y} \rangle$ represent the inner product (5.13). We select $\mathcal{X} \triangleq \mathcal{E}$, the standard basis for $\mathcal{R}^2$. Then using the bilinearity of the inner product,

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= \langle \xi_1(1,0) + \xi_2(0,1), \eta_1(1,0) + \eta_2(0,1) \rangle \\
&= \xi_1 \langle (1,0), \eta_1(1,0) + \eta_2(0,1) \rangle + \xi_2 \langle (0,1), \eta_1(1,0) + \eta_2(0,1) \rangle \\
&= \xi_1 \eta_1 \|(1,0)\|^2 + \xi_1 \eta_2 \langle (1,0), (0,1) \rangle + \xi_2 \eta_1 \langle (0,1), (1,0) \rangle + \xi_2 \eta_2 \|(0,1)\|^2 \\
&= \tfrac{1}{4}\xi_1 \eta_1 - \tfrac{1}{4}\xi_1 \eta_2 - \tfrac{1}{4}\xi_2 \eta_1 + \tfrac{5}{4}\xi_2 \eta_2
\end{aligned}$$

On the surface, we appear to have returned to the defining equation (5.13), but the meaning of the equation is now different; ξ_i and η_i now represent coordinates [or multipliers of the vectors (1,0) and (0,1)] rather than elements of the vectors $\mathbf{x}$ and $\mathbf{y}$. We rewrite the last line of the equation as

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= (\eta_1 \ \ \eta_2) \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \\
&\triangleq [\mathbf{y}]_{\mathcal{E}}^{T} \mathbf{Q}_{\mathcal{E}} [\mathbf{x}]_{\mathcal{E}}
\end{aligned}$$

We have converted the inner product operation to a matrix multiplication. We call $\mathbf{Q}_{\mathcal{E}}$ the matrix of the inner product relative to the basis $\mathcal{E}$. In similar fashion, any inner product on a finite-dimensional space can be represented by a matrix.

Let $\mathcal{X} \triangleq \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for an inner product space $\mathcal{V}$. Then

$$\mathbf{x} = \sum_{k=1}^{n} a_k \mathbf{x}_k \quad \text{and} \quad \mathbf{y} = \sum_{j=1}^{n} b_j \mathbf{x}_j$$

By the argument used for the special case above,

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y}\rangle &= \langle \sum_k a_k \mathbf{x}_k, \sum_j b_j \mathbf{x}_j\rangle \\ &= \sum_j \bar{b}_j \langle \sum_k a_k \mathbf{x}_k, \mathbf{x}_j\rangle \\ &= \sum_j \bar{b}_j \sum_k a_k \langle \mathbf{x}_k, \mathbf{x}_j\rangle \\ &= (\bar{b}_1 \cdots \bar{b}_n) \begin{pmatrix} \langle \mathbf{x}_1, \mathbf{x}_1\rangle & \cdots & \langle \mathbf{x}_n, \mathbf{x}_1\rangle \\ \vdots & & \vdots \\ \langle \mathbf{x}_1, \mathbf{x}_n\rangle & \cdots & \langle \mathbf{x}_n, \mathbf{x}_n\rangle \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ &\triangleq \overline{[\mathbf{y}]}_{\mathcal{X}}^{\mathrm{T}} \mathbf{Q}_{\mathcal{X}} [\mathbf{x}]_{\mathcal{X}} \end{aligned} \tag{5.14}$$

We refer to $\mathbf{Q}_{\mathcal{X}}$ as the **matrix of the inner product $\langle \cdot\ , \cdot \rangle$ relative to the basis $\mathcal{X}$.** It is evident that

$$(\mathbf{Q}_{\mathcal{X}})_{jk} = \langle \mathbf{x}_k, \mathbf{x}_j\rangle \tag{5.15}$$

We can use (5.15) directly to generate the matrix of a given inner product relative to a particular basis. The matrix (5.15) is also known as the **Gram matrix** for the basis $\mathcal{X}$; the matrix consists in the inner products of all pairs of vectors from the basis.

Exercise 2. Use (5.15) to generate the matrix of the inner product (5.13) relative to the standard basis for $\mathcal{R}^2$.

From (5.14), (5.15), and the definition of an inner product we deduce that a Gram matrix, or a matrix of an inner product, has certain special properties which are related to the properties of inner products:

1. Since $\langle \mathbf{x}_k, \mathbf{x}_j\rangle = \overline{\langle \mathbf{x}_j, \mathbf{x}_k\rangle}$, $\mathbf{Q}_{\mathcal{X}} = \overline{\mathbf{Q}}_{\mathcal{X}}^{\mathrm{T}}$.

2. The inner product is positive definite; denoting $\mathbf{z} \triangleq [\mathbf{x}]_{\mathcal{X}}$, we find $\bar{\mathbf{z}}^{\mathrm{T}} \mathbf{Q}_{\mathcal{X}}\, \mathbf{z} \geqslant 0$ for all $\mathbf{z}$ in $\mathcal{M}^{n \times 1}$, with equality if and only if $\mathbf{z} = \boldsymbol{\theta}$.

We describe these matrix properties by saying $\mathbf{Q}_{\mathcal{X}}$ is (1) **hermitian symmetric*** and (2) **positive definite.** For a given basis, the set of all possible

*If $\mathbf{Q}_{\mathcal{X}}$ is real, the complex conjugate is superfluous. Then, if $\mathbf{Q}_{\mathcal{X}} = \mathbf{Q}_{\mathcal{X}}^{\mathrm{T}}$, we say $\mathbf{Q}_{\mathcal{X}}$ is symmetric.

inner products on an n-dimensional space $\mathcal{V}$ is equivalent to the set of positive-definite, hermitian symmetric $n \times n$ matrices. This fact indicates precisely how much freedom we have in picking inner products. In point of fact, (5.14) can be used in defining an inner product for $\mathcal{V}$. We will exploit it in our discussion of orthogonal bases in the next section. A method for determining whether or not a matrix is positive definite is described in P&C 5.9.

Exercise 3. Any inner product on the *real* space $\mathcal{R}^{n\times 1}$ is of the form $\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \mathbf{y}^T\mathbf{Q}\mathbf{x}$ for some symmetric positive-definite matrix $\mathbf{Q}$. The analogous definition for a real function space on the interval $[a, b]$ is

$$\langle \mathbf{f}, \mathbf{g} \rangle \triangleq \int_a^b \int_a^b k(t,s)\mathbf{f}(t)\mathbf{g}(s)\, ds\, dt$$

What properties must the kernel function k possess in order that this equation define a valid inner product (see P&C 5.30)? Show that if $k(t,s) = \omega(t)\delta(t-s)$, then the inner product reduces to (5.12).

5.2 Orthogonality

The thrust of this section is that orthogonal sets of vectors are not only linearly independent, but also lead to independent computation of coordinates. A set $\mathcal{S}$ of vectors is an **orthogonal set** if the vectors are pairwise orthogonal. If, in addition, each vector in $\mathcal{S}$ has unit norm, the set is called **orthonormal.** The two vectors of Example 1 (below) form an orthonormal set relative to the inner product (5.13). The standard basis for $\mathcal{R}^n$ is an orthonormal set relative to the standard inner product. Suppose the set $\mathcal{X} \triangleq \{x_1, \ldots, x_n\}$ is orthogonal. It follows that each vector in $\mathcal{X}$ is orthogonal to (and linearly independent of) the space spanned by the other vectors in the set; for example,

$$\langle \mathbf{x}_1, c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n \rangle = c_2\langle \mathbf{x}_1, \underbrace{\mathbf{x}_2}_{0} \rangle + \cdots + c_n\langle \mathbf{x}_1, \underbrace{\mathbf{x}_n}_{0} \rangle = 0$$

If $\mathcal{X}$ is an orthogonal basis for an n-dimensional space $\mathcal{V}$, then for any vector $\mathbf{x}$ in $\mathcal{V}$, $\mathbf{x} = \sum_{k=1}^{n} c_k\mathbf{x}_k$ and

$$\begin{aligned}\langle \mathbf{x}, \mathbf{x}_k \rangle &= \langle c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n, \mathbf{x}_k \rangle \\ &= c_1\langle \mathbf{x}_1, \mathbf{x}_k \rangle + \cdots + c_k\langle \mathbf{x}_k, \mathbf{x}_k \rangle + \cdots + c_n\langle \mathbf{x}_n, \mathbf{x}_k \rangle \\ &= c_k\langle \mathbf{x}_k, \mathbf{x}_k \rangle\end{aligned}$$

Thus the *k*th coordinate of $\mathbf{x}$ relative to the orthogonal basis $\mathcal{X}$ is

$$c_k = \frac{\langle \mathbf{x}, \mathbf{x}_k \rangle}{\langle \mathbf{x}_k, \mathbf{x}_k \rangle} \tag{5.16}$$

Each coordinate can be determined independently using (5.16). The set of simultaneous equations which, in previous chapters, had to be solved in order to find coordinates is not necessary in this case. Inherent in the "orthogonalizing" inner product is the computational decoupling of the coordinates. If, in fact, the vectors in $\mathcal{X}$ are *orthonormal,* the denominator in (5.16) is 1, and

$$\mathbf{x} = \sum_{k=1}^{n} \langle \mathbf{x}, \mathbf{x}_k \rangle \mathbf{x}_k \tag{5.17}$$

Equation (5.17) is known as a **generalized Fourier series expansion** (or orthonormal expansion) of $\mathbf{x}$ relative to the orthonormal basis $\mathcal{X}$. The *k*th coordinate, $\langle \mathbf{x}, \mathbf{x}_k \rangle$, is called the *k*th **Fourier coefficient** of $\mathbf{x}$ relative to the orthonormal basis $\mathcal{X}$. We will have little need to distinguish between (5.17) and the orthogonal expansion which uses the coefficients (5.16). We will also refer to the latter expansion as a Fourier series expansion, and to (5.16) as a Fourier coefficient.

Example 1. Independent Computation of Fourier Coefficients. From Example 7 of the previous section we know that the vectors $\mathbf{x}_1 \triangleq (1,1)$ and $\mathbf{x}_2 \triangleq (2,0)$ form a basis for $\mathcal{R}^2$ which is orthonormal relative to the inner product (5.13). Let $\mathbf{x} = (2,1)$. Then by (5.17) we know that

$$\mathbf{x} = (2,1) = c_1(1,1) + c_2(2,0)$$

where $c_1 = \langle \mathbf{x}, \mathbf{x}_1 \rangle = \langle (2,1), (1,1) \rangle = 1$ and $c_2 = \langle \mathbf{x}, \mathbf{x}_2 \rangle = \langle (2,1), (2,0) \rangle = \frac{1}{2}$.

Gram-Schmidt Orthogonalization Procedure

The Gram-Schmidt procedure is a technique for generating an orthonormal basis. Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent vectors in the space $\mathcal{R}^2$ with the standard inner product (dot product). (See the arrow space equivalent in Figure 5.3.) We will convert this pair of vectors to an orthogonal pair of vectors which spans the same space. The vector $\mathbf{x}_2$ decomposes uniquely into a pair of components, one collinear with $\mathbf{x}_1$ and the other orthogonal to $\mathbf{x}_1$. The collinear component is $\|\mathbf{x}_2\| \cos\phi$ times the unit vector in the direction of $\mathbf{x}_1$; using the expression (5.4) for the dot

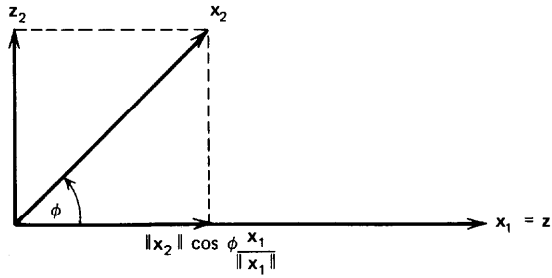


Figure 5.3. Gram-Schmidt orthogonalization in arrow space.

product, we convert this collinear vector to the form

$$\|\mathbf{x}_2\| \cos\phi \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{\mathbf{x}_2 \cdot \mathbf{x}_1}{\|\mathbf{x}_1\|} \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$$

Define $\mathbf{z}_1 \triangleq \mathbf{x}_1$ and $\mathbf{z}_2 \triangleq \mathbf{x}_2 - (\mathbf{x}_2 \cdot \mathbf{x}_1/\|\mathbf{x}_1\|^2)\mathbf{x}_1$. Then $\mathbf{z}_2$ is orthogonal to $\mathbf{z}_1$, and $\{\mathbf{z}_1, \mathbf{z}_2\}$ is an orthogonal set which spans the same space as $\{\mathbf{x}_1, \mathbf{x}_2\}$. We can normalize these vectors to obtain an orthonormal set $\{\mathbf{y}_1, \mathbf{y}_2\}$ which also spans the same space: $\mathbf{y}_1 = \mathbf{z}_1/\|\mathbf{z}_1\|$, and $\mathbf{y}_2 = \mathbf{z}_2/\|\mathbf{z}_2\|$.

The procedure applied to the pair of vectors in $\mathcal{R}^2$ above can be used to orthogonalize a finite number of vectors in any inner product space. Suppose we wish to orthogonalize a set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ from some inner product space $\mathcal{V}$. Assume we have already replaced $\mathbf{x}_1, \ldots, \mathbf{x}_k$ by an orthogonal set $\mathbf{z}_1, \ldots, \mathbf{z}_k$ which spans the same space as $\mathbf{x}_1, \ldots, \mathbf{x}_k$ (imagine $k = 1$). Then $\mathbf{x}_{k+1}$ decomposes uniquely into a pair of components, one in the space spanned by $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ and the other $(\mathbf{z}_{k+1})$ orthogonal to $\mathbf{z}_1, \ldots, \mathbf{z}_k$. Thus $\mathbf{z}_{k+1}$ must satisfy

$$\mathbf{x}_{k+1} = (c_1\mathbf{z}_1 + \cdots + c_k\mathbf{z}_k) + \mathbf{z}_{k+1}$$

Since the set $\{\mathbf{z}_1, \ldots, \mathbf{z}_{k+1}\}$ must be orthogonal, and therefore a basis for the space it spans, the coefficients $\{c_j\}$ are determined by (5.16):

$$c_j = \frac{\langle \mathbf{x}_{k+1}, \mathbf{z}_j \rangle}{\langle \mathbf{z}_j, \mathbf{z}_j \rangle}$$

Therefore,

$$\mathbf{z}_{k+1} = \mathbf{x}_{k+1} - \sum_{j=1}^{k} \frac{\langle \mathbf{x}_{k+1}, \mathbf{z}_j \rangle}{\langle \mathbf{z}_j, \mathbf{z}_j \rangle} \mathbf{z}_j \tag{5.18}$$

Exercise 1. Verify that $\mathbf{z}_{k+1}$ as given in (5.18) is orthogonal to $\mathbf{z}_j$ for $j = 1, \ldots, k$. How do we know the "orthogonal" decomposition of $\mathbf{x}_{k+1}$ is unique?

Starting with $\mathbf{z}_1 = \mathbf{x}_1$ and using (5.18) for $k = 1, \ldots, n\text{-}1$, we generate an orthogonal basis for the space spanned by $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. The procedure can be applied to any finite set of vectors, independent or not; any dependencies will be eliminated (P&C 5.14). Thus we can obtain an orthogonal basis for a vector space by applying (5.18) to any set of vectors which spans the space. The application of (5.18) is referred to as the **Gram-Schmidt orthogonalization procedure.** It requires no additional effort to normalize the vectors at each step, obtaining $\mathbf{y}_j = \mathbf{z}_j/\|\mathbf{z}_j\|$; then (5.18) becomes

$$\mathbf{z}_{k+1} = \mathbf{x}_{k+1} - \sum_{j=1}^{k} \langle \mathbf{x}_{k+1}, \mathbf{y}_j \rangle \mathbf{y}_j \tag{5.19}$$

Numerical accuracy and techniques for retaining accuracy in Gram-Schmidt orthogonalization are discussed in Section 6.6.

Example 2. Gram-Schmidt Orthogonalization in a Function Space. Define $\mathbf{f}_k(t) = t^k$ in the space $\mathcal{P}$ (-1,1) with the standard inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} \mathbf{f}(t)\mathbf{g}(t)\,dt$$

We will apply the Gram-Schmidt procedure to the first few functions in the set $\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \ldots\}$. Using (5.18), with appropriate adjustments in notation, we let $\mathbf{g}_0(t) = \mathbf{f}_0(t) = 1$ and

$$\mathbf{g}_1(t) = \mathbf{f}_1(t) - \frac{\langle \mathbf{f}_1, \mathbf{g}_0 \rangle}{\langle \mathbf{g}_0, \mathbf{g}_0 \rangle} \mathbf{g}_0(t)$$

But $\langle \mathbf{f}_1, \mathbf{g}_0 \rangle = \int_{-1}^{1} (t)(1)\,dt = 0$. Therefore, $\mathbf{g}_1(t) = \mathbf{f}_1(t) = t$, and $\mathbf{g}_1$ is orthogonal to $\mathbf{g}_0$. Again using (5.18),

$$\mathbf{g}_2(t) = \mathbf{f}_2(t) - \frac{\langle \mathbf{f}_2, \mathbf{g}_0 \rangle}{\langle \mathbf{g}_0, \mathbf{g}_0 \rangle} \mathbf{g}_0(t) - \frac{\langle \mathbf{f}_2, \mathbf{g}_1 \rangle}{\langle \mathbf{g}_1, \mathbf{g}_1 \rangle} \mathbf{g}_1(t)$$

The inner products are

$$\langle \mathbf{f}_2, \mathbf{g}_0 \rangle = \int_{-1}^{1} (t^2)(1)\,dt = \tfrac{2}{3}$$

$$\langle \mathbf{g}_0, \mathbf{g}_0 \rangle = \int_{-1}^{1} (1)(1)\,dt = 2$$

$$\langle \mathbf{f}_2, \mathbf{g}_1 \rangle = \int_{-1}^{1} (t^2)(t)\,dt = 0$$

Therefore, $\mathbf{g}_2(t) = t^2 - \frac{1}{3}$, and $\mathbf{g}_2$ is orthogonal to $\mathbf{g}_1$ and $\mathbf{g}_0$. We could continue, if we wished, to generate additional vectors of the orthogonal set $\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \ldots\}$. The functions $\{\mathbf{g}_k\}$ are known as *orthogonal polynomials.* Rather than normalize these orthogonal polynomials, we adjust their length as follows: define $\mathbf{p}_k \triangleq \mathbf{g}_k / \mathbf{g}_k(1)$ so that $\mathbf{p}_k(1) = 1$. The functions $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots\}$ so defined are known as the **Legendre polynomials.** (These polynomials are useful for solving partial *differential equations in spherical coordinates.) Thus* $\mathbf{p}_0(t) = \mathbf{g}_0(t) = 1$, $\mathbf{p}_1(t) = \mathbf{g}_1(t) = t$, and $\mathbf{p}_2(t) = \mathbf{g}_2(t)/\mathbf{g}_2(1) = (3t^2 - 1)/2$. A method of computing orthogonal polynomials which uses less computation than the Gram-Schmidt procedure is described in P&C 5.16.

Orthogonal Projection

The **orthogonal complement** of a set $\mathcal{S}$ of vectors in a vector space $\mathcal{V}$ is the set $\mathcal{S}^\perp$ of all vectors in $\mathcal{V}$ which are orthogonal to every vector in $\mathcal{S}$. For example, the orthogonal complement of the vector $\mathbf{x}_1$ of Figure 5.3 is the subspace spanned by $\mathbf{z}_2$. On the other hand, the orthogonal complement of span$\{\mathbf{z}_2\}$ is not the vector $\mathbf{x}_1$, but rather the space spanned by $\mathbf{x}_1$. An orthogonal complement is always a subspace.

Example 3. An Orthogonal Complement in $\mathcal{C}(0,1)$. Suppose the set $\mathcal{S}$ in the standard inner product space $\mathcal{C}(0,1)$ consists of the single function $\mathbf{f}_1(t) = 1$. Then $\mathcal{S}^\perp$ is the set of all functions whose average is zero; that is, those functions $\mathbf{g}$ for which

$$\langle \mathbf{f}_1, \mathbf{g} \rangle = \int_0^1 (1)\mathbf{g}(t)\,dt = 0$$

As part of our discussion of the decomposition of a vector space $\mathcal{V}$ into a direct sum, $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$, we introduced the concept of a projection on one of the subspaces along the other (Section 4.1). In the derivation of (5.18) we again used this concept of projection. In particular, each time we apply (5.18), we project a vector $\mathbf{x}_{k+1}$ onto the space spanned by $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ along a direction orthogonal to $\mathbf{z}_1, \ldots, \mathbf{z}_k$ (Figure 5.3). Suppose we define $\mathcal{W} \triangleq$ span$\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$. Then any vector which is orthogonal to $\mathbf{z}_1, \ldots, \mathbf{z}_k$ is in $\mathcal{W}^\perp$, the orthogonal complement of $\mathcal{W}$. The only vector which is in both $\mathcal{W}$ and $\mathcal{W}^\perp$ is the vector $\boldsymbol{\theta}$. Since the vector $\mathbf{x}_{k+1}$ of (5.18) can be any vector in $\mathcal{V}$, the derivation of (5.18) constitutes a proof (for finite-dimensional $\mathcal{V}$)* that

$$\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^\perp \tag{5.20}$$

*The projection theorem (5.20) also applies to certain infinite-dimensional spaces. Specifically, it is valid for any (complete) subspace $\mathcal{W}$ of a Hilbert space $\mathcal{V}$. See Bachman and Narici [5.2, p. 172]. These infinite-dimensional concepts (Hilbert space, subspace, and completeness) are discussed in Section 5.3.

That is, any vector in $\mathcal{V}$ can be decomposed uniquely into a pair of components, one in $\mathcal{W}$ and the other orthogonal to $\mathcal{W}$. The projection of a vector $\mathbf{x}$ on a subspace $\mathcal{W}$ along $\mathcal{W}^{\perp}$ is usually referred to as the **orthogonal projection of x on** $\mathcal{W}$. Equation (5.20), which guarantees the existence of orthogonal projections, is sometimes known as the **projection theorem.** This theorem is one of the keys to the solution of the least-square optimization problems explored in Chapter 6.

It is apparent from (5.18) that the orthogonal projection $\mathbf{x}_{\mathcal{W}}$ of an arbitrary vector $\mathbf{x}$ in $\mathcal{V}$ onto the subspace $\mathcal{W}$ spanned by the orthogonal set $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ is

$$\mathbf{x}_{\mathcal{W}} = \sum_{j=1}^{k} \frac{\langle \mathbf{x}, \mathbf{z}_j \rangle}{\|\mathbf{z}_j\|^2} \mathbf{z}_j \tag{5.21}$$

We can also write (5.21) in terms of the normalized vectors $\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ of (5.19):

$$\mathbf{x}_{\mathcal{W}} = \sum_{j=1}^{k} \langle \mathbf{x}, \mathbf{y}_j \rangle \mathbf{y}_j \tag{5.22}$$

Equation (5.22) expresses $\mathbf{x}_{\mathcal{W}}$ as a partial Fourier series expansion, an "attempted" expansion of $\mathbf{x}$ in terms of an orthonormal basis for the subspace on which $\mathbf{x}$ is projected. If $\mathbf{x} - \sum_{j=1}^{k} \langle \mathbf{x}, \mathbf{y}_j \rangle \mathbf{y}_j \neq \boldsymbol{\theta}$, we know that the orthonormal basis for $\mathcal{W}$ is not a basis for the whole space $\mathcal{V}$. It is evident that an orthonormal set $\{\mathbf{y}_i\}$ is a basis for a finite-dimensional space $\mathcal{V}$ if and only if there is no nonzero vector in $\mathcal{V}$ which is orthogonal to $\{\mathbf{y}_i\}$. We can compute the orthogonal projection of $\mathbf{x}$ on $\mathcal{W}$ without concerning ourselves with a basis for the orthogonal complement $\mathcal{W}^{\perp}$. We can do so because a description of $\mathcal{W}^{\perp}$ is inherent in the inner product. Clearly, Gram-Schmidt orthogonalization, orthogonal projection, and Fourier series are closely related. Equation (5.22), or its equivalent, (5.21), is a practical tool for computing orthogonal projections on finite-dimensional subspaces.

Example 4. Computation of an Orthogonal Projection. Let $\mathcal{W}$ be that subspace of the standard inner product space $\mathcal{R}^3$ which is spanned by $\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = (1,0,1)$ and $\mathbf{x}_2 = (0,1,1)$. We seek the orthogonal projection of $\mathbf{x} = (0,0,2)$ on $\mathcal{W}$. We first use the Gram-Schmidt procedure to orthogonalize the set $\{\mathbf{x}_1, \mathbf{x}_2\}$; then we apply (5.21). By (5.18), $\mathbf{z}_1 = \mathbf{x}_1 = (1,0,1)$ and

$$\mathbf{z}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{x}_1 \rangle}{\|\mathbf{x}_1\|^2} \mathbf{x}_1 = (0,1,1) - \tfrac{1}{2}(1,0,1) = \left(-\tfrac{1}{2}, 1, \tfrac{1}{2}\right)$$

By (5.21),

$$\mathbf{x}_{\mathcal{W}} = \frac{\langle \mathbf{x}, \mathbf{z}_1 \rangle}{\|\mathbf{z}_1\|^2}\mathbf{z}_1 + \frac{\langle \mathbf{x}, \mathbf{z}_2 \rangle}{\|\mathbf{z}_2\|^2}\mathbf{z}_2 = \frac{2}{2}(1,0,1) + \frac{1}{(3/2)}\left(-\tfrac{1}{2}, 1, \tfrac{1}{2}\right) = \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{4}{3}\right)$$

Orthonormal Eigenvector Bases for Finite-Dimensional Spaces

In (4.13) we solved the operator equation $\mathbf{Tx} = \mathbf{y}$ by means of spectral decomposition (or diagonalization). By representing the input vector $\mathbf{y}$ in terms of its coordinates relative to a basis of eigenvectors, we converted the operator equation into a set of uncoupled scalar equations, and solution for the output $\mathbf{x}$ became simple. Of course, even when the eigendata were known, a set of simultaneous equations was required in order to decompose $\mathbf{y}$. We now explore the solution of equations by means of an *orthonormal* basis of eigenvectors. The orthonormality allows us to determine independently each eigenvector component of the input; the solution process is then completely decoupled.

Let $\mathbf{T}$ have eigendata $\{\lambda_i\}$ and $\{\mathbf{z}_i\}$, and let $\{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$ be a basis for the space $\mathcal{V}$ on which $\mathbf{T}$ operates. (Then $\mathbf{T}$ must be diagonalizable.) Furthermore, suppose $\mathbf{T}$ is invertible; that is, $\lambda_i \neq 0$. We solve the operator equation $\mathbf{Tx} = \mathbf{y}$ as follows. The vectors $\mathbf{x}$ and $\mathbf{y}$ can be expanded as

$$\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{z}_i \quad \text{and} \quad \mathbf{x} = \sum_{i=1}^{n} d_i \mathbf{z}_i$$

The coordinates $\{c_j\}$ can be determined from $\mathbf{y}$; the numbers $\{d_j\}$ are coordinates of the unknown vector $\mathbf{x}$. Inserting these eigenvector expansions into the operator equation, we obtain

$$\mathbf{Tx} = \sum d_i \mathbf{T}\mathbf{z}_i = \sum d_i \lambda_i \mathbf{z}_i = \sum c_i \mathbf{z}_i = \mathbf{y}$$

or

$$\sum (d_i \lambda_i - c_i)\mathbf{z}_i = \boldsymbol{\theta}$$

Since the vectors $\mathbf{z}_i$ are independent, $d_i = c_i/\lambda_i$, and the solution to the operator equation is

$$\mathbf{x} = \sum_i \left(\frac{c_i}{\lambda_i}\right)\mathbf{z}_i \tag{5.23}$$

Suppose the eigenvector basis $\{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$ is orthonormal relative to the inner product on $\mathcal{V}$. Then the eigenvector expansion of $\mathbf{y}$ can be expressed as the Fourier expansion

$$\mathbf{y} = \sum_{i=1}^{n} \langle \mathbf{y}, \mathbf{z}_i \rangle \mathbf{z}_i$$

and the solution (5.23) becomes

$$\mathbf{x} = \sum_{i=1}^{n} \frac{\langle \mathbf{y}, \mathbf{z}_i \rangle}{\lambda_i} \mathbf{z}_i \tag{5.24}$$

Each component of (5.24) can be evaluated independently.

If $\mathbf{T}$ is not invertible, (5.24) requires division by a zero eigenvalue. The eigenvectors for the zero eigenvalue form a basis for nullspace($\mathbf{T}$). The remaining eigenvectors are taken by $\mathbf{T}$ into range($\mathbf{T}$), and in fact form a basis for range($\mathbf{T}$). To avoid division by zero in (5.24), we split the space: $\mathbf{y}$ = nullspace($\mathbf{T}$) $\oplus$ range($\mathbf{T}$). The equation $\mathbf{Tx} = \mathbf{y}$ has no solution unless $\mathbf{y}$ is in range($\mathbf{T}$) (or $c_j = 0$ for i corresponding to a zero eigenvalue); since the eigenvectors are assumed to be orthonormal, an equivalent statement is that $\mathbf{y}$ must be orthogonal to nullspace($\mathbf{T}$). Treating the eigenvectors corresponding to zero eigenvalues separately, we replace the solution $\mathbf{x}$ in (5.23)-(5.24) by

$$\mathbf{x} = \sum_{\text{nonzero } \lambda_i} d_i \mathbf{z}_i + \sum_{\text{zero } \lambda_i} d_i \mathbf{z}_i$$

$$= \sum_{\text{nonzero } \lambda_i} \frac{\langle \mathbf{y}, \mathbf{z}_i \rangle}{\lambda_i} \mathbf{z}_i + \mathbf{x}_0 \tag{5.25}$$

where $\mathbf{x}_0$ is an arbitrary vector in nullspace($\mathbf{T}$). The first portion of (5.25) is a particular solution to the equation $\mathbf{Tx} = \mathbf{y}$. The second portion, $\mathbf{x}_0$, is the homogeneous solution. The undetermined coefficients d_i in the sum which constitutes $\mathbf{x}_0$ are indicative of the freedom in the solution owing to the noninvertibility of $\mathbf{T}$.

What fortunate circumstances will allow us to find an orthonormal basis of eigenvectors? The eigenvectors are properties of $\mathbf{T}$; they cannot be selected freely. Assume there are enough eigenvectors of $\mathbf{T}$ to form a basis for the space. Were we to orthogonalize an eigenvector basis using the Gram-Schmidt procedure, the resulting set of vectors would not be eigenvectors. However, we have considerable freedom in picking inner products.

In point of fact, since the space is finite dimensional, we can select the inner product to make any particular basis orthonormal.

The key to selection of inner products for finite-dimensional spaces is (5.14), the representation of inner products of vectors in terms of their coordinates. Let the basis $\mathcal{X}$ be $\{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$, the eigenvectors of **T.** We select the matrix of the inner product, $\mathbf{Q}_{\mathcal{X}}$, such that the basis vectors are orthonormal. By (5,15), if $\mathcal{X}$ is to be orthonormal, $\mathbf{Q}_{\mathcal{X}}$ satisfies

$$(\mathbf{Q}_{\mathcal{X}})_{jk} = \langle \mathbf{z}_k, \mathbf{z}_j \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \tag{5.26}$$

or $\mathbf{Q}_{\mathcal{X}} = \mathbf{I}$. By (5.14), this matrix defines the following inner product on $\mathcal{V}$:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \overline{[\mathbf{y}]}_{\mathcal{X}}^{\mathrm{T}} [\mathbf{x}]_{\mathcal{X}} \tag{5.27}$$

The expression (5.27) of an inner product in terms of coordinates relative to an orthonormal basis is called **Parseval's equation.** A basis $\mathcal{X}$ is orthonormal if and only if (5.27) is satisfied; that is, if and only if the inner product between any two vectors equals the standard inner product (in $\mathcal{M}^{n\times 1}$) between their coordinates relative to $\mathcal{X}$.

Example 5. Solution of an Equation by Orthonormal Eigenvector Expansion. Suppose we define $\mathbf{T}: \mathcal{R}^2 \to \mathcal{R}^2$ by

$$\mathbf{T}(\xi_1, \xi_2) \triangleq (2\xi_1 + 3\xi_2, 4\xi_2)$$

[The same operator is used in the decomposition of Example 7, Section 4.1.] The eigendata are $\lambda_1 = 2$, $\mathbf{z}_1 = (1,0)$, $\lambda_2 = 4$, and $\mathbf{z}_2 = (3,2)$. The pair of eigenvectors, $\mathcal{X} \triangleq \{\mathbf{z}_1, \mathbf{z}_2\}$, is a basis for $\mathcal{R}^2$. We define the inner product for $\mathcal{R}^2$ by (5.27):

$$\langle \mathbf{x}, \mathbf{y} \rangle = [\mathbf{y}]_{\mathcal{X}}^{\mathrm{T}} [\mathbf{x}]_{\mathcal{X}}$$

To make this definition more explicit, we find the coordinates of **x** and **y;** let $\mathbf{y} = a_1\mathbf{z}_1 + a_2\mathbf{z}_2$, or

$$\mathbf{y} = (\eta_1, \eta_2) = a_1(1,0) + a_2(3,2)$$

Solution (by row reduction) yields $a_1 = \eta_1 - 3\eta_2/2$ and $a_2 = \eta_2/2$. Similarly, the coordinates of $\mathbf{x} = (\xi_1, \xi_2)$ are $c_1 = \xi_1 - 3\xi_2/2$ and $c_2 = \xi_2/2$. Thus we can express the inner product as

$$\langle (\xi_1, \xi_2), (\eta_1, \eta_2) \rangle = \begin{pmatrix} \eta_1 - 3\eta_2/2 \\ \eta_2/2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \xi_1 - 3\xi_2/2 \\ \xi_2/2 \end{pmatrix}$$

$$= \xi_1\eta_1 - \tfrac{3}{2}\xi_1\eta_2 - \tfrac{3}{2}\xi_2\eta_1 + \tfrac{5}{2}\xi_2\eta_2$$

Relative to this inner product, the basis $\mathfrak{X}$ is orthonormal. We solve the equation $\mathbf{Tx} = \mathbf{T}(\xi_1, \xi_2) = (\eta_1, \eta_2) = \mathbf{y}$ using (5.24):

$$\begin{aligned}\mathbf{x} &= \frac{\langle \mathbf{y}, \mathbf{z}_1 \rangle}{\lambda_1}\mathbf{z}_1 + \frac{\langle \mathbf{y}, \mathbf{z}_2 \rangle}{\lambda_2}\mathbf{z}_2 \\ &= \frac{(\eta_1 - 3\eta_2/2)}{2}(1,0) + \frac{(\eta_2/2)}{4}(3,2) \\ &= (\eta_1/2 - 3\eta_2/8, \eta_2/4)\end{aligned}$$

We have developed two basic approaches for analyzing a finite-dimensional, invertible, diagonalizable, linear equation: (*a*) operator inversion and (*b*) spectral decomposition (or eigenvector expansion). Both methods give explicit descriptions of the input-output relationship of the system for which the equation is a model. The spectral decomposition yields a more detailed description; therefore, it provides more insight than does inversion. If the eigenvector expansion is orthonormal, we also obtain conceptual and computational independence of the individual terms in the expansion.

What price do we pay for the insight obtained by each of these approaches? We take as a measure of computational expense the approximate number of multiplications required to analyze an $n \times n$ matrix equation:

1. Inversion of an $n \times n$ matrix $\mathbf{A}$ (or solution of $\mathbf{Ax} = \mathbf{y}$ for an unspecified $\mathbf{y}$) by use of Gaussian elimination requires $4n^3/3$ multiplications. Actual multiplication of $\mathbf{y}$ by $\mathbf{A}^{-1}$ uses n^2 multiplications for each specific $\mathbf{y}$.

2. Analysis by the nonorthogonal eigenvector expansion (5.23) starts with computation of the eigendata. Determination of the characteristic equation, computation of its roots, and solution for the eigenvectors is considerably more expensive than matrix inversion (see Section 4.2). For each specific $\mathbf{y}$, determination of $\mathbf{x}$ requires $n^3/3$ multiplications to calculate the coordinates of $\mathbf{y}$ relative to the eigenvector basis. The number of multiplications needed to sum up the eigenvector components of $\mathbf{x}$ is relatively unimportant.

3. In order to express the solution $\mathbf{x}$ as the orthonormal eigenvector expansion (5.24), we need to determine the inner product which makes the basis of eigenvectors orthonormal. Determination of that inner product requires the solution of a vector equation with an unspecified right-hand side (see Example 5). Thus to fully define the expression (5.24), we need $4n^3/3$ multiplications in addition to the computation necessary to obtain

the eigendata. It is evident from Example 5 that evaluation of a single inner product in an n-dimensional space can require as few as n multiplications (if no cross-products terms appear and all coefficients are unity) and as many as $2n^2$ multiplications (if all cross-product terms appear). Therefore, for each specific y, computation of x requires between n^2 and $2n^3$ multiplications to evaluate the inner products, and $n^2 + n$ multiplications to perform the linear combination.

The value of orthonormal eigenvector expansion as a vehicle for analyzing equations lies primarily in the insight provided by the complete decomposition (5.24). We pay for this insight by determining the eigendata. For certain classes of problems we are fortunate in that the eigendata is known a priori (e.g., the symmetrical components of (4.27)-(4.28), the Vandermond matrix of P&C 4.16, and the sinusoids or complex exponentials of classical Fourier series). Then the technique is computationally competitive with inversion. We note in Section 5.5 that for (infinite-dimensional) partial differential equations, eigenvector expansion is a commonly used analysis technique.

Infinite OrthonormaI Expansions

We will find that most of the concepts we have discussed in this chapter apply in infinite-dimensional spaces. A significant characteristic of an infinite expansion of a vector (or function) is that the "first few" terms usually dominate. If the infinite expansion is also orthonormal, then we can not only approximate the vector by the first few terms of the expansion, but we can also compute these first few terms, ignoring the remainder -the individual terms of an orthonorrnal expansion are computationally independent. Thus the value of *orthonormal* eigenvector expansion is higher for infinite-dimensional systems than for finite-dimensional systems. Furthermore, for certain classes of models, orthonormal eigendata is standard-it is known a priori. (For example, all constant-coefficient linear differential operators with periodic boundary conditions have an easily determined set of orthogonal sine and cosine functions as eigenfunctions.) For these models, orthonormal eigenvector expansion is a computationally efficient analysis technique (P&C 5.35). In this section we examine briefly a few familiar infinite orthonormal expansions which are useful in the analysis of dynamic systems. A detailed general discussion of infinite orthonormal eigenvector expansions forms the subject of Section 5.5.

We noted in Section 4.3 that models of linear dynamic systems (linear differential operators with initial conditions) have no eigenfunctions because the boundary conditions all occur at one point in time. This fact would seem to preclude the use of eigenfunction expansions in analyzing dynamic systems. However, many practical dynamic systems, electric

power systems for instance, are operated with periodic inputs. The output of a linear *time-invariant* dynamic system with a periodic input quickly approaches a steady-state form which is periodic with the same period as the input. The steady-state form depends only on the periodic input and not on the initial conditions. (Implicit in the term steady-state, however, is a set of **periodic boundary conditions**—the values of the solution **f** and its derivatives must be the same at the beginning and end of the period.) The transition from the initial conditions to the steady-state solution is described by a transient component of the solution. Suppose the system model is a differential equation, denoted by $\mathbf{Lf} = \mathbf{u}$, with initial conditions $\boldsymbol{\beta}_i(\mathbf{f}) = \alpha_i$. The **steady-state solution** $\mathbf{f}_1$ satisfies $\mathbf{Lf}_1 = \mathbf{u}$ (with periodic boundary conditions). Define the **transient solution** $\mathbf{f}_2$ to be the solution of $\mathbf{Lf}_2 = \boldsymbol{\theta}$ with $\boldsymbol{\beta}_i(\mathbf{f}_1 + \mathbf{f}_2) = \alpha_i$ (or $\boldsymbol{\beta}_i(\mathbf{f}_2) = \alpha_i - \boldsymbol{\beta}_i(\mathbf{f}_1)$). Then $\mathbf{f} \triangleq \mathbf{f}_1 + \mathbf{f}_2$ satisfies both the differential equation and the initial conditions.

Example 6. Steady-State and Transient Solutions. The linear time-invariant electrical circuit of Figure 5.4 is described by the differential equation

$$\mathbf{e}(t) = L\frac{d\mathbf{i}(t)}{dt} + R\mathbf{i}(t), \qquad \mathbf{i}(0) = 0 \tag{5.28}$$

Suppose the applied voltage (or input) is the periodic function $\mathbf{e}(t) = E\sin(\omega t + \phi_E)$. We can easily verify that the steady-state solution to the differential equation is

$$\mathbf{i}_1(t) = \frac{E}{\sqrt{(\omega L)^2 + R^2}}\sin(\omega t + \phi_E - \phi_I), \qquad \phi_I = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

Note that $\mathbf{i}_1$ does not satisfy the initial condition $\mathbf{i}_1(0)=0$ unless ϕ_E happens to equal ϕ_I. However, it does satisfy the periodic boundary condition $\mathbf{i}_1(2\pi/\omega) = \mathbf{i}_1(0)$. The transient solution (the solution of the homogeneous differential equation) is of the form

$$\mathbf{i}_2(t) = ce^{-(R/L)t}$$

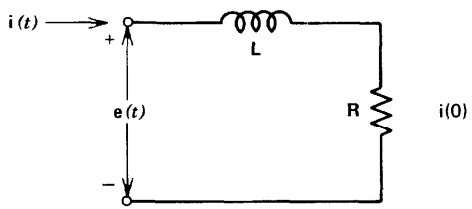


Figure 5.4. A linear time-invariant circuit.

We pick the constant c such that $\mathbf{i}_1(0) + \mathbf{i}_2(0) = 0$:

$$c = -\frac{E\sin(\phi_E - \phi_I)}{\sqrt{(\omega L)^2 + R^2}}$$

Then $\mathbf{i} \triangleq \mathbf{i}_1 + \mathbf{i}_2$ satisfies (5.28).

Exercise 2. Verify that $\mathbf{i}_1$ of Example 6 satisfies the differential equation of (5.28), but not the initial condition. Hint:

$$a\cos\psi + b\sin\psi = \sqrt{a^2 + b^2}\ \sin\left(\psi + \tan^{-1}\left(\frac{a}{b}\right)\right)$$

Steady-state analysis of a dynamic system is analysis of the system with periodic boundary conditions. A linear constant-coefficient differential operator with periodic boundary conditions *does* have eigenfunctions; namely, all sines, cosines, and complex exponentials which have the correct period. In point of fact, the steady-state solution to (5.28) was easy to determine only because the periodic input $\mathbf{e}(t)$ was an eigenfunction of the differential operator for periodic boundary conditions. The eigenvalue corresponding to that eigenfunction is the input impedance $\boldsymbol{Z}$ of the ***R*-*L*** circuit corresponding to the frequency $\boldsymbol{\omega}$ of the applied voltage:

$$Z = R + i\omega L = \sqrt{(\omega L)^2 + R^2}\ \exp\left(i\tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

where $i = \sqrt{-1}$.

It is well known that any "well-behaved" periodic function can be expanded in an orthonormal series of sines and cosines—eigenfunctions of linear constant-coefficient differential operators with periodic boundary conditions. Suppose $\mathbf{f}$ is a periodic function of period p; then*

$$\mathbf{f}(t) = a_0 + a_1\cos\frac{2\pi t}{p} + a_2\cos\frac{4\pi t}{p} + \cdots$$

$$+ b_1\sin\frac{2\pi t}{p} + b_2\sin\frac{4\pi t}{p} + \cdots \qquad (5.29)$$

*This is the classical Fourier series expansion [5.5, p. 312].

where

$$a_0 = \frac{1}{p}\int_0^p \mathbf{f}(t)\,dt$$

$$a_j = \frac{2}{p}\int_0^p \mathbf{f}(t)\cos\frac{2\pi jt}{p}\,dt, \qquad j = 1,2,\ldots$$

$$b_j = \frac{2}{p}\int_0^p \mathbf{f}(t)\sin\frac{2\pi jt}{p}\,dt, \qquad j = 1,2,\ldots$$

We can replace the sinusoidal functions of (5.29) by the normalized functions

$$\begin{aligned}\mathbf{f}_0(t) &\triangleq \sqrt{1/p}\\ \mathbf{f}_k(t) &\triangleq \sqrt{2/p}\,\cos(2\pi kt/p), \qquad k = -1,-2,\ldots\\ &\triangleq \sqrt{2/p}\,\sin(2\pi kt/p), \qquad k = 1,2,\ldots\end{aligned} \tag{5.30}$$

Relative to the inner product

$$\langle \mathbf{f},\mathbf{g}\rangle \triangleq \int_0^p \mathbf{f}(t)\mathbf{g}(t)\,dt \tag{5.31}$$

the functions (5.30) form an orthonormal set. (Since the functions are periodic of period p, we concern ourselves only with values of the functions over a single period.) Therefore, we can write (5.29) in the standard form for a generalized Fourier series:

$$\mathbf{f} = \sum_{k=-\infty}^{\infty} \langle \mathbf{f},\mathbf{f}_k\rangle \mathbf{f}_k \tag{5.32}$$

Exercise 3. Show that the set of functions (5.30) is orthonormal relative to the standard inner product (5.31).

If $\mathbf{f}$ is any periodic function of period p, (5.32) is an orthonormal expansion of $\mathbf{f}$ in terms of the eigenfunctions of any linear constant-coefficient differential operator (assuming periodic boundary conditions of the same period p). Furthermore, since the eigenfunctions are known a priori, they need not be computed. Therefore, the Fourier series described

by (5.29) or (5.32) is valuable in the steady-state analysis of linear time-invariant dynamic systems (P&C 5.35).

A sine or cosine can be expressed as the sum of a pair of complex exponentials with complex coefficients

$$\sin\psi = \frac{e^{i\psi} - e^{-i\psi}}{2i}, \qquad \cos\psi = \frac{e^{i\psi} + e^{-i\psi}}{2}, \qquad i = \sqrt{-1}$$

Therefore, the Fourier series (5.29) can be rewritten in terms of the functions

$$\mathbf{g}_k(t) \triangleq \frac{1}{\sqrt{p}} \exp\left(i\frac{2\pi kt}{p}\right), \qquad k = 0, \pm 1, \pm 2, \ldots \tag{5.33}$$

Assume the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle \triangleq \int_0^p \mathbf{f}(t)\,\overline{\mathbf{g}(t)}\,dt \tag{5.34}$$

(We need the complex conjugation indicated in (5.34) because we are considering the complex-valued functions $\mathbf{g}_k$.) Then

$$\langle \mathbf{g}_k, \mathbf{g}_n \rangle = \frac{1}{p}\int_0^p \exp\left(i\frac{2\pi kt}{p}\right)\exp\left(-i\frac{2\pi nt}{p}\right)dt$$

$$= \frac{\exp\left[i\dfrac{2\pi(k-n)t}{p}\right]}{i2\pi(k-n)}\Bigg|_0^p = 0, \qquad k \neq n$$

$$\langle \mathbf{g}_k, \mathbf{g}_k \rangle = \frac{1}{p}\int_0^p \exp\left(i\frac{2\pi kt}{p}\right)\exp\left(-i\frac{2\pi kt}{p}\right)dt = 1,$$

The set (5.33) is orthonormal, and we can express (5.29) as the exponential Fourier series:

$$\mathbf{f} = \sum_{k=-\infty}^{\infty} \langle \mathbf{f}, \mathbf{g}_k \rangle \mathbf{g}_k$$

or

$$\mathbf{f}(t) = \sum_{k=-\infty}^{\infty} \frac{c_k}{\sqrt{p}} \exp\left(i\frac{2\pi kt}{p}\right), \qquad c_k = \frac{1}{\sqrt{p}} \int_0^p \mathbf{f}(s) \exp\left(-i\frac{2\pi ks}{p}\right) ds \tag{5.35}$$

The exponential series (5.35) is often used in place of (5.29). In some respects it is a more convenient series for use in analyzing constant-coefficient differential equations, because derivatives of exponentials are still exponentials.

We have discussed the applicability of an infinite eigenfunction expansion [the classical Fourier series in either of its forms, (5.29) or (5.35)] for steady-state analysis of dynamic systems. Surprisingly, the approach we have used for steady-state analysis can be applied to a dynamic system even if the system is not operated in a periodic fashion; we merely treat the system as if it were periodic with a single infinite period. We still seek a function $\mathbf{f}_1$ which satisfies the differential equation with periodic boundary conditions, then determine a "transient" solution $\mathbf{f}_2$ to the homogeneous differential system such that $\mathbf{f}_1 + \mathbf{f}_2$ satisfies the initial conditions. Thus it still makes sense to work with exponentials, the eigenfunctions of linear constant-coefficient differential operators (ignoring the initial conditions). We could derive the expansion (in exponentials) of a nonperiodic function by changing variables and letting the period become large. However, we merely state the well-known result, known as the **Fourier integral theorem* :**

$$\mathbf{f}(t) = \int_{-\infty}^{\infty} \mathbf{F}(s) e^{i2\pi st}\, ds \tag{5.36}$$

where

$$\mathbf{F}(s) = \int_{-\infty}^{\infty} \mathbf{f}(t) e^{-i2\pi st}\, dt$$

The expansion (5.36) applies for any "well-behaved" function **f** for which $\int_{-\infty}^{\infty} |\mathbf{f}(t)|\, dt < \infty$. The coefficient function **F** is known as the *Fourier integral* of **f.** The role of the discrete frequency variable k is taken over by the continuous real frequency variable s. The sum in (5.35) becomes an

*Churchill [5.5, pp. 88-90].

integral in (5.36). Let $\mathbf{q}(s, t) \triangleq \exp(i2\pi st)$. Then defining the inner product by

$$\langle \mathbf{f}, \mathbf{g} \rangle \triangleq \int_{-\infty}^{\infty} \mathbf{f}(t)\, \mathbf{g}(t)\, dt \tag{5.37}$$

we can express (5.36) as

$$\mathbf{f}(t) = \int_{-\infty}^{\infty} \langle \mathbf{f}, \mathbf{q}(s, \cdot) \rangle \mathbf{q}(s, t)\, ds \tag{5.38}$$

It can be shown, by a limiting argument, that the infinite set $\{\mathbf{q}(s, \cdot), -\infty < s < \infty\}$ is an orthogonal set; however, $\|\mathbf{q}(s, \cdot)\|$ is not finite. **Parseval's theorem,** a handy tool in connection with Fourier integrals, states that

$$\int_{-\infty}^{\infty} \mathbf{f}(t)\, \overline{\mathbf{g}(t)}\, dt = \int_{-\infty}^{\infty} \mathbf{F}(s)\, \overline{\mathbf{G}(s)}\, ds \tag{5.39}$$

where **F** and **G** are the Fourier integrals of **f** and **g,** respectively. This equation is a direct extension of (5.27). In effect, the "frequency domain" functions **F** and **G** constitute the coordinates of the "time domain" functions **f** and **g,** respectively. Equations analogous to (5.39) can be written for the expansions (5.29) and (5.35).

It is interesting that restricting our concern to periodic functions (or, in effect, to the values of functions on the finite time interval $[0, p]$) reduces (5.36) to (5.35) and allows us to expand these functions in terms of a countable basis (a basis whose members can be numbered using only integer subscripts). Because of the duality exhibited in (5.36) and (5.39) between the time variable t and the frequency variable s, it should come as no surprise that restricting our interest to functions with finite "bandwidth" (functions whose transforms are nonzero only over a finite frequency interval) again allows us to expand the functions in terms of a countable basis. Limited bandwidth functions are fundamental to the analysis of periodic sampling. If $\mathbf{F}(s) = 0$ for $|s| > w$, we say that **f** is band limited to w; or **f** has no frequency components as high as w. For such a function it is well known that the set of samples (values) of **f** at the points $t = k/2w$, $k = 0, \pm 1, \pm 2, \ldots$ contains all the information possessed by **f.** To be more specific, the **sampling theorem** states

$$\mathbf{f}(t) = \sum_{k=-\infty}^{\infty} \mathbf{f}\left(\frac{k}{2w}\right) \frac{\sin 2\pi w(t - k/2w)}{2\pi w(t - k/2w)} \tag{5.40}$$

for any function **f** which is band limited to w [5.18].

We define the functions $\{\mathbf{h}_k\}$ by

$$\mathbf{h}_k(t) \triangleq \sqrt{2w}\,\frac{\sin 2\pi w(t-k/2w)}{2\pi w(t-k/2w)}, \qquad k=0,\pm 1,\pm 2,\ldots$$

The function $\mathbf{h}_0$ is plotted in Figure 5.5; $\mathbf{h}_k$ is just $\mathbf{h}_0$ shifted by $t = k/2w$.

Exercise 4. Use (5.36), (5.39), and the inner product (5.37) to show that (*a*) $\{\mathbf{h}_k\}$ is an orthonormal set, and (*b*) $\langle \mathbf{f}, \mathbf{h}_k \rangle = \mathbf{f}(k/2w)$. Hint: the Fourier integral of $\mathbf{h}_k$ is

$$\mathbf{H}_k(s) = \begin{cases} \dfrac{1}{\sqrt{2w}} \exp\left(-\dfrac{i\pi ks}{w}\right) & |s| < w \\ 0 & |s| \geqslant w \end{cases}$$

As a result of Exercise 4, we can express the sampling theorem as a generalized Fourier series:

$$\mathbf{f} = \sum_{k=-\infty}^{\infty} \langle \mathbf{f}, \mathbf{h}_k \rangle \mathbf{h}_k \tag{5.41}$$

The coefficients of any orthonormal expansion can be computed independently. Thus the fact that the functions $\mathbf{h}_k$ are orthonormal is significant. Each coefficient in (5.41) can be obtained by physically sampling a single point of the function **f.** It is common practice to sample functions in order to process them digitally. The samples of a function are the coordinates of that function relative to the orthonormal basis $\{\mathbf{h}_k\}$. The processes commonly used for physical reconstruction of functions from their samples are all, in some sense, approximations to the sum (5.41).

Extending Parseval's equation (5.27) to the set $\{\mathbf{h}_k\}$, we find that inner products of two band-limited functions can be computed in terms of the

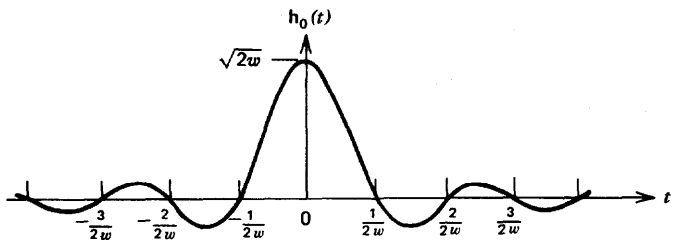


Figure 5.5. The function $\mathbf{h}_0$.

samples of the functions:

$$\int_{-\infty}^{\infty} \mathbf{f}(t)\,\overline{\mathbf{g}(t)}\,dt = \sum_{k=-\infty}^{\infty} \mathbf{f}(k/2w)\,\overline{\mathbf{g}(k/2w)} \tag{5.42}$$

If a function is both periodic and of finite bandwidth, then its Fourier series expansion, (5.29) or (5.35), contains only a finite number of terms; periodicity guarantees that discrete frequencies are sufficient to represent the function, whereas limiting the bandwidth to less than w guarantees that no (discrete) frequencies higher than w are required. Then, although (5.35) and (5.40) express the same function in different coordinates, the first set of coordinates is more efficient in the sense that it converges exactly in a finite number of terms. All the function samples are required in order to reconstruct the full function using (5.40). Yet (5.40) is dominated by its first few terms; only a "few" samples are required to accurately reconstruct the function over its first period. The remaining samples contain little additional information.

Exercise 5. Let $\mathbf{f}(t) = \sin 2\pi t$, a function which is periodic and band limited. ($\mathbf{F}(s) = 0$ for $|s| > 1$). Sample $\mathbf{f}$ at $t = 0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \ldots$ (i.e., let $w = 2$). Then, by (5.40),

$$\mathbf{f}(t) = \sum_{k=-\infty}^{\infty} \mathbf{f}\left(\frac{k}{4}\right)\mathbf{h}_k(t) = \sum_{k=-\infty}^{\infty} \sin\left(\frac{k\pi}{2}\right)\frac{\sin 4\pi(t-k/4)}{4\pi(t-k/4)}$$

The samples are zero for $k = 0, \pm 2, \pm 4, \ldots$. Graphically combine the terms for $k = \pm 1, \pm 3, \pm 5$, and compare the sum with $\mathbf{f}$ over the interval [0, 1].

5.3 Infinite-Dimensional Spaces

We developed the generalized Fourier series expansion (5.17) only for finite-dimensional spaces; yet we immediately recognized its extension to certain well-known infinite-dimensional examples, particularly (5.29). Our goal, ultimately, is to determine how to find orthonormal bases of eigenfunctions for linear operators on infinite-dimensional spaces. A basis of eigenfunctions permits decomposition of an infinite-dimensional operator equation into a set of independent scalar equations, just as in the finite-dimensional case (5.23). Orthogonality of the basis allows independent computation of the coefficients in the expansion as in (5.24). We will find this computational independence particularly valuable for infinite-dimensional problems because the "first few" terms in an infinite

orthonormal expansion dominate that expansion; we can ignore the remaining terms.

To this point, wherever we have introduced infinite expansions of vectors, we have used well-known examples and avoided discussion of the meaning of an infinite sum. Thus we *interpret* the Taylor series expansion

$$\mathbf{f}(t) = \mathbf{f}(0) + \mathbf{f}'(0)t + \frac{\mathbf{f}''(0)t^2}{2!} + \cdots \tag{5.43}$$

of an infinitely differentiable function $\mathbf{f}$ as the expansion of $\mathbf{f}$ in terms of the "basis" $\{1, t, t^2, \ldots\}$. We consider the Fourier series expansion (5.29) as the expansion of a periodic function on the "orthonormal basis"

$$\left\{ \sqrt{\frac{1}{p}}\ ,\ \sqrt{\frac{2}{p}}\ \cos\frac{2\pi kt}{p},\ \sqrt{\frac{2}{p}}\ \sin\frac{2\pi kt}{p},\qquad k = 1,2,\ldots \right\}$$

Yet the definition of linear combination does not pinpoint the meaning of $\sum_{k=1}^{\infty} c_k \mathbf{f}_k$ for an infinite set of functions $\{\mathbf{f}_k\}$. It seems natural and desirable to assume that such an infinite sum implies pointwise convergence of the partial sums. Certainly, the Taylor series (5.43) means that for each t,

$$\mathbf{f}_k(t) \triangleq \mathbf{f}(0) + \mathbf{f}'(0)t + \cdots + \frac{\mathbf{f}^{(k)}(0)t^k}{k!} \to \mathbf{f}(t)$$

as $k \to \infty$. However, it is well-known that the sequence of partial sums in the Fourier series expansion (5.29) of a discontinuous function is not pointwise convergent; the partial sums converge to the midpoints of any discontinuities (P&C 5.18). In an engineering sense, we do not care to which value the series converges at a discontinuity. The actual value of the function is usually defined arbitrarily at that point anyway. We define convergence of the partial sums in a way which ignores the value of the Fourier series at the discontinuities.

Convergence in Norm

Define $\mathbf{y}_n \triangleq \sum_{k=1}^{n} c_k \mathbf{x}_k$, the nth partial sum of the series $\sum_{k=1}^{\infty} c_k \mathbf{x}_k$. We can assign meaning to the infinite sum only if the partial sums $\mathbf{y}_n$ and $\mathbf{y}_m$ become more nearly alike in some sense as $n, m \to \infty$. The natural definition of "likeness" in an inner product space is likeness in norm. That is, $\mathbf{y}_n$ and $\mathbf{y}_m$ are alike if the norm $\| \mathbf{y}_n - \mathbf{y}_m \|$ of their difference is small. An infinite sequence $\{\mathbf{y}_n\}$ from an inner product space $\mathcal{V}$ is called a **Cauchy**

sequence if $\|\mathbf{y}_n - \mathbf{y}_m\| \to 0$ as $n, m \to \infty$; or, rigorously, if for each $\epsilon > 0$ there is an N such that $n,m > N$ implies $\|\mathbf{y}_n - \mathbf{y}_m\| < \epsilon$. Intuitively, a Cauchy sequence is a "convergent" sequence. By means of a Cauchy sequence we can discuss the fact of convergence without explicit reference to the limit vector. We say an infinite sequence $\{\mathbf{y}_n\}$ from an inner product space $\mathcal{V}$ **converges in norm** to the limit $\mathbf{x}$ if $\|\mathbf{x} - \mathbf{y}_n\| \to 0$ as $n \to \infty$.

Exercise 1. Use the triangle inequality (P&C 5.4) to show that a sequence from an inner product space $\mathcal{V}$ can converge in norm to a vector $\mathbf{x}$ in $\mathcal{V}$ only if it is a Cauchy sequence.

Assume the partial sums of a series, $\mathbf{y}_n \triangleq \sum_{k=1}^{n} c_k \mathbf{x}_k$, form a Cauchy sequence; by the infinite sum $\sum_{k=1}^{\infty} c_k \mathbf{x}_k$, we mean the vector $\mathbf{x}$ to which the partial sums converge in norm, We call $\mathbf{x}$ the **limit in norm** of the sequence $\{\mathbf{y}_n\}$. (Note that the limit of a Cauchy sequence need not be in $\mathcal{V}$. The mathematics literature usually does not consider a sequence *convergent* unless the limit *is* in $\mathcal{V}$.)

Let $\mathcal{V}$ be some space of functions defined on [0,1] with the standard function space inner product. One of the properties of inner products guarantees that $\mathbf{f} = \boldsymbol{\theta}$ if $\|\mathbf{f}\| = 0$. We have assumed previously that $\mathbf{f} = \boldsymbol{\theta}$ meant $\mathbf{f}(t) = 0$ for all t in [0,1]. Suppose, however, that $\mathbf{f}$ is the discontinuous function shown in Figure 5.6. Observe that $\|\mathbf{f}\| = 0$, whereas $\mathbf{f}(t) \neq 0$ at $t = 0$, ½ or 1. Changing the value of a function at a few points does not change its integral (or its norm). We are hard pressed to define any inner product for a space containing functions like the one in Figure 5.6 unless we ignore "slight" differences between functions.

We say $\mathbf{f} = \mathbf{g}$ **almost everywhere** if $\mathbf{f}(t) = \mathbf{g}(t)$ except at a finite number of points.* For most practical purposes we can consider convergence in norm to be pointwise convergence almost everywhere. (However, Bachman and

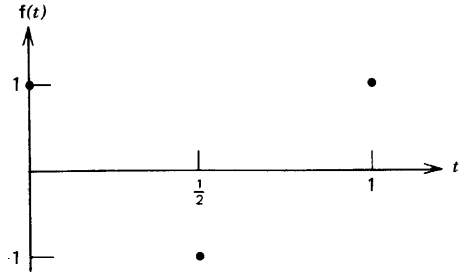


Figure 5.6. A nonzero function with zero norm.

*The definition of "almost everywhere" can be extended to except a countably infinite number of points.

Narici [5.2, p. 173] demonstrate that a sequence of functions can be convergent in norm, yet not converge at all in a pointwise sense.) Convergence in norm is sometimes called **convergence in the mean.** Convergence in norm is precisely the type of convergence which we need for discussion of Fourier series like (5.29). If **f** is a periodic function with period p, the Fourier series expansion (5.29) means

$$\int_0^p \left[\mathbf{f}(t) - a_0 - \sum_{k=1}^{n} \left(a_k \cos\frac{2\pi kt}{p} + b_k \sin\frac{2\pi kt}{p} \right) \right]^2 dt \to 0 \quad \text{as } n \to \infty \tag{5.44}$$

That is, the sequence of partial sums converges in norm to the periodic function **f.** The convergence is pointwise almost everywhere-pointwise except at discontinuities. It makes little practical difference how a function is defined at a finite number of points. Therefore we usually do not distinguish between functions which are equal almost everywhere. Of course, our focus on the convergence in norm of a series of functions does not preclude the possibility that the convergence is actually pointwise and, in fact, uniform.

Infinite-Dimensional Bases

We need to extend the n-dimensional concept of a basis to infinite-dimensional spaces. We naturally think in terms of extending a finite sum to an infinite sum. An infinite set is said to be **countable** if its elements can be numbered using only integer subscripts. We restrict ourselves to a discussion of inner product spaces which have countable bases.*

Definition. Let $\mathcal{V}$ be an infinite-dimensional inner product space. Let $\mathcal{X} \triangleq \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ be a countable set in $\mathcal{V}$. Then $\mathcal{X}$ is said to be a **basis** for $\mathcal{V}$ if every vector $\mathbf{x}$ in $\mathcal{V}$ can be expressed uniquely as a convergent infinite series $\mathbf{x} = \sum_{k=1}^{\infty} c_k \mathbf{x}_k$; that is, if there is a unique set of coordinates $\{c_k\}$ such that $\|\mathbf{x} - \sum_{k=1}^{n} c_k \mathbf{x}_k\|$ can be made arbitrarily small by taking enough terms in the expansion.

Example 1. Bases for $\mathcal{P}(a,b)$. We denote by $\mathcal{P}(a, b)$ the infinite-dimensional space of all real polynomial functions defined on $[a,b]$. Since every polynomial is a (finite) linear combination of functions from the linearly independent set $\mathcal{F} \triangleq \{t^k,\ k = 0,1,2,\ldots\}$, $\mathcal{F}$ is a basis for $\mathcal{P}(a, b)$. Observe that no norm is needed to define a basis for this particular infinite-dimensional space because no infinite sums are required. If we define an inner product on $\mathcal{P}(a, b)$, we can apply the

*A space which has a countable basis is said to be **separable.** Some spaces have only uncountable bases. See Bachman and Narici [5.2, p. 143] for an example.

Gram-Schmidt procedure to the set $\mathcal{F}$, and generate a basis for $\mathcal{P}(a, b)$ consisting of orthogonal polynomials. (See, for instance, the Legendre polynomials of Example 2, Section 5.2.) Each vector in $\mathcal{P}(a,b)$ is a finite linear combination of these orthogonal polynomials. Each different inner product leads to a different orthogonal basis. Of course, each such basis could also be normalized.

Any function that can be expanded in a Taylor series about the origin, as in (5.43), can be represented uniquely by the simple polynomial basis of Example 1. Many familiar functions (e^t, $\sin t$, rational functions, etc.) can be expanded in such a series. These functions are not in $\mathcal{P}(a, b)$, and true infinite sums are required. Thus $\mathcal{F}$ appears to serve as a basis for spaces larger than $\mathcal{P}(a, b)$. How do we tell whether or not $\mathcal{F}$ is a basis for any particular space $\mathcal{V}$ of functions? Of course, the coordinates of the function cannot be unique without independence of the basis vectors. Our previous concept of linear independence, which is based on addition and scalar multiplication, applies only to finite-dimensional spaces. We say an infinite set of vectors $\mathcal{X}$ is **linearly independent** if each finite subset of $\mathcal{X}$ is linearly independent. The vectors in a basis $\mathcal{X}$ must also span $\mathcal{V}$ in the sense that every $\mathbf{x}$ in $\mathcal{V}$ must be representable. But, merely making $\mathcal{X}$ a sufficiently large linearly independent set is not sufficient to guarantee that $\mathcal{X}$ is a basis. The set $\mathcal{F}$ of Example 1 is an infinite linearly independent set. Yet $\mathcal{F}$ is not a basis even for the "nice" space $\mathcal{C}^\infty(-1, 1)$ of infinitely differentiable functions. For example, if we define the function $\mathbf{f}(t) \triangleq \exp(-1/t^2)$ to have the value $\mathbf{f}(0) = 0$ at the origin, it is infinitely differentiable; but it has the Taylor coefficients $\mathbf{f}(0) = \mathbf{f}'(0) = \mathbf{f}''(0)/2 = \ldots = 0$. Thus an attempted Taylor series expansion of $\mathbf{f}$ converges to the wrong (zero) function.

According to a famous theorem of Weierstrass [5.4], any function in $\mathcal{C}(a,b)$ can be represented arbitrarily closely in a pointwise sense (and in norm) by a polynomial. Yet this fact does not imply that $\mathcal{F}$ is a basis for $\mathcal{C}(a,b)$. We must still determine whether or not every $\mathbf{f}$ in $\mathcal{C}(a, b)$ is representable by a unique convergent expansion of the form $\sum_{k=0}^{\infty} c_k t^k$. In general, even though $\{\mathbf{x}_k\}$ is an infinite linearly independent set, there may be no approximation $\sum_{k=1}^{n} c_k \mathbf{x}_k$ that will approach a given vector $\mathbf{x}$ in norm unless the coefficients $\{c_k\}$ are modified as n increases. (See Naylor and Sell [5.17], pp. 315-316.) It is difficult to tell if a specific set is a basis without displaying and examining the coordinates of a general vector in the space. We will find that orthogonality of the vectors in a set eases considerably the task of determining whether or not the set is a basis.

Orthogonal Bases for Infinite-Dimensional Spaces

Actual determination of the coordinates of a specific vector relative to an arbitrary basis is not generally feasible in an infinite-dimensional space. It

requires solving for the numbers c_k in the vector equation $\mathbf{x} = \sum_{k=1}^{\infty} c_k \mathbf{x}_k$; in effect, we must solve an infinite set of simultaneous equations. However, if the basis $\mathcal{X}$ is orthogonal (or orthonormal), the coordinates c_k are the Fourier coefficients, which can be computed independently. This fact is one reason why we work almost exclusively with orthogonal (or orthonormal) bases in infinite-dimensional spaces. If $\mathcal{X} \triangleq \{\mathbf{x}_k\}$ is a countable orthogonal basis for an inner product space $\mathcal{V}$, the Fourier series expansion of a vector $\mathbf{x}$ in $\mathcal{V}$ can be developed by an extension of the process used to obtain the finite-dimensional expansion (5.16)-(5.17). Let $\mathbf{x}_j$ be one of the first n vectors in the infinite dimensional basis $\mathcal{X}$. Let c_i be the ith coordinate of $\mathbf{x}$ relative to $\mathcal{X}$. The Cauchy-Schwartz inequality* shows that

$$\left|\left\langle \mathbf{x} - \sum_{k=1}^{n} c_k \mathbf{x}_k, \mathbf{x}_j \right\rangle\right| \leq \left\| \mathbf{x} - \sum_{k=1}^{n} c_k \mathbf{x}_k \right\| \|\mathbf{x}_j\|$$

The right side of this expression approaches zero as $n \to \infty$. Therefore, for each $j < n$,

$$\left|\left\langle \mathbf{x} - \sum_{k=1}^{n} c_k \mathbf{x}_k, \mathbf{x}_j \right\rangle\right| = \left|\langle \mathbf{x}, \mathbf{x}_j \rangle - \sum_{k=1}^{n} c_k \langle \mathbf{x}_k, \mathbf{x}_j \rangle\right|$$

$$= \left|\langle \mathbf{x}, \mathbf{x}_j \rangle - c_j \|\mathbf{x}_j\|^2\right| \to 0$$

as $n \to \infty$. Since the quantity approaching zero is independent of n, it must equal zero, and

$$c_j = \frac{\langle \mathbf{x}, \mathbf{x}_j \rangle}{\|\mathbf{x}_j\|^2} \tag{5.45}$$

Thus the **Fourier series expansion of x** is

$$\mathbf{x} = \sum_{k=1}^{\infty} \frac{\langle \mathbf{x}, \mathbf{x}_k \rangle}{\langle \mathbf{x}_k, \mathbf{x}_k \rangle} \mathbf{x}_k \tag{5.46}$$

Of course, if the basis is orthonormal, the kth coefficient in (5.46) is just $c_k = \langle \mathbf{x}, \mathbf{x}_k \rangle$.

By an argument similar to the one above, we show that the coefficients $\{c_k\}$ in an orthogonal expansion are unique. Suppose $\mathbf{x} = \sum_{k=1}^{\infty} d_k \mathbf{x}_k$ is a

*P&C 5.4.

second expansion of $\mathbf{x}$. Then by the triangle inequality,*

$$\left\|\left(\mathbf{x}-\sum_{k=1}^{n} c_k\mathbf{x}_k\right)-\left(\mathbf{x}-\sum_{k=1}^{n} d_k\mathbf{x}_k\right)\right\| = \left\|\sum_{k=1}^{n}(d_k-c_k)\mathbf{x}_k\right\|$$

$$\leqslant \left\|\mathbf{x}-\sum_{k=1}^{n} c_k\mathbf{x}_k\right\| + \left\|\mathbf{x}-\sum_{k=1}^{n} d_k\mathbf{x}_k\right\| \to 0$$

as $n\to\infty$. Then if $\mathbf{x}_j$ is one of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, we again employ the Cauchy-Schwartz inequality to find that as $n\to\infty$

$$\left|\left\langle \sum_{k=1}^{n}(d_k-c_k)\mathbf{x}_k, \mathbf{x}_j\right\rangle\right| = \left|\sum_{k=1}^{n}(d_k-c_k)\langle \mathbf{x}_k, \mathbf{x}_j\rangle\right| = |(d_j-c_j)|\|\mathbf{x}_j\|^2$$

$$\leqslant \left\|\sum_{k=1}^{n}(d_k-c_k)\mathbf{x}_k\right\| \|\mathbf{x}_j\| \to 0$$

It follows that $d_j = c_j$, and the coordinates of $\mathbf{x}$ with respect to an orthogonal basis are unique.

Thus the only question of concern, if $\mathfrak{X}$ is an orthogonal set, is whether or not $\mathfrak{X}$ is a large enough set to allow expansion of all vectors $\mathbf{x}$ in $\mathfrak{V}$. If there is a vector $\mathbf{x}$ in $\mathfrak{V}$ for which there is not a convergent expansion, then

$$\mathbf{z} \triangleq \mathbf{x} - \sum_{k=1}^{\infty} \frac{\langle \mathbf{x}, \mathbf{x}_k\rangle}{\langle \mathbf{x}_k, \mathbf{x}_k\rangle}\mathbf{x}_k$$

is nonzero. Furthermore, $\mathbf{z}$ is orthogonal to each vector $\mathbf{x}_j$ in $\mathfrak{X}$, and could be added to $\mathfrak{X}$ to make it more nearly complete (more nearly a basis).

Definition. We say **an orthogonal set is complete in the inner product space** $\mathfrak{V}$ if there is no nonzero vector in $\mathfrak{V}$ which is orthogonal to every vector in $\mathfrak{X}$.

It follows from the discussion above that an *orthogonal set $\mathfrak{X}$ is a basis for $\mathfrak{V}$ if and only if it is complete in $\mathfrak{V}$*. Any orthogonal set in a separable space $\mathfrak{V}$ can be extended (by adding vectors) until it is complete in $\mathfrak{V}$. A practical technique for testing an orthogonal set $\{\mathbf{x}_k\}$ to see if it is a basis consists in showing that the only vector orthogonal to each vector $\mathbf{x}_k$ is the zero vector $\boldsymbol{\theta}$. If $\mathfrak{X}$ is an orthogonal basis for $\mathfrak{V}$, then only for $\mathbf{x} = \boldsymbol{\theta}$ is it true that all the Fourier coefficients $\langle \mathbf{x}, \mathbf{x}_k\rangle$ are equal to zero. Thus this test

*P&C 5.4.

for completeness of the orthogonal set $\mathcal{X}$ is equivalent to a test for validity of the Fourier expansion (5.46) for each x in $\mathcal{V}$.

Example 2. Orthogonal bases for $\mathcal{C}(a,b)$. The Weierstrass approximation theorem [5.4] guarantees that any continuous function can be approximated arbitrarily closely in norm by a polynomial. We noted earlier that this fact is insufficient to guarantee that the set $\mathcal{T} \triangleq \{t^k, \quad k = 0,1,2,\ldots\}$ is a basis for $\mathcal{C}(a,b)$. On the other hand, suppose that $\mathcal{G} \triangleq \{\mathbf{p}_k\}$ is a basis for $\mathcal{P}(a, b)$ consisting in real polynomials $\mathbf{p}_k$ which are orthogonal relative to some inner product. (We could obtain $\mathcal{G}$ from $\mathcal{T}$ by the Gram-Schmidt procedure as in Example 2 of Section 5.2.) We now show that $\mathcal{G}$ is also a basis—an orthogonal basis—for $\mathcal{C}(a,b)$. Let $\mathbf{f}$ be a real continuous function on $[a,b]$. Assume $\langle \mathbf{f}, \mathbf{p}_k \rangle = 0$ for all polynomials $\mathbf{p}_k$ in $\mathcal{G}$. We show that $\mathbf{f}$ must be the zero vector. By the Weierstrass theorem, for each $\epsilon > 0$ there is a polynomial $\mathbf{p}_\epsilon$ such that $\|\mathbf{f} - \mathbf{p}_\epsilon\|^2 < \epsilon$. Furthermore, since $\mathcal{G}$ is a basis for $\mathcal{P}(a, b)$, $\mathbf{p}_\epsilon = \sum_{k=1}^{N} c_k \mathbf{p}_k$ for some finite number N. Then

$$\begin{aligned} \|\mathbf{f} - \mathbf{p}_\epsilon\|^2 &= \|\mathbf{f}\|^2 - 2|\langle \mathbf{f}, \mathbf{p}_\epsilon \rangle| + \|\mathbf{p}_\epsilon\|^2 \\ &= \|\mathbf{f}\|^2 + \|\mathbf{p}_\epsilon\|^2 - 2\left| \sum_{k=1}^{N} c_k \langle \mathbf{f}, \mathbf{p}_k \rangle \right| \\ &= \|\mathbf{f}\|^2 + \|\mathbf{p}_\epsilon\|^2 \end{aligned}$$

Since $\|\mathbf{f}\|^2 + \|\mathbf{p}_\epsilon\|^2 < \epsilon$ for an arbitrarily small number ϵ , $\|\mathbf{f}\| = 0$, and the function $\mathbf{f}$ must be the zero vector. Thus the orthogonal set $\mathcal{G}$ is complete in $\mathcal{C}(a, b)$, and all orthogonal polynomial bases for $\mathcal{P}(a,b)$ are bases for $\mathcal{C}(a,b)$ as well.

Harmuth [5.13] describes an interesting orthogonal basis for $\mathcal{C}(a,b)$—the set of Walsh functions. These functions, which take on only the values 1 and -1, are extremely useful in digital signal processing; only additions and subtractions are needed to compute the Fourier coefficients.

The classical Fourier series expansion (5.29) for periodic functions applies to functions $\mathbf{f}$ in the standard inner product space $\mathcal{C}(a,$ b); we merely repeat the values of $\mathbf{f}$ on $[a,b]$ periodically outside of $[a,b]$ with period $p = b - a$. If we denote the set of sinusoidal functions (5.30) by $\mathcal{K}$, then the orthonormal set $\mathcal{K}$ is complete in $\mathcal{C}(a,b)$; it is an orthonormal basis for $\mathcal{C}(a,b)$.

Exercise 2. The Fourier series expansion (5.29) of a periodic function $\mathbf{f}$ contains only sine terms if $\mathbf{f}$ is an odd function and only cosine terms if $\mathbf{f}$ is an even function. Show that in addition to the sine-cosine expansion mentioned in Example 2, a function in $\mathcal{C}(0,b)$ can be expanded in two additional series of period $p = 2b$, one involving only sines (the Fourier sine series), the other involving only cosines (the Fourier cosine series).

If $\{\mathbf{x}_k\}$ is an orthonormal basis for $\mathcal{V}$, the set of Fourier coefficients (or coordinates) $\{\langle \mathbf{x}, \mathbf{x}_k \rangle\}$ is equivalent to the vector $\mathbf{x}$ itself, and operations on $\mathbf{x}$ can be carried out in terms of operations on the Fourier coefficients. For

instance, we can compute inner products by means of **Parseval's equation:**

$$\langle \mathbf{x}, \mathbf{y} \rangle = \left\langle \sum_{k=1}^{\infty} \langle \mathbf{x}, \mathbf{x}_k \rangle \mathbf{x}_k, \sum_{j=1}^{\infty} \langle \mathbf{y}, \mathbf{x}_j \rangle \mathbf{x}_j \right\rangle$$

$$= \sum_{k=1}^{\infty} \langle \mathbf{x}, \mathbf{x}_k \rangle \langle \mathbf{y}, \mathbf{x}_k \rangle \tag{5.47}$$

(Because we are concerned primarily with real spaces, we usually drop the complex conjugate.) If $\mathbf{y} = \mathbf{x}$, (5.47) becomes **Parseval's identity:**

$$\|\mathbf{x}\|^2 = \sum_{k=1}^{\infty} |\langle \mathbf{x}, \mathbf{x}_k \rangle|^2 \tag{5.48}$$

Equation (5.48) is also a special case of the Pythagorean theorem. Furthermore, it is the limiting case (equality) of Bessel's inequality (P&C 5.4). In point of fact, Bessel's inequality becomes the identity (5.48) for each $\mathbf{x}$ in $\mathcal{V}$ if and only if the orthonormal set $\{\mathbf{x}_k\}$ is a basis for $\mathcal{V}$.

Of course, not all bases for infinite-dimensional spaces are orthogonal bases. Naylor and Sell [5.17, p. 317] describe one set of conditions which guarantees that a nonorthogonal countable set is a basis. However, rarely do we encounter in practical analysis the use of a nonorthogonal basis for an infinite-dimensional space.

In a finite-dimensional space we can pick an inner product to orthonormalize any basis; specifically, we pick the inner product defined by Parseval's equation (5.27). The infinite-dimensional equivalent (5.47) is less useful for this purpose because the unknown inner product is needed to find the coordinates in the equation. In an infinite-dimensional space, the choice of inner' product still determines the orthonormality of a set of vectors; but the norm associated with the inner product also determines whether the vectors of an orthonormal set are complete in the space. Given a basis for an inner product space, what changes can we make in the inner product (in order to orthonormalize the basis) and still have a basis? For spaces of functions defined on a finite interval, a positive reweighting of the inner product does not destroy convergence. For example, if $\{\mathbf{f}_k\}$ is a basis for $\mathcal{C}(a,b)$ with the standard function space inner product, then for any $\mathbf{f}$ in $\mathcal{C}(a, b)$ (with unique coordinates $\{c_k\}$ relative to $\{\mathbf{f}_k\}$) and any $\epsilon > 0$ there is a number N such that $\int_a^b |\mathbf{f}(t) - \sum_{k=1}^{n} c_k \mathbf{f}_k(t)|^2 dt < \epsilon$ for $n > N$. Suppose we define a new inner product for the same space of continuous functions:

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\omega} \triangleq \int_a^b \omega(t) \mathbf{f}(t) \overline{\mathbf{g}(t)}\, dt \tag{5.49}$$

where $\omega(t)$ is bounded and positive for t in $[a,b]$. Then, using the same basis $\{\mathbf{f}_k\}$ and the same coefficients $\{c_k\}$,

$$\int_a^b \omega(t)\left|\mathbf{f}(t)-\sum_{k=1}^{n} c_k\mathbf{f}_k(t)\right|^2 dt \leqslant M\int_a^b \left|\mathbf{f}(t)-\sum_{k=1}^{n} c_k\mathbf{f}_k(t)\right|^2 dt < M\epsilon$$

where M is a positive bound on $\omega(t)$. Since ϵ is arbitrarily small, $M\epsilon$ is also arbitrarily small. Thus for large enough n the partial sum is still arbitrarily close to $\mathbf{f}$ in the new norm. We represent by $\mathcal{C}(\omega;a,b)$ the space of continuous functions with the inner product (5.49). It is evident that the choice of ω affects the definition of orthogonality, but does not affect the convergence or nonconvergence of sequences of vectors. Of course, the weighted inner product (5.49) does not represent all possible inner products on the function space $\mathcal{C}(a,b)$; it does not allow for "cross products" analogous to those in (5.13). Yet it is general enough to allow us to orthogonalize many useful bases.

Example 3. Orthogonalizing a Basis by Weighting the Inner Product The shaft position ϕ of an armature-controlled motor as a function of armature voltage $\mathbf{u}$ is described by

$$(\mathbf{L}\phi)(t) \triangleq \frac{d^2\phi(t)}{dt^2} + \frac{d\phi(t)}{dt} = \mathbf{u}(t)$$

The eigenfunctions of $\mathbf{L}$ with the boundary conditions $\phi(0) = \phi(b) = 0$ are given by (4.38):

$$\mathbf{f}_k(t) = e^{-t/2}\sin\left(\frac{\pi kt}{b}\right), \qquad k = 1,2,\ldots$$

We pick the weight ω in the inner product (5.49) so that the set $\{\mathbf{f}_k\}$ is orthogonal:

$$\begin{aligned}\langle \mathbf{f}_k,\mathbf{f}_m\rangle_\omega &= \int_0^b \omega(t)e^{-t/2}\sin\left(\frac{\pi kt}{b}\right)e^{-t/2}\sin\left(\frac{\pi mt}{b}\right)dt\\ &= \int_0^b \omega(t)e^{-t}\sin\left(\frac{\pi kt}{b}\right)\sin\left(\frac{\pi mt}{b}\right)dt\\ &= 0\end{aligned}$$

for $m \neq k$. The functions $\{\sin(\pi kt/b)\}$ form a well-known orthogonal basis for $\mathcal{C}(0,b)$ using the standard function space inner product, as we noted in Example 2. Therefore, the weight $\omega(t) = e^t$ makes the functions $\{\mathbf{f}_k\}$ orthogonal with respect to the weighted inner product. (The choice $\omega(t) = 2e^t/b$ would make the set orthonormal. However, it is more convenient to normalize the eigenfunctions, multiplying each by $\sqrt{2/b}$).

We now demonstrate that the eigenfunctions $\{\mathbf{f}_k\}$ are a basis [complete in $\mathcal{C}(0,b)$] by showing that the only function orthogonal to all functions in the set is the zero function. Suppose $\langle \mathbf{f},\mathbf{f}_k\rangle_\omega = 0$ for all k. Then

$$\langle \mathbf{f},\mathbf{f}_k\rangle_\omega = \int_0^b e^t \mathbf{f}(t) e^{-t/2} \sin\left(\frac{\pi k t}{b}\right) dt = \int_0^b \mathbf{f}(t) e^{t/2} \sin\left(\frac{\pi k t}{b}\right) dt = 0$$

Since $\{\sin(\pi kt/b)\}$ is an orthogonal basis with respect to the standard function space inner product, and since the Fourier coefficients of $\mathbf{f}e^{t/2}$ relative to this basis are all zero, $\mathbf{f}e^{t/2} = \boldsymbol{\theta}$ and $\mathbf{f} = \boldsymbol{\theta}$. Therefore, $\{\mathbf{f}_k\}$ is an orthogonal basis for the space $\mathcal{C}(e^t;0,b)$. This orthogonal basis of eigenfunctions is used in Example 4, Section 5.5 to diagonalize and solve the differential equation described above.

Hilbert Spaces

From Example 2 it is evident that a single infinite set can be a basis for several different infinite-dimensional spaces. Suppose $\{\mathbf{x}_k\}$ is an orthonormal basis for an infinite-dimensional inner product space $\mathcal{V}$. Presumably there are vectors $\mathbf{x}$, not in $\mathcal{V}$, which can be expanded uniquely in terms of $\{\mathbf{x}_k\}$ (assuming we extend the inner product space operations to the additional vectors). What is the largest, most inclusive space for which $\{\mathbf{x}_k\}$ is a basis? We refer to the largest space $\mathcal{H}$ of vectors which can be represented in the form of $\mathbf{x} = \sum_{k=1}^{\infty} c_k \mathbf{x}_k$ as the space **spanned** (or **generated)** by the basis $\{\mathbf{x}_k\}$. (Because $\{\mathbf{x}_k\}$ is orthonormal, the coefficients in the expansion of $\mathbf{x}$ are necessarily unique.) We show that $\mathcal{H}$ is precisely the space of vectors $\mathbf{x}$ which are **square-summable** combinations of the basis vectors; that is, $\mathbf{x}$ such that $\mathbf{x} = \sum_{k=1}^{\infty} c_k \mathbf{x}_k$ with $\sum_{k=1}^{\infty} |c_k|^2 < \infty$.

Suppose a vector $\mathbf{x}$ in $\mathcal{H}$ can be expressed as $\mathbf{x} = \sum_{k=1}^{\infty} c_k \mathbf{x}_k$, where $\{\mathbf{x}_k\}$ is an orthonormal basis for the inner product space $\mathcal{V}$. Define $\mathbf{y}_n \overset{\Delta}{=} \sum_{k=1}^{n} c_k \mathbf{x}_k$. Then $\{\mathbf{y}_n,\ n = 1,2,\ldots\}$ is a Cauchy sequence which approaches $\mathbf{x}$, and $\|\mathbf{y}_m - \mathbf{y}_n\| \to 0$ as $m, n \to \infty$. If we assume $n > m$ and use the orthonormality of $\{\mathbf{x}_k\}$, we find $\|\mathbf{y}_n - \mathbf{y}_m\|^2 = \|\sum_{k=1}^{n} c_k \mathbf{x}_k - \sum_{k=1}^{m} c_k \mathbf{x}_k\|^2 = \|\sum_{k=m+1}^{n} c_k \mathbf{x}_k\|^2 = \sum_{k=m+1}^{n} |c_k|^2$. Therefore, $\sum_{k=m+1}^{n} |c_k|^2 \to 0$ as $m, n \to \infty$. It follows that $\sum_{k=m+1}^{\infty} |c_k|^2 \to 0$ as $m \to \infty$; in other words, for each $\epsilon > 0$ there is a positive number M such that $m > M$ implies $\sum_{k=m+1}^{\infty} |c_k|^2 < \epsilon$. Pick a value of ϵ, and let m be a finite number greater than M. Then

$$\sum_{k=1}^{m} |c_k|^2 < \infty \quad \text{and} \quad \sum_{k=m+1}^{\infty} |c_k|^2 < \epsilon$$

Consequently, $\sum_{k=1}^{\infty} |c_k|^2 < \infty$, and $\mathbf{x}$ can be expanded on the basis $\{\mathbf{x}_k\}$ only if $\mathbf{x}$ is a square-summable combination of the basis vectors. Conversely, square summability of the coefficients $\{c_k\}$ implies that $\|\mathbf{y}_m - \mathbf{y}_n\|^2$

$\rightarrow 0$ as $m, n \rightarrow \infty$, and the sequence $\{\mathbf{y}_n\}$ is a Cauchy (convergent) sequence. Thus any square-summable combination of $\{\mathbf{x}_k\}$ must converge to some vector $\mathbf{x}$ in the space which we have denoted $\mathcal{H}$.

It is apparent that $\mathcal{H}$ may be more complete than $\mathcal{V}$. If we were to associate a single inner product space with the basis $\{\mathbf{x}_k\}$, the natural choice would be the largest space for which $\{\mathbf{x}_k\}$ is a basis, the space $\mathcal{H}$. If $\mathcal{V} \neq \mathcal{H}$, then $\mathcal{V}$ and $\mathcal{H}$ differ only in their "limit vectors." Suppose $\mathbf{x}$ satisfies $\mathbf{x} = \sum_{k=1}^{\infty} c_k \mathbf{x}_k$, and again denote the nth partial sum by $\mathbf{y}_n = \sum_{k=1}^{n} c_k \mathbf{x}_k$. The sequence of partial sums $\{\mathbf{y}_n\}$ is a Cauchy sequence with limit $\mathbf{x}$. Thus each $\mathbf{x}$ in $\mathcal{H}$ is the limit of a Cauchy sequence in $\mathcal{V}$. In point of fact, $\mathcal{H}$ differs from $\mathcal{V}$ only in that $\mathcal{H}$ contains the limits of more Cauchy sequences from $\mathcal{V}$ than does $\mathcal{V}$.

Example 4. A Cauchy Sequence in $\mathcal{C}(0,1)$ with no Limit in $\mathcal{C}(0,1)$. The functions $\{\mathbf{f}_k\}$ of Figure 5.7 form a Cauchy sequence in $\mathcal{C}(0,1)$ with the standard function space inner product (5.16); that is,

$$\int_0^1 (\mathbf{f}_m(t) - \mathbf{f}_n(t))^2 \, dt \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

The limit in norm of the sequence $\{\mathbf{f}_k\}$ is the discontinuous function

$$\mathbf{f}(t) = \begin{cases} 1, & t \leqslant \frac{1}{2} \\ 0, & t > \frac{1}{2} \end{cases}$$

which is not in $\mathcal{C}(0,1)$. The limit vector $\mathbf{f}$ is a member of a space which is larger and more complete than $\mathcal{C}(0,1)$. Yet $\mathbf{f}$ can be expanded uniquely in the sine-cosine basis (5.30) for $\mathcal{C}(0,1)$.

Definition. Let $\mathcal{S}$ be set in an inner product space $\mathcal{V}$. A vector $\mathbf{x}$ in $\mathcal{V}$ is called a **point of closure** of $\mathcal{S}$ if for each $\epsilon > 0$ there is a vector $\mathbf{y}$ in $\mathcal{S}$ such

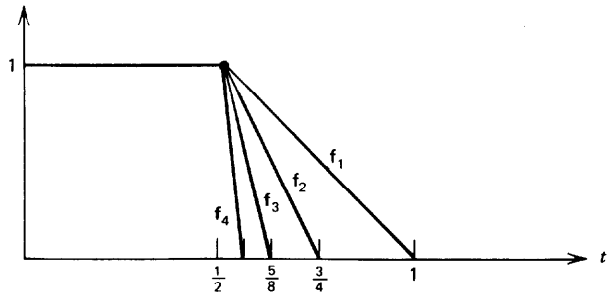


Figure 5.7. A Cauchy sequence in $\mathcal{C}(0,1)$.

that $\|\mathbf{x} - \mathbf{y}\| < \epsilon$; that is, $\mathbf{x}$ can be approximated arbitrarily closely in norm by vectors $\mathbf{y}$ in $\mathcal{S}$. The **closure** of $\mathcal{S}$, denoted $\bar{\mathcal{S}}$, consists in $\mathcal{S}$ together with all its points of closure. If $\mathcal{S}$ contains all its points of closure, it is said to be **closed.** A set $\mathcal{S}_1$ in $\mathcal{S}$ is said to be **dense** in $\mathcal{S}$ if $\mathcal{S}$ is the closure of $\mathcal{S}_1$; that is, if every vector in $\mathcal{S}$ can be approximated arbitrarily closely in norm by a vector in $\mathcal{S}_1$.

Definition. An inner product space $\mathcal{H}$ is said to be **complete** if every Cauchy (convergent) sequence from $\mathcal{H}$ converges in norm to a limit in $\mathcal{H}$. A complete inner product space is called a **Hilbert space.**

The terms closed and complete, as applied to inner product spaces, are essentially equivalent concepts. The inner product space $\mathcal{V}$ discussed above is not complete, whereas the "enlarged" space $\mathcal{H}$ is complete; $\mathbf{x}$ is a Hilbert space. The space $\mathcal{V}$ is dense in $\mathcal{H}$; that is, $\mathcal{H}$ is only a slight enlargement of $\mathcal{V}$. We can complete any inner product space by extending its definition to include all of its limit vectors. Of course, the definitions of addition, scalar multiplication, and inner product must be extended to these additional limit vectors [5.11, p. 17].

Example 5. Finite-Dimensional Hilbert Spaces. Every finite-dimensional inner product space is complete [5.23, p. 143]. For instance, we cannot conceive of an infinite sequence of real n-tuples converging to anything but another real n-tuple; the ith components of a sequence of n-tuples constitute a sequence of real numbers, and the real numbers are complete.

Example 6. The Hilbert Space l_2. We denote by l_2^c the space of square-summable sequences of complex numbers with the inner product $\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \sum_{k=1}^{\infty} \xi_k \bar{\eta}_k$, where ξ_k and η_k are the kth elements of $\mathbf{x}$ and $\mathbf{y}$, respectively. (A square-summable sequence is a sequence for which $\|\mathbf{x}\|^2 = \sum_{k=1}^{\infty} |\xi_k|^2 < \infty$.) We use the symbol l_2 to represent the space of *real* square-summable sequences; then the complex conjugate in the inner product is superfluous. Both the real l_2 and the complex l_2^c are complete [5.23, p. 48]. The standard basis $\{\boldsymbol{\epsilon}_i\}$, where $\boldsymbol{\epsilon}_i = (0, \ldots, 0, 1_i, 0, \ldots)$, is an orthonormal basis for both the real and complex cases.

Example 7. The Hilbert Space $\mathfrak{L}_2(a,b)$. Let $\mathfrak{L}_2^c(a,b)$ be the space of complex square-integrable* functions defined on the finite interval $[a,b]$ with the inner product $\langle \mathbf{f}, \mathbf{g} \rangle \triangleq \int_a^b \mathbf{f}(t)\mathbf{g}(t)\,dt$. (A square-integrable function is one for which $\|\mathbf{f}\|^2 = \int_a^b |\mathbf{f}(t)|^2\,dt$ is finite.) The symbol $\mathfrak{L}_2(a,b)$ is used to represent the space of *real* square-integrable functions; then the complex conjugate is unnecessary. We usually concern ourselves only with the real space. Both the real $\mathfrak{L}_2(a,b)$ and the complex $\mathfrak{L}_2^c(a,b)$ are complete [5.2, p. 115]. The space $\mathfrak{L}_2(a,b)$ contains no delta functions. However, it does contain certain discontinuous functions, for example,

*The integral used in this definition is the Lebesgue integral. For all practical purposes, we can consider Lebesgue integration to be the same as the usual Riemann integration. Where the Riemann integral exists, the two integrals are equal. See Royden [5.21].

step functions. (Recall from the definition of equality in norm that we ignore isolated discontinuities. As a practical matter, we seldom encounter a function with more than a few discontinuities in a finite interval.) We can think of $\mathcal{L}_2(a,b)$ as essentially a space of functions which are piecewise continuous, but perhaps unbounded, in the finite interval $[a,b]$. Any set $\mathcal{G} \triangleq \{\mathbf{p}_k\}$ of orthogonal polynomials which forms a basis for $\mathcal{P}(a,b)$ is a basis for both the real and complex $\mathcal{L}_2(a,b)$. An orthonormal basis for both the real and complex $\mathcal{L}_2(a, b)$ is the set of sinusoids (5.30), with $p = b - a$. Another orthonormal basis for the complex $\mathcal{L}_2^c(a, b)$ is the set of complex exponentials (5.33) with $p = b - a$.

Example 8. The Hilbert Space $\mathcal{L}_2(\omega; a, b)$. Let $\mathcal{L}_2(\omega; a, b)$ represent the set of all ω-square-integrable functions with the inner product (5.49). That is, $\mathcal{L}_2(\omega; a, b)$ contains those functions that have finite norm under the inner product (5.49). From the discussion associated with (5.49) it is apparent that $\mathcal{L}_2(\omega; a, b)$ differs from $\mathcal{L}_2(a, b)$ only in the inner product. Both spaces contain precisely the same functions, and completeness of $\mathcal{L}_2(\omega; a, b)$ follows from the completeness of $\mathcal{L}_2(a,b)$.

A Hilbert space possesses many subsets that are themselves inner product spaces (using the same inner product). These subsets may or may not be complete. If a subset is a complete inner product space, it is itself a Hilbert space, and we refer to it as a **subspace.** If a subset is a vector space, but is not necessarily complete, it is properly termed a **linear manifold.** Since all finite-dimensional vector spaces are complete, all finite-dimensional linear manifolds of $\mathcal{L}_2(a, b)$ are subspaces. However, $\mathcal{P}(a,b)$, $\mathcal{C}(a,b)$, $\mathcal{C}^n(a,b)$, $\mathcal{C}^\infty(a,b)$, and the space of piecewise-continuous functions on $[a, b]$ are (incomplete) linear manifolds of $\mathcal{L}_2(a, b)$. Each of these spaces is dense in $\mathcal{L}_2(a, b)$, and thus is nearly equal to $\mathcal{L}_2(a, b)$.

We note that if $\mathcal{H}$ is a Hilbert space and $\mathcal{S}$ is any set in $\mathcal{H}$, then the orthogonal complement $\mathcal{S}^\perp$ must be a subspace. For if $\{\mathbf{x}_n\}$ is a Cauchy sequence in $\mathcal{S}^\perp$ with limit $\mathbf{x}$ in $\mathcal{H}$, then $\langle \mathbf{x}_n, \mathbf{y} \rangle = 0$ for each $\mathbf{y}$ in $\mathcal{S}$; it follows that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \lim_{n\to\infty} \mathbf{x}_n, \mathbf{y} \rangle = \lim_{n\to\infty} \langle \mathbf{x}_n, \mathbf{y} \rangle = 0$$

and the limit vector $\mathbf{x}$ is also orthogonal to $\mathcal{S}$. [In order to take the limit outside the inner product, we have relied on the continuity of inner products. See (5.56).]

Example 9. An Infinite-Dimensional (Complete) Subspace of $\mathcal{L}_2(a,b)$. Let $\mathcal{W}$ be the (one-dimensional) subspace of constant functions in $\mathcal{L}_2(a, b)$. By the previous paragraph, the orthogonal complement $\mathcal{W}^\perp$ is complete. But $\mathcal{W}^\perp$ consists in those functions $\mathbf{f}$ in $\mathcal{L}_2(a, b)$ which satisfy $\int_a^b c\mathbf{f}(t)\,dt = 0$ for all constants c. Thus the functions in $\mathcal{L}_2(a, b)$ whose average value is zero form a complete subspace of $\mathcal{L}_2(a, b)$. This subspace is itself a Hilbert space.

Why do we care whether or not a vector space is complete? One reason is that we wish to extend finite-dimensional concepts to infinite-dimensional cases. Some of these concepts extend only for a Hilbert space, the natural generalization of a finite-dimensional space. (Recall that finite-dimensional spaces are Hilbert spaces.) The only concept we have discussed thus far which applies only for Hilbert spaces is the **projection theorem** (5.20), $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^{\perp}$. The proof of (5.20) depends on the fact that repeated application of the Gram-Schmidt procedure can generate no more than n orthogonal vectors in an n-dimensional space. The theorem is valid in an infinite-dimensional space $\mathcal{V}$ if and only if $\mathcal{V}$ is a Hilbert space and $\mathcal{W}$ is a (complete) subspace of $\mathcal{V}$ [5.2, p. 172]; of course, $\mathcal{W}^{\perp}$ is always complete.

Fortunately, the question of completeness of an inner product space is seldom of practical concern, since we can complete any inner product space by extending its definition. Suppose a linear transformation $\mathbf{T}$ has its domain and range in a separable Hilbert space $\mathcal{H}$ (a Hilbert space with a countable basis). Then if domain($\mathbf{T}$) is dense in $\mathcal{H}$, we refer to $\mathbf{T}$ as a *linear operator on the Hilbert space* $\mathcal{H}$. For instance, the completion of the space $\mathcal{C}^{\infty}(ab)$ of real, infinitely differentiable functions on $[a,b]$ is just $\mathcal{L}_2(a,b)$ of Example 7. We apply a differential operator $\mathbf{L}$ to any space of "sufficiently differentiable" functions and still refer to $\mathbf{L}$ as a differential operator on $\mathcal{L}_2(a,b)$.

In our examination of finite-dimensional vector spaces we used the process of taking coordinates to equate every n-dimensional space to the matrix space $\mathcal{M}^{n \times 1}$. We now equate inner product spaces, both finite and infinite-dimensional. Two inner product spaces, $\mathcal{V}$ with inner product $\langle\ ,\ \rangle_{\mathcal{V}}$ and $\mathcal{W}$ with inner product $\langle\ ,\ \rangle_{\mathcal{W}}$, are **isomorphic** (or **equivalent)** if there is an invertible linear transformation $\mathbf{T}$: $\mathcal{V} \rightarrow \mathcal{W}$ which preserves inner products; that is, for which $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{V}} = \langle \mathbf{Tx}, \mathbf{Ty} \rangle_{\mathcal{W}}$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{V}$. The process of taking coordinates relative to any orthonormal basis is just such a transformation.

Example 10. Coordinates for Real n-Dimensional Inner Product Spaces. For n-dimensional spaces, we take $\mathcal{M}^{n \times 1}$ with its standard inner product as our space of coordinates. Let $\mathcal{V}$ be any real n-dimensional inner product space; let $\mathcal{X}$ be an orthonormal basis for $\mathcal{V}$. Define $\mathbf{T}$: $\mathcal{V} \rightarrow \mathcal{M}^{n \times 1}$ as the invertible linear transformation which assigns to each vector $\mathbf{x}$ in $\mathcal{V}$ its set of Fourier coefficients (or coordinates) in $\mathcal{M}^{n \times 1}$:

$$\mathbf{Tx} \triangleq (\langle \mathbf{x}, \mathbf{x}_1 \rangle_{\mathcal{V}} \cdots \langle \mathbf{x}, \mathbf{x}_n \rangle_{\mathcal{V}})^{\mathrm{T}} = [\mathbf{x}]_{\mathcal{X}}$$

Since $\mathcal{X}$ is orthonormal, Parseval's equation (5.27) is satisfied:

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{V}} = [\mathbf{y}]_{\mathcal{X}}^{\mathrm{T}} [\mathbf{x}]_{\mathcal{X}} = \langle \mathbf{Tx}, \mathbf{Ty} \rangle_{\mathcal{M}^{n \times 1}}$$

This is the standard inner product in the real space $\mathfrak{M}^{n\times 1}$. Clearly, each real n-dimensional inner product space (a Hilbert space) is equivalent to $\mathfrak{M}^{n\times 1}$ with its standard inner product. (By inserting a complex conjugate in the inner product, we can show that every complex n-dimensional inner product space is equivalent to $\mathfrak{M}_c^{n\times 1}$, the space of complex $n \times 1$ matrices with its standard inner product.)

Example 11. Coordinates for Real Separable Infinite-Dimensional Hilbert Spaces. The logic of Example 10 applies to all separable Hilbert spaces (Hilbert spaces which have countable bases). For separable infinite-dimensional spaces, we take l_2 with its standard inner product as our space of coordinates. A separable space has a countable basis. Any such basis can be orthonormalized by the Gram-Schmidt procedure. Suppose $\mathcal{X} = \{\mathbf{x}_k\}$ is an orthonormal basis for a real separable Hilbert space $\mathcal{H}$. We define $\mathbf{T}$: $\mathcal{H} \rightarrow l_2$ as the process of assigning Fourier coefficients relative to this basis :

$$\mathbf{Tx} \triangleq (\langle \mathbf{x}, \mathbf{x}_1 \rangle_{\mathcal{H}}, \langle \mathbf{x}, \mathbf{x}_2 \rangle_{\mathcal{H}}, \ldots) = [\mathbf{x}]_{\mathcal{X}} \tag{5.50}$$

From our discussion of the space spanned by an orthonormal basis, we know that the coordinates (Fourier coefficients) of vectors in $\mathcal{H}$ consist in the square-summable sequences which constitute l_2. Since Fourier expansions exist and are unique for each $\mathbf{x}$ in $\mathcal{H}$, $\mathbf{T}$ is invertible. Because the set $\{\mathbf{x}_k\}$ is orthonormal, Parseval's equation (5.47) applies:

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}} &= \sum_{k=1}^{\infty} \langle \mathbf{x}, \mathbf{x}_k \rangle_{\mathcal{H}} \langle \mathbf{y}, \mathbf{x}_k \rangle_{\mathcal{H}} \\ &= \langle \mathbf{Tx}, \mathbf{Ty} \rangle_{l_2}\end{aligned}$$

This is the standard inner product between the coordinates of $\mathbf{x}$ and $\mathbf{y}$ (in l_2). Therefore, every real separable infinite-dimensional Hilbert space is equivalent to the real space l_2 with its standard inner product. Thus the somewhat mysterious space $\mathcal{L}_2(a,b)$ is, in essence, no more complicated than l_2. In Example 11 of Section 5.4 we introduce $\mathcal{L}_2(\Omega)$, an inner product space of square-integrable functions defined on a finite two-dimensional domain Ω; $\mathcal{L}_2(\Omega)$ is also equivalent to l_2. (By inserting a complex conjugate in each inner product, we find that every complex separable infinite-dimensional Hilbert space is equivalent to the complex space l_2^c with its standard inner product.)

5.4 Adjoint Transformations

In the preceding section we developed separable Hilbert spaces as natural generalizations of n-dimensional inner product spaces. We know that we can generate a countable orthonormal basis for any such space. In (5.25) we diagonalized and computationally decoupled an operator equation in an n-dimensional space by means of an orthonormal basis of eigenvectors. We have also discussed the applicability of orthonormal eigenvectors to an infinite-dimensional example, the steady-state analysis of a dynamic sys-

tem. Can we diagonalize a general linear operator on an infinite-dimensional space, a differential operator, for instance? We can if there is a countable orthonormal basis for the infinite-dimensional space which is composed of eigenvectors of the operator. In Example 3, Section 5.3 we orthogonalized a set of eigenfunctions by careful choice of the inner product. We would like to be able to make general statements which clearly characterize the existence of an orthonormal basis of eigenvectors for a given operator on a given infinite-dimensional space. Given a set of eigenvectors for an operator **T** on a given *vector space* $\mathcal{V}$, for what inner products are the eigenvectors an orthogonal (or orthonormal) basis? For what operators **T** on a given *inner product space* $\mathcal{V}$ do there exist orthonormal bases of eigenvectors? In sum, under what conditions can we diagonalize an operator equation by means of an orthonormal basis? The answers to these questions are to a great extent answered in the concept of "self-adjointness." We introduce the adjoint transformation in this section, and return to a discussion of orthonormal bases of eigenvectors for solving operator equations in Section 5.5.

We observed in Chapter 1 that we can interpret a matrix multiplication **Ax** either as a linear combination of the columns of **A** or as a set of standard inner products of **x** with the rows of **A.** Furthermore, we know that the number of independent rows of **A** equals the number of independent columns. Since the rows and columns of **A** possess much common information, we would be surprised if multiplication by the transposed matrix $\mathbf{A}^{\mathbf{T}}$ did not describe a transformation closely related to multiplication by **A.** Let $\mathbf{T}: \mathfrak{M}^{2\times 1} \to \mathfrak{M}^{3\times 1}$ be defined by

$$\mathbf{Tx} \triangleq \mathbf{Ax} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x}$$

We define the "transpose" transformation $\mathbf{T}^{\mathbf{T}}: \mathfrak{M}^{3\times 1} \to \mathfrak{M}^{2\times 1}$ by

$$\mathbf{T}^{\mathbf{T}}\mathbf{y} \triangleq \mathbf{A}^{\mathbf{T}}\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{y}$$

The range and nullspace of **T** are important indicators of its structure. They display the nonsolvability and nonuniqueness of solutions of the equation **Tx** = **y.** Observe that

$$\text{range}(\mathbf{T}) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}, \quad \text{nullspace}(\mathbf{T}) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

$$\text{range}(\mathbf{T}^{\mathbf{T}}) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \quad \text{nullspace}(\mathbf{T}^{\mathbf{T}}) = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Range($\mathbf{T}$) and nullspace($\mathbf{T}$) are, of course, in different spaces. However, range($\mathbf{T}$) and nullspace($\mathbf{T}^{\mathrm{T}}$) are in the same space; in fact,

$$\mathcal{M}^{3\times 1} = \mathrm{range}(\mathbf{T}) \oplus \mathrm{nullspace}(\mathbf{T}^{\mathrm{T}})$$

Similarly,

$$\mathcal{M}^{2\times 1} = \mathrm{range}(\mathbf{T}^{\mathrm{T}}) \oplus \mathrm{nullspace}(\mathbf{T})$$

Furthermore, these direct sums are orthogonal relative to the standard inner products for the two spaces. It is evident, at least for this example, that $\mathbf{T}$ and $\mathbf{T}^{\mathrm{T}}$ together characterize the transformation $\mathbf{T}$ more explicitly than does $\mathbf{T}$ alone.

We extend the transpose concept to general linear transformations. We find that the orthogonal decomposition illustrated above still applies. Recall that orthogonal decomposition is closely related to Fourier series expansion and, therefore, to orthonormal bases. The generalization of $\mathbf{T}^{\mathrm{T}}$, together with $\mathbf{T}$ itself, characterizes the existence or nonexistence of orthonormal bases of eigenvectors.

Bounded Linear Transformations

The generalization of the transpose matrix exists only for transformations which satisfy certain restrictions. We now define the concepts which we use to express these restrictions.

Definition. Let $\mathbf{T}$ be a (possibly nonlinear) transformation from an inner product space $\mathcal{V}$ into an inner product space $\mathcal{W}$. Then $\mathbf{T}$ **is bounded** if there is a positive number α such that

$$\|\mathbf{Tx}\|_{\mathcal{W}} \leqslant \alpha \|\mathbf{x}\|_{\mathcal{V}} \quad \text{for all } \mathbf{x} \text{ in } \mathcal{V} \tag{5.51}$$

We define the **norm of T** by*

$$\|\mathbf{T}\| \triangleq \inf\{\, a: \|\mathbf{Tx}\|_{\mathcal{W}} \leqslant \alpha \|\mathbf{x}\|_{\mathcal{V}} \quad \text{for all } \mathbf{x} \text{ in } \mathcal{V}\} \tag{5.52}$$

We can think of $\|\mathbf{T}\|$ as the tightest bound for $\mathbf{T}$. It follows that

$$\|\mathbf{Tx}\|_{\mathcal{W}} \leqslant \|\mathbf{T}\| \, \|\mathbf{x}\|_{\mathcal{V}} \tag{5.53}$$

*The term "inf" means **infimum** or greatest lower bound. If the bound is actually reached, the infimum is just the minimum. The term "sup" means **supremum** or least upper bound; if the bound is attained, the supremum is the maximum.

If **T** is linear, (5.52) can be expressed as

$$\|\mathbf{T}\| = \inf\left\{\alpha: \frac{\|\mathbf{Tx}\|_{\mathcal{W}}}{\|\mathbf{x}\|_{\mathcal{V}}} \leqslant \alpha\right\}$$

$$= \inf\{\alpha: \|\mathbf{Tx}\|_{\mathcal{W}} \leqslant \alpha,\ \|\mathbf{x}\|_{\mathcal{V}} = 1\}$$

$$= \sup_{\|\mathbf{x}\|_{\mathcal{V}}=1} \{\|\mathbf{Tx}\|_{\mathcal{W}}\} \tag{5.54}$$

Example 1. A Norm is a Bounded Functional. Define $\mathbf{T}: \mathcal{V} \to \mathcal{R}$ by $\mathbf{Tx} \triangleq \|\mathbf{x}\|_{\mathcal{V}}$. Then $\|\mathbf{Tx}\|_{\mathcal{R}} = |\mathbf{Tx}| = \|\mathbf{x}\|_{\mathcal{V}}$. Clearly, the number 1 is a bound for **T** and $\|\mathbf{T}\| = 1$.

Example 2. Matrix Transformations are Bounded. Define $\mathbf{T}: \mathcal{M}_c^{n\times 1} \to \mathcal{M}_c^{m\times 1}$ by $\mathbf{Tx} \triangleq \mathbf{Ax}$, where **A** is a (possibly complex) $m \times n$ matrix. Assuming standard inner products, $\|\mathbf{Tx}\|^2 = \|\mathbf{Ax}\|^2 = \overline{\mathbf{x}}^T\overline{\mathbf{A}}^T\mathbf{Ax}$. Then, by (5.54), $\|\mathbf{T}\|^2 = \max_{\mathbf{x}^T\mathbf{x}=1} \overline{\mathbf{x}}^T\overline{\mathbf{A}}^T\mathbf{Ax}$. It can be shown that $\|\mathbf{T}\| = \sqrt{\lambda_m}$, where λ_m is the largest eigenvalue of the matrix $\overline{\mathbf{A}}^T\mathbf{A}$ (see P&C 5.29). We call $\sqrt{\lambda_m}$ the **spectral radius of A** and denote it by $\sigma(\mathbf{A})$. We also refer to $\sigma(\mathbf{A})$ as the **norm of A,** denoted $\|\mathbf{A}\|$. Thus

$$\|\mathbf{T}\| = \|\mathbf{A}\| = \sigma(\mathbf{A}) = \sqrt{\lambda_m}$$

It is apparent that the bound $\sqrt{\lambda_m}$ is attained for **x** equal to a normalized eigenvector of $\overline{\mathbf{A}}^T\mathbf{A}$ corresponding to the eigenvalue λ_m. The fact that matrix transformations are bounded implies that *all transformations on finite-dimensional spaces are bounded.*

If **A** is square, it makes sense to speak of the eigenvalues of **A** itself. If **A** is also real and symmetric, its spectral radius is just the largest eigenvalue of **A.** That this statement is not true for every square matrix is demonstrated by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

for which the largest eigenvalue is $\mathbf{A} = 1$, but for which $\|\mathbf{A}\| = \sigma(\mathbf{A}) = \sqrt{2}$. The bound $\|\mathbf{A}\|$ is attained for $\mathbf{x} = (1\ 1)^T$.

Example 3. Integral Operators are Bounded Define the linear operator **T** on $\mathcal{L}_2(a,b)$ by $(\mathbf{Tf})(t) = \int_a^b k(t,s)\mathbf{f}(s)\,ds$, where the kernel k satisfies $\int_a^b \int_a^b k^2(t,s)\,ds\,dt < \infty$. Such a kernel is called a **Hilbert-Schmidt kernel** and **T** is known as a **Hilbert-Schmidt integral operator.** If k is bounded for t and s in $[a, b]$, for instance, then **T** is Hilbert-Schmidt. Many operators are of this type; for example, the inverses of most differential operators defined on a finite interval. We apply the

Cauchy-Schwartz inequality (P&C 5.4) to find

$$\|\mathbf{Tf}\|^2 = \int_a^b \left[\int_a^b k(t,s)\mathbf{f}(s)\,ds \right]^2 dt$$

$$\leqslant \int_a^b \left[\int_a^b k^2(t,s)\,ds \int_a^b \mathbf{f}^2(s)\,ds \right] dt$$

$$= \int_a^b \int_a^b k^2(t,s)\,ds\,dt \|\mathbf{f}\|^2$$

Therefore, by (5.54),

$$\|\mathbf{T}\| \leqslant \left[\int_a^b \int_a^b k^2(t,s)\,ds\,dt \right]^{1/2} \tag{5.55}$$

and **T** is bounded. Under what conditions is the bound (5.55) actually the norm of **T** ? The Cauchy-Schwartz inequality becomes an equality if and only if the two arguments $k(t,s)$ and $\mathbf{f}(s)$ are dependent functions of s. Therefore, if there is an **f** in $\mathfrak{L}_2(a, b)$ such that $k(t,s) = \mathbf{g}(t)\mathbf{f}(s)$, then the bound which we have exhibited is actually $\|\mathbf{T}\|$. For many integral operators, $\|\mathbf{T}\|$ is equal to the magnitude of the largest eigenvalue of **T** (P&C 5.29).

Example 4. Differential Operators are not Bounded Differential operators are among the most useful transformations, yet they are seldom bounded. For instance, let **D** operate on $\mathfrak{L}_2(0,1)$. Let $\{\mathbf{f}_k\}$ be the sequence of functions shown in Figure ***5.8a***. In the $\mathfrak{L}_2$ norm, $\|\mathbf{f}_1\|^2 = \int_0^1 t^2\,dt = \frac{1}{3}$ and $\|\mathbf{f}_\infty\|^2 = \int_0^1 dt = 1$. Therefore, the functions in the sequence satisfy

$$\|\mathbf{f}_1\|^2 = \tfrac{1}{3} \leqslant \|\mathbf{f}_k\|^2 < 1 = \|\mathbf{f}_\infty\|^2$$

Yet we recognize from Figure ***5.8b*** that

$$\|\mathbf{Df}_k\|^2 = 2^{k-1} \to \infty$$

There is no number α which "bounds" **D** for all **f** in $\mathfrak{L}_2(0,1)$. In the limit as $k \to \infty$, an equivalent statement is that the derivative of a discontinuous function contains a delta function, but delta functions are not square integrable; they are not in $\mathfrak{L}_2(0, 1)$.

Definition. A (possibly nonlinear) transformation $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ is said to be **continuous at $\mathbf{x}_0$** if for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|\mathbf{x} - \mathbf{x}_0\|_{\mathcal{V}} < \delta \Rightarrow \|\mathbf{Tx} - \mathbf{Tx}_0\|_{\mathcal{W}} < \epsilon$$

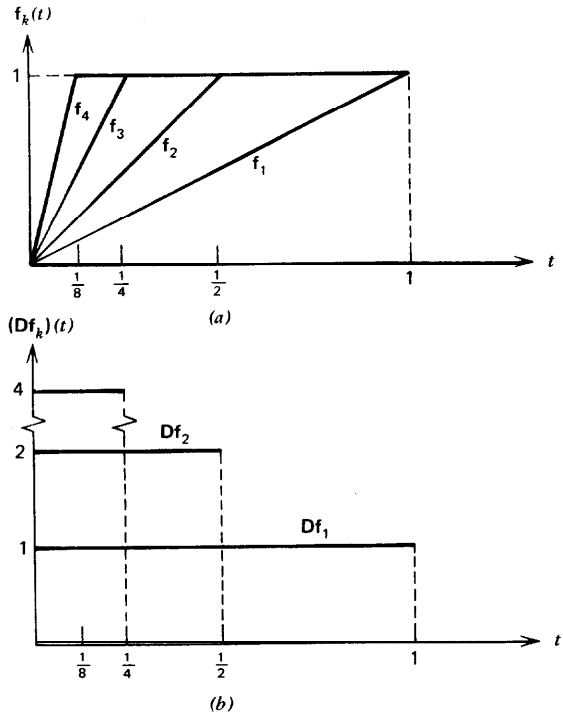


Figure 5.8. Differentiation of a sequence of functions.

That is, $\mathbf{T}$ is continuous at $\mathbf{x}_0$ if making $\|\mathbf{x} - \mathbf{x}_0\|_{\mathcal{V}}$ small will guarantee that $\|\mathbf{Tx} - \mathbf{Tx}_0\|_{\mathcal{W}}$ is small. If $\mathbf{T}$ is continuous for each $\mathbf{x}$ in $\mathcal{V}$, we just say **$\mathbf{T}$ is continuous.**

The nonlinear transformation $\mathbf{Tx} \triangleq \|\mathbf{x}\|_{\mathcal{V}}$, for example, is continuous. Suppose $\mathbf{T}$ is continuous and $\mathbf{x}_0 = \lim_{n\to\infty} \mathbf{x}_n$ in $\mathcal{V}$. If $\mathbf{x}_n$ approaches $\mathbf{x}_0$, then $\mathbf{Tx}_n$ approaches $\mathbf{Tx}_0$; in other words,

$$\lim_{n\to\infty} (\mathbf{Tx}_n) = \mathbf{Tx}_0 = \mathbf{T}\left(\lim_{n\to\infty} \mathbf{x}_n\right) \tag{5.56}$$

We will find this fact useful in the decoupling of equations on infinite-dimensional spaces. It is easy to show that *a linear transformation is continuous if and only if it is bounded.* Thus the linear transformations of Examples 2 and 3 are continuous transformations. It is apparent that bounded (or continuous) linear transformations are "well behaved." The linear differential operators of Example 4 are not continuous and are

"poorly behaved." It is for bounded (continuous) linear transformations that we will show most of the useful results of this chapter. It is usually difficult, if not impossible, to extend these concepts to unbounded linear transformations in a rigorous manner. Yet the concepts will be shown, by example, to extend in certain instances.

Completely Continuous Transformations

We now introduce briefly the concept of "complete continuity" of a linear transformation in order to specify conditions which guarantee the existence of a *countable* basis of eigenvectors.

Definition. A set $\mathcal{S}$ in an inner product space $\mathcal{V}$ is said to be **bounded** if there is a constant M such that $\|\mathbf{x}\| \leqslant M$ for all $\mathbf{x}$ in $\mathcal{S}$. A set $\mathcal{S}$ is **compact** if each infinite sequence of vectors from $\mathcal{S}$ contains a subsequence that converges to a vector of $\mathcal{S}$. Every compact set is closed and bounded. In finite-dimensional spaces the converse is also true: a set is compact if and only if it is closed and bounded, and every bounded set is closed [5.22, p. 185].

It is easy to see that a linear transformation $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ is bounded (continuous) if and only if it maps bounded sets in $\mathcal{V}$ into bounded sets in $\mathcal{W}$. A stronger restriction on **T,** which guarantees the countability of the eigenvalues and eigenvectors of **T**, is that of complete continuity.

Definition. A linear transformation **T** is **completely continuous** if it maps bounded sets into compact sets.

A completely continuous transformation is continuous, but the converse is not necessarily true. On an infinite-dimensional space, even the (continuous) identity operator is not completely continuous. Any bounded linear transformation whose range is finite-dimensional is completely continuous; thus any operator on a finite-dimensional space is completely continuous. The Hilbert-Schmidt integral operators of Example 3 are also completely continuous. Suppose **T** and **U** are completely continuous transformations mapping $\mathcal{V}$ into $\mathcal{W}$; then the linear combination $a\mathbf{T} + b\mathbf{U}$ is completely continuous. If **T** and **U** are linear operators on $\mathcal{V}$, one of which is bounded and the other completely continuous, then **TU** and **UT** are completely continuous. If $\mathbf{T}_k$ is the kth member of a sequence of completely continuous linear transformations mapping $\mathcal{V}$ into $\mathcal{W}$, then the limit operator **T** defined by

$$\lim_{k \to \infty} \|\mathbf{T}_k - \mathbf{T}\| = \mathbf{0}$$

is completely continuous. If a completely continuous transformation **T** is

defined on a infinite-dimensional space, then $\mathbf{T}^{-1}$, if it exists, is unbounded. Detailed discussions of completely continuous transformations can be found in Bachman and Narici [5.2] and Stakgold [5.22]. We content ourselves with this brief introduction; however, we make occasional reference to the consequences of complete continuity.

Bounded Linear Functionals

The key theorem in the development of a generalization of the transpose matrix is the Riesz-Frechet theorem. This theorem relates inner products and "bounded linear functionals." As indicated in Section 2.3, a functional is a scalar-valued transformation. Suppose $\mathcal{V}$ is an inner product space. The transformation $\mathbf{B}: \mathcal{V} \rightarrow \mathcal{C}$ defined by $\mathbf{Bx} \triangleq \langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{V}}$ for a fixed vector $\mathbf{y}$ in $\mathcal{V}$ is a linear functional. (Recall that an inner product is linear on the left.) By the Cauchy-Schwartz inequality (P&C 5.4), $\|\mathbf{Bx}\|_{\mathcal{C}} = |\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{V}}| \leqslant \|\mathbf{y}\|_{\mathcal{V}} \|\mathbf{x}\|_{\mathcal{V}}$ and $\|\mathbf{y}\|_{\mathcal{V}}$ is a bound for the linear functional $\mathbf{B}$. Furthermore, the bound is attained with the normalized vector $\mathbf{x} = \mathbf{y}/\|\mathbf{y}\|_{\mathcal{V}}$, and thus $\|\mathbf{B}\| = \|\mathbf{y}\|_{\mathcal{V}}$. Each different $\mathbf{y}$ in $\mathcal{V}$ specifies a different bounded (or continuous) linear functional. If $\mathcal{V}$ is a Hilbert space, we can say more—any bounded linear functional on $\mathcal{V}$ can be represented by an inner product.

Riesz-Fréchet Theorem. Corresponding to any bounded linear functional $\mathbf{B}$ on a Hilbert space $\mathcal{H}$ there is a unique vector $\mathbf{y}$ in $\mathcal{H}$ such that

$$\mathbf{Bx} = \langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}} \quad \text{for all } \mathbf{x} \text{ in } \mathcal{H} \tag{5.57}$$

Furthermore, $\|\mathbf{B}\| = \|\mathbf{y}\|_{\mathcal{H}}$.

Proof. If $\mathbf{B}$ is the zero functional, it is obvious that $\mathbf{y} = \boldsymbol{\theta}$. Assume $\mathbf{B}$ is not the zero functional, and let $\mathcal{W}$ = nullspace($\mathbf{B}$). ($\mathcal{W}$ consists of all $\mathbf{x}$ in $\mathcal{H}$ for which $\mathbf{Bx} = \langle \mathbf{x}, \mathbf{y} \rangle = 0$. Thus the vector $\mathbf{y}$ which we seek spans $\mathcal{W}^{\perp}$.) Let $\mathbf{y}_0$ be a unit vector in $\mathcal{W}^{\perp}$, and $\mathbf{x}$ any vector in $\mathcal{H}$. The vector $(\mathbf{Bx})\mathbf{y}_0 - (\mathbf{By}_0)\mathbf{x}$ is in $\mathcal{W}$ (verified by substitution into $\mathbf{Bx} = 0$).* Therefore,

$$\langle (\mathbf{Bx})\mathbf{y}_0 - (\mathbf{By}_0)\mathbf{x}, \mathbf{y}_0 \rangle = (\mathbf{Bx}) - (\mathbf{By}_0)\langle \mathbf{x}, \mathbf{y}_0 \rangle = 0$$

or $\mathbf{Bx} = \langle \mathbf{x}, (\mathbf{By}_0)\mathbf{y}_0 \rangle$. The vector $\mathbf{y} = (\mathbf{By}_0)\mathbf{y}_0$ in $\mathcal{W}^{\perp}$ represents $\mathbf{B}$ as required by (5.57). To see that $\mathbf{y}$ is uniquely determined by $\mathbf{B}$, we assume both $\mathbf{y}$ and $\mathbf{z}$ will do. Then $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle$ or $\langle \mathbf{x}, \mathbf{y} - \mathbf{z} \rangle = 0$ for all $\mathbf{x}$ in $\mathcal{H}$ including

*Since range($\mathbf{B}$) is one-dimensional, $\mathbf{Bx}$ and $\mathbf{By}_0$ are scalars; they can be used as scalar multipliers.

$\mathbf{x} = \mathbf{y} - \mathbf{z}$. It follows that $\mathbf{y} - \mathbf{z} = \boldsymbol{\theta}$ or $\mathbf{z} = \mathbf{y}$. We showed that $\|\mathbf{B}\| = \|\mathbf{y}\|_{\mathcal{Y}}$ in the discussion prior to the theorem.

Example 5. Bounded Linear Functionals on $\mathcal{M}_c^{n \times 1}$. Any linear functional $\mathbf{B}$ on the standard inner product space $\mathcal{M}_c^{n \times 1}$ is bounded and representable as $\mathbf{Bx} = \bar{\mathbf{y}}^T\mathbf{x}$ for some $\mathbf{y}$ in $\mathcal{M}_c^{n \times 1}$. That is, a linear functional acting on $n \times 1$ matrices is necessarily the taking of a specific linear combination of the n elements of the matrices. Furthermore, $\|\mathbf{B}\| = \|\mathbf{y}\| = \sqrt{\bar{\mathbf{y}}^T\mathbf{y}} = \sigma(\mathbf{y})$, the spectral radius of the $n \times 1$ matrix $\mathbf{y}$.

Example 6. Bounded Linear Functionals on $\mathcal{L}_2(a,b)$. The most general bounded linear functional $\mathbf{B}$ on the standard Hilbert space $\mathcal{L}_2(a, b)$ is $\mathbf{Bu} \triangleq \int_a^b \mathbf{u}(t)\mathbf{g}(t)\,dt$ for some specific $\mathbf{g}$ in $\mathcal{L}_2(a, b)$, and $\|\mathbf{B}\| = [\int_a^b |\mathbf{g}(t)|^2 dt]^{1/2}$. For example, the response $\mathbf{f}$ of a single-input linear time-invariant dynamic system with zero initial conditions is the convolution of the system input $\mathbf{u}$ with the impulse response $\mathbf{g}$*:

$$\mathbf{f}(t) = \int_0^t \mathbf{u}(s)\mathbf{g}(t-s)\,ds$$

For the interval $0 \leqslant t \leqslant b$, this function can be written

$$\mathbf{f}(t) = \int_0^b \mathbf{u}(s)\mathbf{k}_t(s)\,ds = \langle \mathbf{u}, \mathbf{k}_t \rangle$$

where

$$\mathbf{k}_t(s) = \mathbf{g}(t-s) \quad s \leqslant t$$
$$= 0 \qquad\qquad s > t$$

Thus for each instant t, $\mathbf{f}(t)$ is a bounded linear functional on $\mathcal{L}_2(0, b)$. The function in $\mathcal{L}_2(0,b)$ that represents the linear functional is $\mathbf{k}_t$. Treated as a function of t and s, $\mathbf{k}_t(s)$ is the Green's function for the dynamic system (see Chapter 3). A crude measure of the effect of the linear functional is the norm of the functional, $\|\mathbf{k}_t\| = [\int_0^t \mathbf{g}^2(t-s)\,ds]^{1/2}$.

Example 7. Function Evaluation, an Unbounded Linear Functional. Define $\mathbf{B}$: $\mathcal{L}_2(a, b) \to \mathcal{R}$ by $\mathbf{Bf} \triangleq \mathbf{f}(t_0)$, where t_0 is in $[a, b]$. This linear functional, evaluation at t_0, is not bounded. For if $\mathbf{f}_k$ is a pulse of height k and width $1/k^2$, centered at t_0, then $\|\mathbf{f}_k\| = 1$, but $\|\mathbf{Bf}_k\| = k \to \infty$. It is well known that $\mathbf{f}(t_0) = \int_a^b \mathbf{f}(t)\delta(t - t_0)\,dt$, where $\delta(t - t_0)$ is a Dirac delta function centered at t_0.† Thus in a sense the Riesz-Fréchet theorem extends to at least this unbounded linear functional. However, $\delta(t - t_0)$ does not have a finite norm and therefore is not in $\mathcal{L}_2(a, b)$.

*See Appendix 2 for a discussion of convolution and impulse response.

† See Appendix 2 for an introduction to the properties of delta functions.

The Adjoint

Let $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ be a bounded linear transformation between *Hilbert spaces* $\mathcal{V}$ and $\mathcal{W}$. We now introduce the adjoint transformation **T*,** a generalization of the transposed matrix multiplication with which we introduced this section. The vector $\mathbf{Tx}$ is in $\mathcal{W}$. Since an inner product is a bounded linear functional of its left argument, $\mathbf{Ux} \triangleq \langle \mathbf{Tx}, \mathbf{z} \rangle_{\mathcal{W}}$ is a bounded linear functional of the variable $\mathbf{x}$ in $\mathcal{V}$. In fact, by means of the Cauchy-Schwartz inequality (P&C 5.4) and the inequality (5.53), we can exhibit a bound:

$$\|\mathbf{Ux}\|_{\mathcal{W}} = |\langle \mathbf{Tx}, \mathbf{z} \rangle_{\mathcal{W}}| \leqslant \|\mathbf{Tx}\|_{\mathcal{W}} \|\mathbf{z}\|_{\mathcal{W}} \leqslant \|\mathbf{T}\| \|\mathbf{x}\|_{\mathcal{V}} \|\mathbf{z}\|_{\mathcal{W}}$$

or $\|\mathbf{U}\| \leqslant \|\mathbf{T}\| \|\mathbf{z}\|_{\mathcal{W}}$. The Riesz-Fréchet theorem (5.57) guarantees that there exists a unique vector $\mathbf{y}$ in $\mathcal{V}$ which represents this bounded linear functional $\mathbf{U}$ in the sense that

$$\mathbf{Ux} = \langle \mathbf{Tx}, \mathbf{z} \rangle_{\mathcal{W}} = \langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{V}}$$

It is evident from this equation that $\mathbf{y}$ in $\mathcal{V}$ and $\mathbf{z}$ in $\mathcal{W}$ are related. We define this relation to be the adjoint transformation, $\mathbf{y} = \mathbf{T^*z}$.

Definition. The **adjoint transformation T*** is defined by

$$\langle \mathbf{Tx}, \mathbf{z} \rangle_{\mathcal{W}} = \langle \mathbf{x}, \mathbf{T^*z} \rangle_{\mathcal{V}} \tag{5.58}$$

for all $\mathbf{x}$ in $\mathcal{V}$ and $\mathbf{z}$ in $\mathcal{W}$.

The existence of **T*** is guaranteed by the bounded linear nature of **T** and the completeness of $\mathcal{V}$. Uniqueness and linearity of **T*** are easily verified. Furthermore, **T*** is bounded; we recognize that

$$\|\mathbf{T^*z}\|_{\mathcal{V}} = \|\mathbf{y}\|_{\mathcal{V}} = \|\mathbf{U}\| \leqslant \|\mathbf{T}\| \|\mathbf{z}\|_{\mathcal{W}}$$

By (5.54),

$$\|\mathbf{T^*}\| = \sup_{\|\mathbf{z}\|_{\mathcal{W}} = 1} \|\mathbf{T^*z}\|_{\mathcal{V}} \leqslant \|\mathbf{T}\|$$

Since (5.58) is symmetric in **T** and **T*,** reversing the roles of **T** and **T*** shows that $\|\mathbf{T}\| \leqslant \|\mathbf{T^*}\|$. Thus

$$\|\mathbf{T}\| = \|\mathbf{T^*}\| \tag{5.59}$$

An explicit description of **T*** can be obtained from the defining equation (5.58) and a description of **T.** The basic technique for obtaining the

description of $\mathbf{T}^*$ is to write $\langle \mathbf{Tx}, \mathbf{z} \rangle_{\mathcal{W}}$ and manipulate it into the form of $\langle \mathbf{x}, \mathbf{T}^*\mathbf{z} \rangle_{\mathcal{V}}$; in effect, we work operations off of $\mathbf{x}$ and onto $\mathbf{z}$.

Example 8. The Adjoint of a Matrix Transformation. Let $\mathcal{V} = \mathcal{C}_c^{n \times 1}$ and $\mathcal{W} = \mathcal{C}_c^{m \times 1}$, each with its standard inner product. Define $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ by $\mathbf{Tx} \triangleq \mathbf{Ax}$, where $\mathbf{A}$ is an $m \times n$ matrix. Then

$$\begin{aligned}\langle \mathbf{Tx}, \mathbf{z} \rangle_{\mathcal{W}} &= \langle \mathbf{Ax}, \mathbf{z} \rangle_{\mathcal{W}} \\ &= \overline{\mathbf{z}}^{\mathrm{T}} \mathbf{Ax} \\ &= \left(\overline{\overline{\mathbf{A}}^{\mathrm{T}} \mathbf{z}} \right)^{\mathrm{T}} \mathbf{x} \\ &= \langle \mathbf{x}, \overline{\mathbf{A}}^{\mathrm{T}} \mathbf{z} \rangle_{\mathcal{V}} = \langle \mathbf{x}, \mathbf{T}^*\mathbf{z} \rangle_{\mathcal{V}}\end{aligned}$$

Clearly,

$$\mathbf{T}^*\mathbf{z} = \overline{\mathbf{A}}^{\mathrm{T}} \mathbf{z} \tag{5.60}$$

Of course, if $\mathbf{A}$ is real (or if $\mathcal{V}$ and $\mathcal{W}$ are real) the conjugate is superfluous. It is apparent that the transposed matrix example with which we introduced Section 5.4 is just a special case of the general adjoint concept. If the inner products are not standard, multiplication by the conjugated transposed matrix is not the adjoint (P&C 5.19).

Example 9. The Adjoint of an Integral Operator. Let $\mathcal{V} = \mathcal{W} = \mathcal{L}_2^c(a,b)$. Define $\mathbf{T}$ by $(\mathbf{Tf})(t) \triangleq \int_a^b k(t,s)\mathbf{f}(s)\,ds$, where k is a Hilbert-Schmidt kernel; by Example 3, $\mathbf{T}$ is bounded. Then

$$\begin{aligned}\langle \mathbf{Tf}, \mathbf{g} \rangle &= \int_a^b (\mathbf{Tf})(t)\, \overline{\mathbf{g}(t)}\, dt \\ &= \int_a^b \int_a^b k(t,s)\mathbf{f}(s)\, ds\, \overline{\mathbf{g}(t)}\, dt \\ &= \int_a^b \mathbf{f}(s) \int_a^b k(t,s)\, \overline{\mathbf{g}(t)}\, dt\, ds \\ &= \int_a^b \mathbf{f}(s)\, \overline{\int_a^b \overline{k(t,s)}\, \mathbf{g}(t)\, dt}\, ds \\ &= \int_a^b \mathbf{f}(s)\, \overline{(\mathbf{T}^*\mathbf{g})(s)}\, ds\end{aligned}$$

Therefore,

$$(\mathbf{T}^*\mathbf{g})(t) \triangleq \int_a^b \overline{k(s,t)}\, \mathbf{g}(s)\, ds \tag{5.61}$$

Whereas **T** requires integration with respect to the first variable in the kernel k, **T*** requires integration with respect to the second variable. The kernel $\overline{k(s,t)}$ is the analogue of the conjugated transposed matrix of Example 8. Once again, if the spaces are real, the conjugations are superfluous.

Exercise 1. Let $\mathcal{V} = \mathcal{W} = \mathcal{L}_2(\omega;a,b)$, for which the inner product is (5.49). Assume the scalars are real. Define $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ by

$$(\mathbf{Tf})(t) \triangleq \int_a^b k(t,s)\mathbf{f}(s)\,ds$$

where k is a Hilbert-Schmidt kernel. Show that

$$(\mathbf{T}^*\mathbf{g})(t) = \int_a^b \frac{\omega(s)k(s,t)}{\omega(t)}\mathbf{g}(s)\,ds \tag{5.62}$$

Adjoints of Differential Operators

Only for a bounded linear transformation on a Hilbert space does the preceding discussion guarantee the existence of an adjoint transformation which satisfies (5.58) and (5.59). In point of fact, if **T** is not bounded, it makes no sense to speak of $\|\mathbf{T}\|$. Yet among the most useful transformations are the linear differential operators, which are unbounded. Thus if the adjoint concept is useful, we have reason to attempt to apply the concept to differential operators. We shall see that differential operators do have adjoints.

Consider the simple differential operator **D** defined by $(\mathbf{Df})(t) \triangleq \mathbf{f}'(t)$ acting on functions defined over the interval $[a,b]$. Assume the standard function space inner product for both the domain and range of definition. Nothing prevents us from using the approach of Examples 8 and 9 to try to generate an adjoint for **D.** The natural technique for working differentiations off of one argument and onto another is integration by parts:

$$\begin{aligned}\langle \mathbf{Df}, \mathbf{g}\rangle &= \int_a^b \mathbf{f}'(t)\mathbf{g}(t)\,dt \\ &= \mathbf{f}(t)\mathbf{g}(t)\Big|_a^b - \int_a^b \mathbf{f}(t)\mathbf{g}'(t)\,dt\end{aligned} \tag{5.63}$$

It seems logical to define **D*** by $(\mathbf{D}^*\mathbf{g})(t) \triangleq -\mathbf{g}'(t) = (-\mathbf{Dg})(t)$. However, this definition does not quite agree with the defining equation (5.58) unless the boundary term, $\mathbf{f}(b)\,\mathbf{g}(b) - \mathbf{f}(a)\,\mathbf{g}(a)$, is zero. We must not lose sight of

the fact that differential operators usually have associated boundary conditions. Suppose the boundary condition associated with **D** is $\mathbf{f}(a) - \mathbf{f}(b) = 0$. Then in order that the boundary term of (5.63) be zero, we must have $\mathbf{f}(b)[\mathbf{g}(b) - \mathbf{g}(a)] = 0$, or $\mathbf{g}(b) - \mathbf{g}(a) = 0$.

It should be apparent that for any ordinary linear differential operator **L** with a set of accompanying homogeneous boundary conditions we can use integration by parts to generate an adjoint differential operator **L*** with accompanying adjoint homogeneous boundary conditions. The operators **L** and **L*** satisfy the defining equation, (5.58), for all **f** and **g** in $\mathfrak{L}_2(a,b)$ which satisfy the respective boundary conditions. [Of course, we have given up on (5.59).] If we were to change the homogeneous boundary conditions associated with **L,** we would obtain a different set of adjoint boundary conditions, but the same adjoint differential operator **L*.** We refer to the adjoint differential operator **L*** as the **formal adjoint of L.** The formal adjoint is independent of boundary conditions. The definition of an operator always includes a definition of its domain. We use the homogeneous boundary conditions associated with **L** to restrict the domain of **L.** The **adjoint boundary conditions,** which arise naturally out of the integration by parts, determine the restrictions on the domain of the formal adjoint **L*** in order that it be a true adjoint of **L** [in the sense that it obeys (5.58)]. Thus the formal adjoint of **D** is $\mathbf{D}^* = -\mathbf{D}$. If the boundary condition associated with **D** is an initial condition, $\mathbf{f}(a) = 0$, then by (5.63) the adjoint boundary condition is a final condition which requires that $\mathbf{g}(b) = 0$ for each **g** in the domain of **D*.**

If we wished, we could further restrict the domains of **L** and **L*** to include only differentiable functions, thereby eliminating delta functions and their derivatives from range(**L**) and range(**L***). However, formal use of integration by parts works for delta functions. Therefore, as a practical matter, we do not concern ourselves with this restriction.

Example 10. The Adjoint of D^n. Let the *n*th derivative operator $\mathbf{D}^n$ act on a space of real functions defined over $[a,b]$. Assuming the standard inner product,

$$\begin{aligned}
\langle \mathbf{D}^n\mathbf{f}, \mathbf{g}\rangle &= \int_a^b \mathbf{f}^{(n)}(t)\mathbf{g}(t)\,dt \\
&= \mathbf{f}^{(n-1)}(t)\mathbf{g}(t)\big|_a^b - \int_a^b \mathbf{f}^{(n-1)}(t)\mathbf{g}'(t)\,dt \\
&= \mathbf{f}^{(n-1)}(t)\mathbf{g}(t)\big|_a^b - \mathbf{f}^{(n-2)}(t)\mathbf{g}'(t)\big|_a^b + \int_a^b \mathbf{f}^{(n-2)}(t)\mathbf{g}^{(2)}(t)\,dt \\
&= \mathbf{f}^{(n-1)}(t)\mathbf{g}(t)\big|_a^b - \mathbf{f}^{(n-2)}(t)\mathbf{g}'(t)\big|_a^b + \cdots \\
&\quad + (-1)^{n-1}\mathbf{f}(t)\mathbf{g}^{(n-1)}(t)\big|_a^b + (-1)^n \int_a^b \mathbf{f}(t)\mathbf{g}^{(n)}(t)\,dt
\end{aligned}$$

(The intermediate terms are indicated only to show the pattern which the terms follow. Some of the terms shown are extraneous if $n = 1$ or 2.) The formal adjoint of $\mathbf{D}^n$ is clearly $(-1)^n \mathbf{D}^n$. The adjoint boundary conditions depend upon the boundary conditions associated with $\mathbf{D}^n$. A specific set of boundary conditions does not determine a unique set of adjoint boundary conditions. However, the domain defined by the adjoint boundary conditions is unique.

Exercise 2. Show that the formal adjoint of the differential operator $\mathbf{L} \triangleq \mathbf{D}^n + a_1\mathbf{D}^{n-1} + \cdots + a_n\mathbf{I}$ acting on a space of functions with the *standard inner product* is $\mathbf{L}^* \triangleq (-1)^n\mathbf{D}^n + (-1)^{n-1}a_1\mathbf{D}^{n-1} + \cdots + a_n\mathbf{I}$.

Example 11. The Adjoint of a Partial Differential Operator. Let $\mathfrak{L}_2(\Omega)$ be the space of real functions which are defined on a two-dimensional region Ω and which have finite norm under the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle \triangleq \int_\Omega \mathbf{f}(\mathbf{p})\mathbf{g}(\mathbf{p})\, d\mathbf{p} \tag{5.64}$$

where $\mathbf{p} = (s,t)$, an arbitrary point in Ω. We define the Laplacian operator ∇^2 on $\mathfrak{L}_2^2(\Omega)$ by

$$(\nabla^2\mathbf{f})(s,t) \triangleq \frac{\partial^2\mathbf{f}(s,t)}{\partial s^2} + \frac{\partial^2\mathbf{f}(s,t)}{\partial t^2} \qquad \text{for } (s,t) \text{ in } \Omega$$

For this problem, the **symmetric form of Green's theorem** is the equivalent of integration by parts; it states

$$\int_\Omega (\mathbf{f}\nabla^2\mathbf{g} - \mathbf{g}\nabla^2\mathbf{f})\, d\mathbf{p} = \oint_\Gamma (\mathbf{f}\mathbf{g}_n - \mathbf{g}\mathbf{f}_n)\, d\mathbf{p} \tag{5.65}$$

where Γ represents the boundary of the region Ω, and the subscript n indicates the derivative in a direction normal to Γ and directed out of Ω.* Using this theorem, we find

$$\begin{aligned}\langle \nabla^2\mathbf{f}, \mathbf{g} \rangle &= \int_\Omega (\nabla^2\mathbf{f})(\mathbf{p})\mathbf{g}(\mathbf{p})\, d\mathbf{p} \\ &= \int_\Omega \mathbf{f}(\mathbf{p})(\nabla^2\mathbf{g})(\mathbf{p})\, d\mathbf{p} + \oint_\Gamma [\mathbf{g}(\mathbf{p})\mathbf{f}_n(\mathbf{p}) - \mathbf{f}(\mathbf{p})\mathbf{g}_n(\mathbf{p})]\, d\mathbf{p}\end{aligned}$$

Clearly, the formal adjoint of ∇^2 is just ∇^2 itself.

Not all boundary conditions are appropriate for a partial differential operator. For the Laplacian operator, one appropriate homogeneous boundary condition is $a\mathbf{f}(\mathbf{p}) + b\mathbf{f}^n(\mathbf{p}) = 0$ on the boundary. The adjoint boundary condition is selected such

*See Wylie [5.24, p. 575].

that the boundary integral in Green's theorem is zero. It is sufficient to make the integrand zero for each $\mathbf{p}$ on Γ:

$$\mathbf{g}(\mathbf{p})\mathbf{f}_n(\mathbf{p}) - \left(-\frac{b}{a}\mathbf{f}_n(\mathbf{p})\right)\mathbf{g}_n(\mathbf{p}) = \mathbf{f}_n(\mathbf{p})\left[\mathbf{g}(\mathbf{p}) + \frac{b}{a}\mathbf{g}_n(\mathbf{p})\right] = 0$$

Thus the adjoint boundary condition is $a\,\mathbf{g}(\mathbf{p}) + b\,\mathbf{g}_n(\mathbf{p}) = 0$ on Γ, the same as the original boundary condition associated with ∇^2.

Properties of Adjoints

Taking adjoints is similar to conjugation of complex numbers. Let $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{U}$ be Hilbert spaces, and $\mathbf{I}$ the identity operator on $\mathcal{V}$. Suppose $\mathbf{T}$ and $\mathbf{U}$ are bounded linear transformations from $\mathcal{V}$ into $\mathcal{W}$, and $\mathbf{S}$ is a bounded linear transformation from $\mathcal{W}$ into $\mathcal{U}$. Then it is easy to show:

$$\begin{aligned}
&(a)\quad \mathbf{I}^* = \mathbf{I}\\
&(b)\quad (\mathbf{T}^*)^* = \mathbf{T}\\
&(c)\quad (a\mathbf{T} + b\mathbf{U})^* = \bar{a}\mathbf{T}^* + \bar{b}\mathbf{U}^*\\
&(d)\quad (\mathbf{ST})^* = \mathbf{T}^*\mathbf{S}^*\\
&(e)\quad \text{If } \mathbf{T} \text{ has a bounded inverse, } \mathbf{T}^* \text{ is invertible and } (\mathbf{T}^*)^{-1} = (\mathbf{T}^{-1})^*
\end{aligned} \tag{5.66}$$

In fact, property (e) of (5.66) may be valid even if $\mathbf{T}^{-1}$ is not bounded. For example, let us define $\mathbf{T}$ on $\mathcal{L}_2(0,1)$ to be the bounded integral operator

$$(\mathbf{Tf})(t) \triangleq \int_0^t \mathbf{f}(s)\,ds$$

Then $\mathbf{T}^{-1}$ is the unbounded differential operator $\mathbf{Lf} \triangleq \mathbf{Df}$ with the homogeneous boundary condition $\mathbf{f}(0) = 0$. By (5.61) we know that $\mathbf{T}^*$ differs from $\mathbf{T}$ only in an interchange of the roles of t and s in the kernel function. In this instance the kernel function for $\mathbf{T}$ is

$$\begin{aligned}
k(t,s) &= 1, \qquad s < t\\
&= 0, \qquad s > t
\end{aligned}$$

Therefore,

$$(\mathbf{T}^*\mathbf{g})(t) \triangleq \int_0^1 k(s,t)\mathbf{g}(s)\,ds = \int_t^1 \mathbf{g}(s)\,ds$$

On the other hand, it follows from (5.63) that $(\mathbf{T}^{-1})^*$, the adjoint of $\mathbf{Lf} \triangleq \mathbf{Df}$ with its homogeneous boundary condition $\mathbf{f}(0) = 0$, is $\mathbf{L}^*\mathbf{g} \triangleq -\mathbf{Dg}$ with the adjoint boundary condition $\mathbf{g}(1) = 0$. But this differential system is also $(\mathbf{T}^*)^{-1}$, which we verify by acting on it with $\mathbf{T}^*$ to get the identity operator:

$$[\mathbf{T}^*(-\mathbf{Dg})](t)\Big|_{\mathbf{g}(1)=0} = \int_t^1 (-\mathbf{Dg})(s)\,ds\Big|_{\mathbf{g}(1)=0} = \mathbf{g}(t)$$

or $\mathbf{T}^*(\mathbf{T}^*)^{-1}\mathbf{g} = \mathbf{g}$.

One of the most valuable characteristics of the adjoint transformation is that it generates orthogonal decompositions of the domain and range of definition of **T.** These decompositions are central to all forms of least-square optimization, to many iterative techniques for optimizing functionals, and to iterative techniques for solving nonlinear equations (Chapters 6-8).

Orthogonal Decomposition Theorem. If $\mathcal{V}$ and $\mathcal{W}$ are Hilbert spaces, and $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ is a bounded linear transformation, then

$$\begin{aligned} \mathcal{V} &= \text{nullspace}(\mathbf{T}) \overset{\perp}{\oplus} \overline{\text{range}(\mathbf{T}^*)} \\ \mathcal{W} &= \text{nullspace}(\mathbf{T}^*) \overset{\perp}{\oplus} \overline{\text{range}(\mathbf{T})} \end{aligned} \tag{5.67}$$

The symbol $\overset{\perp}{\oplus}$ implies that these direct sums are orthogonal; the nullspaces and ranges are orthogonal complements. Theorem (5.67) is illustrated abstractly in Figure 5.9. The bars over range($\mathbf{T}$) and range($\mathbf{T}^*$) indicate the completion (or closure) of these linear manifolds. We have already seen this orthogonal decomposition demonstrated in the matrix example that introduced Section 5.4.

By the projection theorem, $\mathcal{V} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{U}^\perp$ for any subspace $\mathcal{U}$ of a Hilbert space $\mathcal{V}$. Therefore, if we can show that $\overline{\text{nullspace}(\mathbf{T})} = [\text{range}(\mathbf{T}^*)]^\perp$, it follows that $[\text{nullspace}(\mathbf{T})]^\perp = [\text{range}(\mathbf{T}^*)]^{\perp\perp} = \overline{\text{range}(\mathbf{T}^*)}$, and the first orthogonal decomposition of the theorem is proved. Let $\mathbf{y}$ be an arbitrary vector in $\mathcal{W}$; then $\mathbf{T}^*\mathbf{y} = \mathbf{z}$ is an arbitrary vector in range($\mathbf{T}^*$). The orthogonal complement of range($\mathbf{T}^*$) consists in all vectors $\mathbf{x}$ that are orthogonal to all $\mathbf{z}$ in range($\mathbf{T}^*$); that is, all $\mathbf{x}$ such that

$$0 = \langle \mathbf{x}, \mathbf{z} \rangle_{\mathcal{V}} = \langle \mathbf{x}, \mathbf{T}^*\mathbf{y} \rangle_{\mathcal{V}} = \langle \mathbf{Tx}, \mathbf{y} \rangle_{\mathcal{W}}$$

for all $\mathbf{y}$ in $\mathcal{W}$. Therefore, $\mathbf{Tx} = 0$, the vectors $\mathbf{x}$ constitute the nullspace of $\mathbf{T}$, and nullspace($\mathbf{T}$) = $[\text{range}(\mathbf{T}^*)]^\perp$. The proof of the second orthogonal decomposition is parallel to that above.

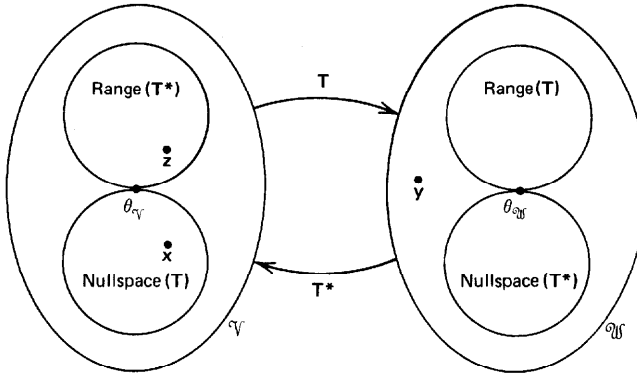


Figure 5.9. Orthogonal direct-sum decomposition described by **T***.

Since an orthogonal complement is always a (complete) subspace, we can see that the nullspace of any *bounded* linear transformation will be complete. On the other hand, the range need not be complete. For instance, let $\mathbf{T}$ on $\mathcal{L}_2(0, 1)$ be defined by $(\mathbf{Tf})(t) \triangleq \int_0^t \mathbf{f}(s)\,ds$. Then since $\mathcal{L}_2(0, 1)$ contains no delta functions, range($\mathbf{T}$) contains only continuous functions; but the space of continuous functions [and thus, range($\mathbf{T}$)] is not complete under the $\mathcal{L}_2(0, 1)$ norm. We usually assume range($\mathbf{T}$) and range($\mathbf{T}^*$) are complete, or ignore the difference between the ranges and their closures.

Although we have proved the orthogonal decomposition theorem (5.67) only for bounded linear transformations, it holds for many unbounded linear transformations as well. We use the theorem wherever the adjoint operator is defined. In particular, we apply the theorem to differential operators, even though they are not bounded.

Example 12. Orthogonal Decomposition for a Partial Differential Operator. Define ∇^2 on $\mathcal{L}_2(\Omega)$ as in Example 11. Let the boundary condition be $\mathbf{f}_n(\mathbf{p}) = 0$ on the boundary Γ. Then, by Example 11, the adjoint operator and adjoint boundary conditions are identical to the original operator and boundary conditions. The nullspace of ∇^2 with the boundary condition $\mathbf{f}_n = \boldsymbol{\theta}$ is the set of functions which are constant over Ω; that is, only if $\mathbf{f}(\mathbf{p}) = c$ for $\mathbf{p}$ in Ω do we have $(\nabla^2\mathbf{f})(\mathbf{p}) = 0$ for $\mathbf{p}$ in Ω and $\mathbf{f}_n(\mathbf{p}) = 0$ for $\mathbf{p}$ on Γ. By the orthogonal decomposition theorem, we expect range(∇^2) to be the orthogonal complement of nullspace(∇^2). Therefore, if we wish to solve $\nabla^2\mathbf{f} = \mathbf{u}$ with $\mathbf{f}_n = \boldsymbol{\theta}$ on the boundary, we must be sure $\mathbf{u}$ is orthogonal to nullspace(∇^2), or

$$\langle \mathbf{u}, \mathbf{f} \rangle = \int_\Omega \mathbf{u}(\mathbf{p})\mathbf{f}(\mathbf{p})\,d\mathbf{p}$$

$$= c \int_\Omega \mathbf{u}(\mathbf{p})\,d\mathbf{p} = 0$$

This result can be given a physical interpretation. If $\mathbf{u}(\mathbf{p})$ is the rate at which heat is introduced at the point $\mathbf{p}$ [with units of (heat)/(time)(area)], then the steady-state temperature distribution satisfies Poisson's equation $\nabla^2\mathbf{f} = \mathbf{u}$. The boundary condition $\mathbf{f}_n(\mathbf{p}) = 0$ says no heat is leaving Ω at the point $\mathbf{p}$ on the boundary. The orthogonal decomposition shows that we cannot find a steady-state temperature distribution such that no heat leaves the region unless the total heat generated per unit time, $\int_\Omega \mathbf{u}(\mathbf{p})\,d\mathbf{p}$, is zero.

It is apparent from Example 12 that the orthogonal decomposition does apply to at least some unbounded linear transformations. It often provides a useful way of checking whether or not a differential equation is solvable. If range($\mathbf{T}$) is closed, the operator equation $\mathbf{Tx} = \mathbf{y}$ is solvable if and only if $\mathbf{y}$ is orthogonal to nullspace($\mathbf{T}^*$). This nullspace is often easier to explore than is range($\mathbf{T}$); if $\mathbf{T}^* = \mathbf{T}$, as in Example 12, we find nullspace($\mathbf{T}^*$) by solving the homogeneous equation $\mathbf{Tx} = \boldsymbol{\theta}$. If the boundary condition in Example 12 were $\mathbf{f}(\mathbf{p}) = 0$ on the boundary Γ, the nullspace of ∇^2 would be empty. Then the orthogonal decomposition theorem would show that the operator was invertible.

If an operator $\mathbf{T}$ is not invertible, then the equation $\mathbf{Tx} = \mathbf{y}$ may have no solution or it may have many. The orthogonal decomposition theorem finds considerable use in solving such equations uniquely in a least square sense (Chapter 6). In point of fact, the decomposition pervades essentially all least-square optimization.

Let $\mathbf{T}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Suppose $\mathbf{T}$ has eigenvalues and eigenvectors. We discovered in Section 4.3 that if $\mathbf{T}^{-1}$ exists, the eigendata for $\mathbf{T}$ and $\mathbf{T}^{-1}$ are related; $\mathbf{T}$ and $\mathbf{T}^{-1}$ have identical eigenvectors and inverse eigenvalues. Given the close relationship between $\mathbf{T}$ and $\mathbf{T}^*$, we also expect the eigendata for $\mathbf{T}$ to provide some information about the eigendata for $\mathbf{T}^*$. Let $\mathbf{x}_i$ be an eigenvector for $\mathbf{T}$ corresponding to the eigenvalue λ_i. Then, for any $\mathbf{y}$ in $\mathcal{H}$,

$$\langle \mathbf{Tx}_i, \mathbf{y}\rangle = \langle \lambda_i \mathbf{x}_i, \mathbf{y}\rangle = \langle \mathbf{x}_i, \bar{\lambda}_i \mathbf{y}\rangle = \langle \mathbf{x}_i, \mathbf{T}^*\mathbf{y}\rangle$$

Thus $\langle \mathbf{x}_i, (\mathbf{T}^* - \bar{\lambda}_i\mathbf{I})\mathbf{y}\rangle = 0$ or range$(\mathbf{T}^* - \bar{\lambda}_i\mathbf{I})$ is orthogonal to $\mathbf{x}_i$. Consequently, range$(\mathbf{T}^* - \bar{\lambda}_i\mathbf{I})$ does not fill $\mathcal{H}$, and nullspace$(\mathbf{T}^* - \bar{\lambda}_i\mathbf{I})$ must be nonempty. In the finite-dimensional case we express this fact as

$$\text{nullity}(\mathbf{T}^* - \bar{\lambda}_i\mathbf{I}) = \dim \mathcal{H} - \text{rank}(\mathbf{T}^* - \bar{\lambda}_i\mathbf{I})$$

We see that *if* λ_i *is an eigenvalue for* $\mathbf{T}$, $\bar{\lambda}_i$ *is an eigenvalue for* $\mathbf{T}^*$.

We will show that the eigenvectors of $\mathbf{T}$ and $\mathbf{T}^*$ are related as well. Suppose $\mathbf{x}_i$ and $\mathbf{x}_j$ are eigenvectors of $\mathbf{T}$ corresponding to the eigenvalues λ_i and λ_j, respectively; let $\mathbf{y}_i$ and $\mathbf{y}_j$ be the corresponding eigenvectors of $\mathbf{T}^*$.

Then

$$0 = \langle \mathbf{T}\mathbf{x}_i, \mathbf{y}_j \rangle - \langle \mathbf{x}_i, \mathbf{T}^*\mathbf{y}_j \rangle$$
$$= (\lambda_i - \lambda_j)\langle \mathbf{x}_i, \mathbf{y}_j \rangle \tag{5.68}$$

Clearly, *if* λ_i *and* λ_j *are different eigenvalues of* **T**, *the eigenvectors* $\mathbf{x}_i$ *and* $\mathbf{y}_j$ *(of* **T** *and* **T***, *respectively) are orthogonal.*

As a general rule, we expect **T** to be diagonalizable. That is, there is usually a countable (and perhaps orthonormal) basis for $\mathcal{H}$ composed of eigenvectors for **T.** Nondiagonalizability is the exception. (Of course, in infinite-dimensional spaces we sometimes find there are no eigenvectors; any dynamic system with initial conditions is an example.) Suppose **T** is diagonalizable; let $\{\lambda_i\}$ be the eigenvalues of **T** and $\{x_i\}$ a corresponding set of eigenvectors, a basis for $\mathcal{H}$. Then the numbers $\{\bar{\lambda}_i\}$ are the eigenvalues of **T*;** we denote the corresponding eigenvectors of **T*** by $\{\mathbf{y}_i\}$. The eigenvectors $\mathbf{y}_i$ can be chosen such that

$$\langle \mathbf{x}_i, \mathbf{y}_j \rangle = \delta_{ij} \tag{5.69}$$

For those eigenspaces of **T** which are one-dimensional, (5.69) requires only normalization of the one available eigenvector $\mathbf{y}_i$ so that $\langle \mathbf{x}_i, \mathbf{y}_i \rangle = 1$. If **T** has several independent eigenvectors, say, $\mathbf{x}_i, \ldots, \mathbf{x}_m$ for a single eigenvalue λ_i, then the eigenvalue $\bar{\lambda}_i$ for **T*** also has *m* independent eigenvectors, $\mathbf{y}_1, \ldots, \mathbf{y}_m$, which we choose by solving m^2 independent linear equations in m^2 unknowns, $\langle \mathbf{x}_k, \mathbf{y}_j \rangle = \delta_{kj}$, for $k, j = 1, \ldots, m$. The eigenvectors $\{\mathbf{y}_i\}$ of **T*,** chosen to satisfy (5.69), form a basis for $\mathcal{H}$ which we say is **biorthogonal** to the basis $\{\mathbf{x}_i\}$. We call $\{\mathbf{y}_i\}$ the **reciprocal basis** (P&C 5.31).

Since the eigenvectors of **T,** $\{\mathbf{x}_i\}$, have been assumed to form a basis for $\mathcal{H}$, we can express any **x** in $\mathcal{H}$ in the form $\mathbf{x} = \sum_k c_k \mathbf{x}_k$. Then for any vector $\mathbf{y}_i$ in the reciprocal basis,

$$\langle \mathbf{x}, \mathbf{y}_i \rangle = \sum_k c_k \langle \mathbf{x}_k, \mathbf{y}_i \rangle = \sum_k c_k \delta_{ki} = c_i$$

where we have used the continuity of the inner product to take the infinite sum outside the inner product. Therefore any **x** in $\mathcal{H}$ has the representation

$$\mathbf{x} = \sum_k \langle \mathbf{x}, \mathbf{y}_k \rangle \mathbf{x}_k \tag{5.70}$$

The "biorthogonal" eigenvector expansion (5.70) is very much like an orthonormal eigenvector expansion. It can be used to diagonalize the

operator equation, **Tx = y.** Furthermore, because of the biorthogonal nature of the reciprocal bases, the coefficients are computationally independent. Given a basis of eigenvectors $\{\mathbf{x}_i\}$, it is evident that finding the reciprocal eigenvector basis $\{\mathbf{y}_i\}$ is an alternative to finding the inner product which orthonormalizes $\{\mathbf{x}_i\}$. *

Exercise 3. Show that every **x** in $\mathcal{H}$ also has a biorthogonal expansion in the eigenvectors of **T* :**

$$\mathbf{x} = \sum_k \langle \mathbf{x}, \mathbf{x}_k \rangle \mathbf{y}_k \tag{5.71}$$

5.5 Spectral Decomposition in Infinite-Dimensional Spaces

Because differential equations appear so frequently as models for real phenomena, we have a keen interest in the analysis of such equations. Motivated by the insight that comes from the decoupling of finite-dimensional equations, we seek to perform a similar decoupling of equations involving infinite-dimensional spaces. Suppose **T** is a linear transformation on an infinite-dimensional Hilbert space $\mathcal{H}$. In order to diagonalize (or decouple) the equation **Tx** = **y**, we search for a basis for $\mathcal{H}$ composed of eigenvectors of **T.** Because the space is infinite dimensional, we naturally want to work only with an orthonormal basis; we will find that orthonormality of the eigenvectors requires that **T** be self-adjoint (**T*** = **T**). Furthermore, we wish the orthonormal eigenvectors of **T** to be countable and complete in $\mathcal{H}$ in order that we can expand any vector in $\mathcal{V}$ as a unique, infinite sum of eigenvectors. If **T** has a countable, orthonormal set of eigenvectors $\{\mathbf{x}_k\}$ which is a basis for $\mathcal{H}$, then we can express any vector **x** in $\mathcal{H}$ uniquely as a Fourier series expansion in the eigenvectors:

$$\mathbf{x} = \sum_{k=1}^{\infty} \langle \mathbf{x}, \mathbf{x}_k \rangle \mathbf{x}_k \tag{5.72}$$

Equivalent to the statement that any vector **x** in $\mathcal{H}$ can be expanded uniquely as in (5.72) is the following orthogonal direct-sum decomposition of $\mathcal{H}$ into one-dimensional eigenspaces:

$$\mathcal{H} = \text{span}(\mathbf{x}_1) \overset{\perp}{\oplus} \text{span}(\mathbf{x}_2) \overset{\perp}{\oplus} \cdots \tag{5.73}$$

If we sum those subspaces which are associated with identical eigenvalues,

*See Lamarsh [5.15, p. 549] for a practical function space example.

we can rewrite (5.73) as

$$\mathcal{H} = \sum_{j=1}^{\infty} \overset{\perp}{\oplus} \text{nullspace}(\mathbf{T} - \lambda_j \mathbf{I}) \tag{5.74}$$

where the set $\{\lambda_j\}$ consists of the distinct eigenvalues of **T** (which are not necessarily numbered in correspondence to the eigenvectors $\{\mathbf{x}_k\}$). Equation (5.74) is known as the **spectral theorem.** We will use (5.72) and its equivalents (5.73) and (5.74), to analyze (diagonalize or decouple) operator equations in infinite-dimensional spaces; in particular, differential equations.

Orthonormal Eigenvectors

Assume the bounded linear operator **T** on the Hilbert space $\mathcal{H}$ is diagonalizable; that is, $\mathcal{H}$ has a countable basis of eigenvectors of **T.** A logical place to begin exploration of *orthonormal* eigenvector bases for $\mathcal{H}$ is (5.69)-(5.70). It is clear that if $\mathbf{T}^* = \mathbf{T}$, the eigenvalues and eigenvectors of **T** and $\mathbf{T}^*$ are identical. Then the eigenvalues λ_i are real, the eigenvectors corresponding to different eigenvalues are orthogonal, and the eigenvectors $\mathbf{x}_i$ can be selected so that they form a countable orthonormal basis. A linear operator for which $\mathbf{T}^* = \mathbf{T}$ is said to be **self-adjoint.** Self-adjointness is the key to orthonormality of eigenvectors. If the eigenvalues are real, self-adjointness of **T** is, in fact, necessary in order that there exist eigenvectors of **T** which form an orthonormal basis for $\mathcal{H}$. For if **T** is diagonalizable and $\mathcal{X} \triangleq \{\mathbf{x}_i\}$ is an orthonormal eigenvector basis, then $[\mathbf{T}]_{\mathcal{X}\mathcal{X}}$ is a (possibly infinite) diagonal matrix;

$$[\mathbf{T}]_{\mathcal{X}\mathcal{X}} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$$

But for any orthonormal basis, $[\mathbf{T}^*] = \overline{[\mathbf{T}]}^{\mathrm{T}}$ (P&C 5.27). Therefore,

$$[\mathbf{T}^*]_{\mathcal{X}\mathcal{X}} = \begin{pmatrix} \bar{\lambda}_1 & & & \\ & \bar{\lambda}_2 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$$

and orthonormal eigenvectors for **T** are also orthonormal eigenvectors for **T***. It follows that if the eigenvalues $\{\lambda_i\}$ are real, **T* = T.** In sum, if the linear operator **T** is diagonalizable (a basis of eigenvectors exists) and the eigenvalues of **T** are real, then *there exists an orthonormal basis for the Hilbert space* $\mathcal{H}$ *consisting in eigenvectors of* **T** *if and only if* **T** *is self-adjoint.*

Exercise 1. Show that if **T** is diagonalizable (and the eigenvalues are not necessarily real) there is an orthonormal basis for $\mathcal{H}$ consisting in eigenvectors of **T** if and only if **TT* = T*T.** A linear operator such that **TT*** = **T*T** is said to be a **normal operator.** Show that a normal operator which has real eigenvalues is self-adjoint.

In Section 5.2 we determined how to pick an inner product to orthonormalize a basis for a finite-dimensional space. The result, (5.27), was applied in Example 5 of that section to make the eigenvectors of a particular diagonalizable transformation orthonormal. Since the inner product was chosen to orthonormalize the eigenvector basis, it must also have made the operator self-adjoint. To see that this is the case, we find the adjoint of the operator **T** of Example 5, Section 5.2 relative to the orthonormalizing inner product. The operator **T** on $\mathcal{R}^2$ was defined by

$$\mathbf{T}(\xi_1, \xi_2) \triangleq (2\xi_1 + 3\xi_2, 4\xi_2)$$

The orthonormalizing inner product was

$$\langle (\xi_1, \xi_2), (\eta_1, \eta_2) \rangle \triangleq \xi_1\eta_1 - \tfrac{3}{2}\xi_1\eta_2 - \tfrac{3}{2}\xi_2\eta_1 + \tfrac{5}{2}\xi_2\eta_2$$

Let $\mathbf{x} \triangleq (\xi_1, \xi_2)$ and $\mathbf{y} \triangleq (\eta_1, \eta_2)$. Then

$$\begin{aligned}\langle \mathbf{Tx}, \mathbf{y} \rangle &= \langle (2\xi_1 + 3\xi_2, 4\xi_2), (\eta_1, \eta_2) \rangle \\ &= (2\xi_1 + 3\xi_2)\eta_1 - \tfrac{3}{2}(2\xi_1 + 3\xi_2)\eta_2 - \tfrac{3}{2}(4\xi_2)\eta_1 \\ &= 2\xi_1\eta_1 - 3\xi_1\eta_2 - 3\xi_2\eta_1 - \tfrac{11}{2}\xi_2\eta_2\end{aligned}$$

Since $\langle \mathbf{Tx}, \mathbf{y} \rangle$ is real and symmetric in **x** and **y**,

$$\langle \mathbf{Tx}, \mathbf{y} \rangle = \langle \mathbf{Ty}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{Ty} \rangle$$

Hence **T** is self-adjoint. In general, if we can pick an inner product to make a linear operator self-adjoint, we automatically guarantee that we can find an orthonormal basis consisting of eigenvectors of that operator (P&C 5.32).

Self-Adjoint Linear Operators on Real Function Spaces

Most models of physical systems have real eigenvalues. For these models, orthonormal bases of eigenvectors require self-adjointness of the model. It is because of the usefulness of orthonormal bases of eigenvectors for infinite-dimensional spaces that there is so much emphasis in the literature of physics and mathematics on self-adjoint differential and integral operators.

Let $(\mathbf{Tu})(t) \triangleq \int_a^b k(t,s)\mathbf{u}(s)\,ds$ define an integral operator on a real function space. The adjoint of **T** relative to the standard inner product is given by (5.61). If the integral operator is to be self-adjoint, it is apparent that the kernel of the integral operator must be symmetric:

$$k(t,s) = k(s,t) \tag{5.75}$$

Suppose $\mathbf{Tu} = \mathbf{f}$. We can interpret $k(t,s)$ as a measure of the influence of the value, $\mathbf{u}(s)$, of the "input" function at point s on the value, $\mathbf{f}(t)$, of the "output" function at point t. Self-adjointness, (5.75), implies that the source point and observation point can be interchanged. That is, if $\mathbf{u}(s) = \delta(t_1 - s)$ then $\mathbf{f}(t_2) = \int_a^b k(t_2,s)\delta(t_1 - s)\,ds = k(t_2,t_1)$; interchanging the source and observation points, we find, $\mathbf{f}(t_1) = \int_a^b k(t_1,s)\delta(t_2 - s)\,ds = k(t_1,t_2) = k(t_2,t_1) = \mathbf{f}(t_2)$. This interchangeability of source and observation points is called **reciprocity.** Any system which can be described by an integral operator that is self-adjoint in the standard inner product exhibits reciprocity.

Example 1. A Self-Adjoint Integral Operator. The differential equation $-\mathbf{D}^2\mathbf{f} = \mathbf{u}$ with $\mathbf{f}(0) = \mathbf{f}(b) = 0$ describes the steady-state temperature distribution **f** along the length of an insulated bar of length b. The input function **u** represents the rate of heat generation, perhaps from induction heating, as a function of position within the bar. The temperature is fixed at the bar ends. We inverted this differential equation in Chapter 3. The Green's function (the kernel of the inverse operator), as given in (3.14), is

$$k(t,s) = \frac{(b-s)t}{b}, \qquad 0 \leqslant t \leqslant s$$

$$= \frac{(b-t)s}{b}, \qquad s \leqslant t \leqslant b$$

for $0 \leqslant s \leqslant b$. By (5.61), the adjoint Green's function is

$$k(s,t) = \frac{(b-t)s}{b}, \qquad 0 \leqslant s \leqslant t$$

$$= \frac{(b-s)t}{b}, \qquad t \leqslant s \leqslant b$$

for $0 \leqslant t \leqslant b$. Clearly, $k(t,s) = k(s,t)$, and the integral operator is self-adjoint. It is well known that steady-state heat flow problems exhibit reciprocity.

Suppose we use the weighted inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b \omega(s)\mathbf{f}(s)\mathbf{g}(s)\,ds$ rather than the standard inner product. Then, by (5.62), in order that the integral operator $\mathbf{T}$ be self-adjoint we must have

$$k(t,s) = \frac{\omega(s)k(s,t)}{\omega(t)} \tag{5.76}$$

We will employ (5.76) when we discuss techniques for picking inner products for function spaces.

The adjoint of a differential system ($\mathbf{L}$ with its boundary conditions) consists in the formal adjoint $\mathbf{L}^*$ with the adjoint boundary conditions (or adjoint domain). Recall that the formal adjoint is independent of boundary conditions. If $\mathbf{L}^* = \mathbf{L}$, we say $\mathbf{L}$ is **formally self-adjoint.** We say *the differential system* ($\mathbf{L}$ *with its boundary conditions*) *is self-adjoint if* $\mathbf{L}$ *is formally self-adjoint and the adjoint boundary conditions are identical to the boundary conditions for* $\mathbf{L}$.[†] Exercise 2 of Section 5.4 shows that for the standard inner product the differential operators that are formally self-adjoint are those that contain only even derivatives. In the next example we explore various boundary conditions for the simplest differential operator which is formally self-adjoint with respect to the standard inner product.

Example 2. Self-Adjoint Boundary Conditions for $-\mathbf{D}^*$**.** The operator $\mathbf{L} = -\mathbf{D}^2$ is formally self-adjoint. From Example 10 of Section 5.4, using the standard inner product, we find that

$$\begin{aligned}\langle -\mathbf{D}^2\mathbf{f}, \mathbf{g} \rangle &= \langle \mathbf{f}, -\mathbf{D}^2\mathbf{g} \rangle - \mathbf{f}'(t)\mathbf{g}(t)\big|_a^b + \mathbf{f}(t)\mathbf{g}'(t)\big|_a^b \\ &= \langle \mathbf{f}, -\mathbf{D}^2\mathbf{g} \rangle + \mathbf{f}(b)\mathbf{g}'(b) - \mathbf{f}(a)\mathbf{g}'(a) - \mathbf{f}'(b)\mathbf{g}(b) + \mathbf{f}'(a)\mathbf{g}(a)\end{aligned}$$

Suppose the boundary conditions associated with $\mathbf{L}$ are $\mathbf{f}(a) = \mathbf{f}(b) = 0$. Then the adjoint boundary conditions are $\mathbf{g}(a) = \mathbf{g}(b) = 0$, and $\mathbf{L} = -\mathbf{D}^2$ is self-adjoint. This result is consistent with Example 1, wherein we showed the self-adjointness of the integral operator which is the inverse of this differential system (for the case where $a = 0$). On the other hand, let the boundary conditions associated with $\mathbf{L}$ be the initial conditions $\mathbf{f}(a) = \mathbf{f}'(a) = 0$. Then the adjoint boundary conditions are the final conditions $\mathbf{g}(b) = \mathbf{g}'(b) = 0$, and $\mathbf{L}$ is not self-adjoint. We found in Chapter 3 that the Green's function $k(t,s)$ for an initial condition problem is always zero for $s \geqslant t$. Thus the integral operator which is the inverse of this initial condition problem is

[†]The adjoint boundary conditions are not necessarily unique, but the domain which they define is unique. To be precise, for self-adjointness we require the domain of $\mathbf{L}^*$ to be identical to the domain of $\mathbf{L}$.

not self-adjoint either. Furthermore, we found in Section 4.1 that initial condition problems have no eigenvalues, much less orthonormal eigenfunctions.

Exercise 2. Let $\mathbf{L} = -\mathbf{D}^2$. Verify the following adjoint boundary conditions. Assume $c_1 \neq c_2$.

Boundary Conditions on **L**	Boundary Conditions on **L***
$\mathbf{f}(a) + c_1\mathbf{f}'(a) = \mathbf{f}(b) + c_2\mathbf{f}'(b) = 0$ (separated conditions)	$\mathbf{g}(a) + c_1\mathbf{g}'(a) = \mathbf{g}(b) + c_2\mathbf{g}'(b) = 0$
$\mathbf{f}(a) + c_1\mathbf{f}(b) = \mathbf{f}'(a) + c_2\mathbf{f}'(b) = 0$ (mixed conditions)	$\mathbf{g}(a) + \frac{1}{c_2}\mathbf{g}(b) = \mathbf{g}'(a) + \frac{1}{c_1}\mathbf{g}'(b) = 0$
$\mathbf{f}(a) - \mathbf{f}(b) = \mathbf{f}'(a) - \mathbf{f}'(b) = 0$ (periodic conditions)	$\mathbf{g}(a) - \mathbf{g}(b) = \mathbf{g}'(a) - \mathbf{g}'(b) = 0$
No boundary conditions	$\mathbf{g}(a) = \mathbf{g}(b) = \mathbf{g}'(a) = \mathbf{g}'(b) = 0$

Example 2 and Exercise 2 demonstrate a few general conclusions that can be drawn concerning self-adjointness of second-order differential operators, assuming the differential operator is formally self-adjoint:

1. Separated end-point conditions (wherein each boundary condition involves only one point of $[a,b]$) always yield a self-adjoint operator.
2. Mixed end-point conditions (wherein more than one point of $[a,b]$ can be involved in each boundary condition) seldom yield a self-adjoint operator.
3. Periodic boundary conditions (wherein the conditions at a equal the conditions at b) always yield a self-adjoint operator.
4. Initial conditions always lead to final adjoint boundary conditions. Thus dynamic initial condition problems are never self-adjoint.

We speak loosely of **self-adjoint boundary conditions** when we mean boundary conditions that lead to a self-adjoint operator.

Example 3. A Self-Adjoint Partial Differential Operator. In Example 11 of Section 5.4 we obtained the adjoint of the operator ∇^2 as it acted on the space of two-dimensional functions $\mathcal{L}_2(\Omega)$ with its standard inner product (5.64). We found that ∇^2 is formally self-adjoint. Furthermore, the boundary condition $a\mathbf{f}(\mathbf{p}) + b\mathbf{f}_n(\mathbf{p}) = 0$ for $\mathbf{p}$ on the boundary Γ is also self-adjoint. (This boundary condition is an extension of the separated endpoint conditions illustrated above.) The inverse of

the equation $\nabla^2 \mathbf{f} = \mathbf{u}$ together with the above boundary condition is of the form

$$\mathbf{f}(\mathbf{p}) = \int_{\Omega} k(\mathbf{p}, \mathbf{q})\mathbf{u}(\mathbf{q})d\mathbf{q}$$

Since the differential system is self adjoint, we expect the Green's function k (which defines the inverse system) to be symmetric in $\mathbf{p}$ and $\mathbf{q}$. Consequently the inverse equation will exhibit reciprocity. Since the differential operator (and its inverse) is self adjoint, there exists an orthonormal basis for $\mathfrak{L}_2(\Omega)$ consisting of eigenfunctions of ∇^2 which satisfy the above homogeneous boundary condition on Γ. We use these eigenfunctions later (Example 6) to derive the Green's function and solve the partial differential equation.

Exercise 3. Let Γ be the boundary of the rectangle $0 \leqslant s \leqslant a$, $0 \leqslant t \leqslant b$ in the (s,t) plane. The eigenfunctions corresponding to Example 3 are given in (4.63) for the boundary condition $\mathbf{f}(\mathbf{p}) = 0$ on Γ. Show that these eigenfunctions are orthogonal with respect to the standard inner product for $\mathfrak{L}_2(\Omega)$.

Choosing Inner Products to Orthonormalize Eigenfunctions

Suppose $\mathbf{T}$ is a diagonalizable operator which acts on $\mathfrak{L}_2(a, b)$. Then there is a basis for $\mathfrak{L}_2(a,b)$ consisting in eigenfunctions for $\mathbf{T}$. The discussion associated with (5.49) showed that we can modify the inner product with a bounded positive weight function ω without changing the convergence of sequences of vectors; therefore, the eigenfunctions of $\mathbf{T}$ are also a basis for $\mathfrak{L}_2(\omega; a, b)$. Although the weighted inner product (5.49) does not represent all possible inner products on the function space, in some circumstances we would expect to be able to make an eigenvector basis orthonormal (or at least orthogonal) by choice of the weight function. A given eigenvector basis can be orthogonal only if $\mathbf{T}$ is self-adjoint with respect to the weighted inner product. (Of course, we cannot make $\mathbf{T}$ self-adjoint unless the eigenvalues are real.)

In Example 3 of Section 5.3 we orthogonalized the eigenfunctions, $\mathbf{f}_k(t) = e^{-t/2}\sin(\pi kt/b)$, of the differential operator $\mathbf{D}^2 + \mathbf{D}$ with the boundary conditions $\mathbf{f}(0) = \mathbf{f}(b) = 0$ by choosing the weight function $\omega(t) = \mathrm{e}^t$. Since the eigenfunction basis can be orthogonal only if the operator is self-adjoint, we could as well pick ω to assure self-adjointness. The adjoint of $\mathbf{D}^2 + \mathbf{D}$ is determined by

$$\begin{aligned}\langle(\mathbf{D}^2+\mathbf{D})\mathbf{f},\mathbf{g}\rangle_\omega &\triangleq \int_0^b \omega(\mathbf{f}''+\mathbf{f}')\mathbf{g}\,dt \\ &= \int_0^b \mathbf{f}[\omega\mathbf{g}''+(2\omega'-\omega)\mathbf{g}'+(\omega''-\omega')\mathbf{g}]\,dt \\ &\quad +(\omega\mathbf{g}\mathbf{f}'-\omega\mathbf{g}'\mathbf{f}-\omega'\mathbf{g}\mathbf{f}+\omega\mathbf{g}\mathbf{f})\big|_0^b \\ &\triangleq \langle\mathbf{f},(\mathbf{D}^2+\mathbf{D})^*\mathbf{g}\rangle_\omega + \text{boundary terms}\end{aligned}$$

In order that the operator be formally self-adjoint, it must satisfy

$$\langle \mathbf{f}, (\mathbf{D}^2+\mathbf{D})^*\mathbf{g}\rangle_\omega = \int_0^b \omega \mathbf{f}(\mathbf{g}''+\mathbf{g}')\,dt$$

We choose ω so that the integrands in the above two expressions for $\langle \mathbf{f}, (\mathbf{D}^2+\mathbf{D})^*\mathbf{g}\rangle$ are identical; that is, so $2\omega' - \omega = \omega$ and $\omega'' - \omega' = \theta$. The common solutions to these two differential equations are the multiples of $\omega(t) = e^t$, the same weight function found earlier. Note that there is no additional freedom in the choice of ω with which to produce self-adjointness of the boundary conditions; the self-adjointness of the boundary conditions can be investigated after ω is determined.

Exercise 4. The Green's function for the differential operator $\mathbf{D}^2 + \mathbf{D}$ with the boundary conditions $\mathbf{f}(0) = \mathbf{f}(b) = 0$ is the function $k(t,s)$ of (3.42). Use the self-adjointness condition (5.76) to show again that $\omega(t) = e^t$.

We will demonstrate that every "nice" second-order differential operator is *formally* self-adjoint with respect to some weight function ω. As a consequence, since so many physical systems are representable by second-order differential equations, we can use orthonormal bases of eigenfunctions in analyzing an appreciable fraction of the differential equations which appear in practice. Suppose the differential operator $\mathbf{L}$ is defined for functions $\mathbf{f}$ which are twice continuously differentiable on $[a,b]$ by

$$(\mathbf{Lf})(t) \triangleq g_0(t)\mathbf{f}''(t) + g_1(t)\mathbf{f}'(t) + g_2(t)\mathbf{f}(t) \tag{5.77}$$

Assume $g_i(t)$ is continuous and $g_0(t) < 0$ in the interval. We define the new variables p, q, and ω by

$$p(t) \triangleq \exp\int_a^t \frac{g_1(s)}{g_0(s)}\,ds, \qquad \omega(t) \triangleq -\frac{p(t)}{g_0(t)}, \qquad q(t) \triangleq g_2(t) \tag{5.78}$$

From (5.78) it follows that $g_1/g_0 = p'/p$. (Furthermore, p', q, and ω are continuous; p and ω are bounded and positive.) Then the general second-order differential operator (5.77) can be expressed as

$$\begin{aligned}\mathbf{Lf} &= g_0\left(\mathbf{f}'' + \frac{g_1}{g_0}\mathbf{f}'\right) + g_2\mathbf{f} \\ &= -\frac{p}{\omega}\left(\mathbf{f}'' + \frac{p'}{p}\mathbf{f}'\right) + q\mathbf{f} \\ &= -\frac{1}{\omega}(p\mathbf{f}')' + q\mathbf{f}\end{aligned} \tag{5.79}$$

The operator (5.79) is commonly referred to as a **regular Sturm-Liouville operator.*** We now show via the form (5.79) that *the general second-order differential operator* $\mathbf{L}$ *is formally self-adjoint with respect to the positive weight function* ω *given in* (5.78):

$$\begin{aligned}
\langle \mathbf{Lf}, \mathbf{g} \rangle_\omega &= \int_a^b \omega \left[-\frac{1}{\omega}(p\mathbf{f}')' + q\mathbf{f} \right] \mathbf{g}\, dt \\
&= \int_a^b [-(p\mathbf{f}')'\mathbf{g} + \omega q \mathbf{f}\mathbf{g}]\, dt \\
&= -(p\mathbf{f}')\mathbf{g}|_a^b + \int_a^b p\mathbf{f}'\mathbf{g}'\, dt + \int_a^b \omega q \mathbf{f}\mathbf{g}\, dt \\
&= -p\mathbf{f}'\mathbf{g}|_a^b + \mathbf{f}p\mathbf{g}'|_a^b - \int_a^b \mathbf{f}(p\mathbf{g}')'\, dt + \int_a^b \omega q \mathbf{f}\mathbf{g}\, dt \\
&= p(\mathbf{f}\mathbf{g}' - \mathbf{f}'\mathbf{g})|_a^b + \int_a^b \omega \mathbf{f} \left[-\frac{1}{\omega}(p\mathbf{g}')' + q\mathbf{g} \right] dt \\
&= \text{boundary terms} + \langle \mathbf{f}, \mathbf{Lg} \rangle_\omega
\end{aligned} \tag{5.80}$$

If the boundary conditions are also self-adjoint, we expect to find an orthonormal set of eigenfunctions for $\mathbf{L}$. In point of fact, this orthonormal set of eigenfunctions is complete in $\mathcal{L}_2(\omega; a, b)$, and we can diagonalize equations which involve the general second-order differential operator (5.77). See Birkhoff and Rota [5.3].

We experimented previously with the differential operator $\mathbf{L} = \mathbf{D}^2 + \mathbf{D}$ and boundary conditions $\mathbf{f}(0) = \mathbf{f}(b) = 0$, finding that self-adjointness of the operator and orthogonality of the eigenfunctions both require the weight function $\omega(t) = e^t$. We now treat this operator by means of our general result, (5.80). In order that $g_0(t)$ be negative as required for (5.77), we work with $-\mathbf{L} \triangleq -\mathbf{D}^2 - \mathbf{D}$ (which has the same eigenfunctions as does $\mathbf{L}$). By the substitution (5.78) we find $p(t) = e^t$ and, once again, $\omega(t) = e^t$.

The eigenfunctions of a differential operator $\mathbf{L}$ satisfy the equation: $\mathbf{Lf} - \lambda\mathbf{f} = \boldsymbol{\theta}$. An equivalent equation for the second-order differential operator $\mathbf{L}$ of (5.79) is

$$-\omega(\mathbf{Lf} - \lambda\mathbf{f}) = (p\mathbf{f}')' + (\lambda\omega - \hat{q})\mathbf{f} = \boldsymbol{\theta} \tag{5.81}$$

* If the interval $[a,b]$ were infinite, if p or ω were equal to zero at some point, or if q were discontinuous, (5.79) would be a singular Sturm-Liouville operator. See Birkhoff and Rota [5.3] for examples.

where $\hat{q} \triangleq \omega q$. Equation (5.81) is known as a regular Sturm-Liouville equation. For certain boundary conditions on **f**, **L** will have eigendata. If the boundary conditions are self-adjoint with weight ω, the eigenfunctions can be chosen so they are orthonormal with weight ω. We can obtain the eigendata from (5.81) and the boundary conditions. We call (5.81) together with self-adjoint boundary conditions a **regular Sturm-Liouville system.**

Decoupling of Equations By Means of Eigenvector Expansion

We wish to analyze the linear equation $\mathbf{Tx} = \mathbf{y}$, wherein **x** and **y** are members of a separable infinite-dimensional Hilbert space $\mathcal{H}$. We have found that we can have orthonormal eigenvectors of **T** only if **T** is self-adjoint. It can be shown that complete continuity of **T** is sufficient (but not necessary) to guarantee that the eigenvalues and eigenvectors of **T** are countable. Furthermore, complete continuity together with self-adjointness guarantees that eigendata exist and, that the eigenvectors are complete in $\mathcal{H}$.* We assume **T** is self-adjoint and completely continuous; then the spectral theorem (5.74) applies, and **T** is diagonalizable by means of an orthonormal eigenvector basis $\{\mathbf{x}_k\}$. The vectors **x** and **y** can be expanded using (5.72):

$$\mathbf{y} = \sum_{k=1}^{\infty} c_k \mathbf{x}_k \quad \text{and} \quad \mathbf{x} = \sum_{k=1}^{\infty} d_k \mathbf{x}_k \tag{5.82}$$

where $c_k = \langle \mathbf{y}, \mathbf{x}_k \rangle$, and can be computed using the known vector **y**; d_k is the kth Fourier coefficient of the unknown solution vector **x.** We substitute the expansions (5.82) into the equation $\mathbf{Tx} = \mathbf{y}$ to find

$$\begin{aligned} \boldsymbol{\theta} = \mathbf{y} - \mathbf{Tx} &= \sum_{k=1}^{\infty} c_k \mathbf{x}_k - \mathbf{T}\left(\sum_{k=1}^{\infty} d_k \mathbf{x}_k \right) \\ &= \sum_{k=1}^{\infty} c_k \mathbf{x}_k - \sum_{k=1}^{\infty} d_k \mathbf{T}\mathbf{x}_k \\ &= \sum_{k=1}^{\infty} (c_k - \lambda_k d_k) \mathbf{x}_k \end{aligned}$$

where we have relied on the continuity of **T** and (5.56) to take **T** inside the infinite sum. Then using the orthonormality of the basis $\{\mathbf{x}_k\}$, we find

$$0 = \|\mathbf{y} - \mathbf{Tx}\|^2 = \| \sum_{k=1}^{\infty} (c_k - \lambda_k d_k) \mathbf{x}_k \|^2 = \sum_{k=1}^{\infty} |c_k - \lambda_k d_k|^2$$

* See Bachman and Narici [5.2, Chapter 24] and Stakgold [5.22, Chapter 3].

Since each term in the sum is non-negative, $c_k = \lambda_k d_k, k = 1,2,\ldots$. Then, if **T** is invertible (i.e., has no zero eigenvalues),

$$\mathbf{x} = \sum_{k=1}^{\infty} \frac{\langle \mathbf{y}, \mathbf{x}_k \rangle}{\lambda_k} \mathbf{x}_k \tag{5.83}$$

Equation (5.83) is an explicit expression of the solution **x** to the equation **Tx** = **y** in terms of the eigendata for **T.** The fact that **T** is assumed to act on a Hilbert space is not really a restriction. Were $\mathcal{H}$ an incomplete inner product space, it could be completed and the definition of **T** extended to the complete space. Furthermore, we are not significantly hampered by the boundedness (or complete continuity) used in the derivation of (5.83). Suppose, for example, that **T** represents a self-adjoint differential operator with its boundary conditions (an unbounded operator). If the boundary conditions are appropriate, **T** is invertible and $\mathbf{T}^{-1}$ is typically bounded (and perhaps completely continuous). We simply replace **Tx** = **y** by $\mathbf{T}^{-1}\mathbf{y}$ = **x**, and repeat the above argument to find again that $d_k = c_k/\lambda_k$ and (5.83) is valid.

Example 4. Analysis of a Differential System by Eigenfunction Expansion. The shaft position $\phi(t)$ of a dc motor (with prescribed initial and final shaft positions) is related to the armature voltage **u** of the motor by the following differential equation and boundary conditions:

$$\mathbf{L}\phi \triangleq \ddot{\phi} + \dot{\phi} = \mathbf{u}, \qquad \phi(0) = \phi(b) = 0$$

The Green's function (3.42) for this system is bounded. Therefore, the inverse operator is Hilbert-Schmidt and, consequently, completely continuous. Furthermore, the differential system is self-adjoint relative to the weight function $\omega(t) = e^t$, as noted in Exercise 4. The eigenvalues and eigenfunctions for this differential system are given by (4.37) and (4.38):

$$\lambda_k = -\frac{1}{4} - \left(\frac{k\pi}{b}\right)^2, \qquad \mathbf{f}_k(t) = \sqrt{2/b}\, e^{-t/2} \sin\left(\frac{\pi k t}{b}\right), \qquad k = 1,2,\ldots$$

We determined in Example 3 of Section 5.3 that these eigenfunctions are orthogonal relative to the weight function which makes the differential system self-adjoint. (We have added the multiplier *m* in order to make the functions orthonormal.) We also showed in that example that these eigenfunctions form a basis for $\mathcal{C}(e^t; 0, b)$. [Of course, it is also a basis for the completion of that space, $\mathcal{L}_2(e^t; 0, b)$.] We now use (5.83) to express the solution to the differential system as an expansion in the eigenfunctions of the system; ϕ takes the role of **x**, **y** becomes

u, and $\mathbf{x}_k$ becomes $\mathbf{f}_k$:

$$\phi(t) = \frac{2}{b}\sum_{k=1}^{\infty} \frac{\int_0^b \mathbf{u}(s)e^{s/2}\sin(\pi ks/b)\,ds}{-\frac{1}{4}-\left(\frac{k\pi}{b}\right)^2} e^{-t/2}\sin\left(\frac{\pi kt}{b}\right) \tag{5.84}$$

The solution to this differential system was obtained by inversion in Section 3.3. Following (3.42) we used the inverse to determine the solution for the input $\mathbf{u}(t) = 1$:

$$\phi(t) = t - \frac{be^b}{e^b - 1}(1 - e^{-t})$$

For this same input, $\mathbf{u}(t) = 1$, (5.84) becomes

$$\phi(t) = \frac{2}{b}\sum_{k=1}^{\infty} \frac{(\pi k/b)\left[(-1)^k e^{b/2} - 1\right]}{\left[\frac{1}{4} + (\pi k/b)^2\right]^2} e^{-t/2}\sin\left(\frac{\pi kt}{b}\right)$$

In Figure 5.10, we compare the exact solution with the first two terms of the eigenfunction expansion for $b = \pi$. It is apparent from the figure that for all practical purposes the first few terms of the series determine the solution.

Example 5. Using the eigenfunctions of Example 4, compute the first two terms of the eigenfunction expansion of the input $\mathbf{u}(t) = 1$ with $b = \pi$. (Hint: multiply each curve of Figure 5.10 by the appropriate eigenvalue.) Note that the convergence of this series is considerably slower than the convergence of the output function $\phi(t)$. Furthermore, the series does not converge at the endpoints; convergence in the $\mathcal{L}_2$ norm does not imply convergence everywhere.

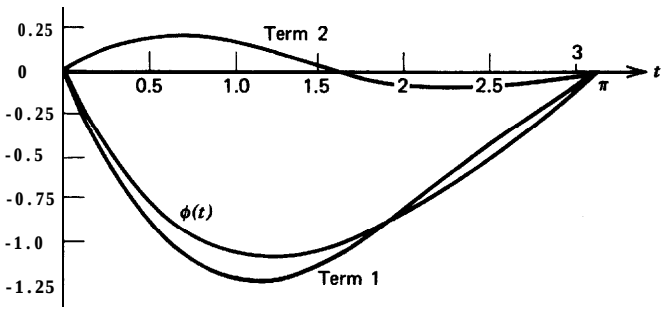


Figure 5.10. Convergence of (5.84) for $\mathbf{u}(t) = 1$ and $b = \pi$.

Suppose that **T** is self-adjoint and completely continuous, but is not invertible. Then the equation **Tx** = **y** has no solution unless **y** is in range(**T**). Furthermore, if the equation has a solution, we can add to it any vector in nullspace to obtain another solution. Formal application of (5.83) would require division by a zero eigenvalue. To resolve this difficulty, we decompose the equation. By the orthogonal decomposition theorem (5.67) and the self-adjointness of **T,** $\mathcal{V}$ = nullspace(**T**) $\overset{\perp}{\oplus}$ $\overline{\text{range}(\mathbf{T})}$. Those eigenvectors in the orthonormal basis which are associated with the zero eigenvalue form an orthonormal basis for nullspace(**T**). The remaining eigenvectors form an orthonormal basis for range(**T**). Moreover, the action of **T** on range(**T**) is one-to-one (the zero eigenvalues have been removed). We use (5.83) to obtain a particular solution to the equation **Tx** = **y** by summing over only the nonzero eigenvalues; the resulting solution lies in range(**T**). Assuming **y** is in range(**T**), the general solution to the equation **Tx** = **y,** for a self-adjoint completely continuous transformation **T,** is expressed in terms of the eigendata for **T** by

$$\mathbf{x} = \sum_{\substack{\text{nonzero} \\ \lambda_k}} \frac{\langle \mathbf{y}, \mathbf{x}_k \rangle}{\lambda_k} \mathbf{x}_k + \mathbf{x}_0 \tag{5.85}$$

where $\mathbf{x}_0$ is an arbitrary vector in nullspace(**T**). As with (5.83), (5.85) may be valid even though **T** is not completely continuous. This fact is illustrated by the unbounded differential operator of Example 5. Rather than dwell further on conditions wherein (5.85) is valid, we adopt the (possibly risky) course of assuming its validity whenever the equation is useful.

In principle, in order to determine whether or not **Tx** = **y** has solutions, we must solve explicitly for range(**T**) and see whether or not **y** is in range(**T**). Finding range(**T**) directly can be difficult for, say, a differential operator. It is simpler to apply the orthogonal decomposition which was introduced in the previous paragraph. If **T** is self-adjoint and range(**T**) is closed,

$$\mathcal{V} = \text{nullspace}(\mathbf{T}) \overset{\perp}{\oplus} \text{range}(\mathbf{T})$$

We solve **Tx** = $\boldsymbol{\theta}$ for the vectors in nullspace(**T**). Then, rather than explicitly determine the vectors in range(**T**), we simply check to see whether or not **y** is orthogonal to all vectors in nullspace(**T**); **y** will be orthogonal to nullspace(**T**) if and only if it lies in range(**T**). Although the range of a differential operator is not closed, this orthogonality test for existence of a solution is used most often for differential equations. See Friedman [5.8, Chapter 3].

Example 5. Analysis by Eigenfunction Expansion—a Noninvertible Case. Let **T** represent the differential operator $\mathbf{L} \triangleq \mathbf{D}^2$ with the homogeneous boundary conditions $\mathbf{f}(1) - \mathbf{f}(0) = \mathbf{f}'(1) - \mathbf{f}'(0) = 0$. The operator **T** is self-adjoint with respect to the standard function space inner product. However, **T** is degenerate; the differential system

$$\mathbf{f}'' = \mathbf{u}, \qquad \mathbf{f}(1) = \mathbf{f}(0), \qquad \mathbf{f}'(1) = \mathbf{f}'(0)$$

has solutions only for a restricted set of functions **u.** Nullspace consists in the solutions to the completely homogeneous system,

$$\mathbf{f}''(t) = 0, \qquad \mathbf{f}(1) = \mathbf{f}(0), \qquad \mathbf{f}'(1) = \mathbf{f}'(0)$$

Thus nullspace consists in the "constant functions," $\mathbf{f}_0(t) = c$. By the discussion above, **u** can be in range(**T**) only if **u** is orthogonal (with respect to the standard inner product) to nullspace(**T**). Therefore, in order that the differential system have a solution, **u** must satisfy

$$\langle \mathbf{u}, \mathbf{f}_0 \rangle = \int_0^1 \mathbf{u}(s)\, c\, ds = 0$$

for all constants c; that is, $\int_0^1 \mathbf{u}(s)\, ds = 0$. The constant functions are eigenfunctions of **T** for the eigenvalue $\lambda_0 = 0$. The nonzero eigenvalues and the corresponding eigenfunctions can be determined by the techniques of Section 4.3; they are

$$\lambda_k = -(2\pi k)^2, \qquad \mathbf{f}_k(t) = \sqrt{2} \cos 2\pi k t, \qquad \mathbf{g}_k(t) = \sqrt{2} \sin 2\pi k t$$

for $k = 1,2,\ldots$. Note that there are two eigenfunctions, $\mathbf{f}_k$ and $\mathbf{g}_k$, for each eigenvalue. The orthogonality of the eigenfunctions for different eigenvalues follows from the self-adjointness of **T.** Each pair of eigenfunctions has been selected such that it forms an orthogonal pair. The whole set of eigenfunctions is essentially (5.30), the basis for the classical Fourier series; the constant function is missing since it is not in range(**T**). If $\int_0^1 \mathbf{u}(s)\, ds = 0$, the solution to the differential system can be expressed by the eigenfunction expansion (5.85):

$$\mathbf{f}(t) = \mathbf{f}_0(t) + \sum_{k=1}^{\infty} \frac{\langle \mathbf{u}, \mathbf{f}_k \rangle}{\lambda_k} \mathbf{f}_k(t) + \sum_{k=1}^{\infty} \frac{\langle \mathbf{u}, \mathbf{g}_k \rangle}{\lambda_k} \mathbf{g}_k(t)$$

$$= c + 2 \sum_{k=1}^{\infty} \frac{\int_0^1 \mathbf{u}(s) \cos(2\pi k s)\, ds}{-(2\pi k)^2} \cos(2\pi k t)$$

$$+ 2 \sum_{k=1}^{\infty} \frac{\int_0^1 \mathbf{u}(s) \sin(2\pi k s)\, ds}{-(2\pi k)^2} \sin(2\pi k t)$$

The arbitrary constant c expresses the freedom (or nonuniqueness) in the solution. As in Example 4. comparison of the exact solution with the sum of the first few terms of the series for a specific $\mathbf{u}$ demonstrates that convergence of the series is rapid.

We found earlier that the general second-order differential operator (5.77) is formally self-adjoint with respect to the weight function ω of (5.78). If the boundary conditions are also self-adjoint, the second-order differential operator (or regular Sturm-Liouville operator) has eigendata; these eigendata are determined by (5.81) together with the boundary conditions. It can be shown that the solutions (eigenfunctions) of any regular Sturm-Liouville system are complete; they form a countable basis, orthonormal with respect to weight ω, for the space $\mathfrak{L}_2(a,b)$.* As exemplified by Example 4, it can also be shown that any regular Sturm-Liouville system has an infinite sequence of real eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. (If the differential operator is invertible, its inverse is a Hilbert-Schmidt operator; that is, the inverse is completely continuous.) Furthermore, the eigenfunctions for a regular Sturm-Liouville system are similar to the sinusoidal functions in that the eigenfunctions for the nth eigenvalue have n zero crossings in the interval $[a,b]$. Sturm-Liouville problems are typically steady state (or standing-wave) problems. Examples of physical systems modeled by regular Sturm-Liouville operators and self-adjoint boundary conditions are vibrating strings, beams, and membranes. Steady-state heat flow in one dimension is another example. Less typical is the motor control problem introduced in (3.40) and solved by eigenfunction expansion in Example 4. In this problem, the standing-wave nature arises because conditions are placed on the future position of the motor shaft.

It follows from Parseval's identity (5.48) that the Fourier coefficients of a vector $\mathbf{x}$ relative to a countably infinite orthonormal basis $\{\mathbf{x}_k\}$ must approach zero: $\langle \mathbf{x}, \mathbf{x}_k \rangle \to 0$ as $k \to \infty$. It is evident from Examples 4 and 5 that the convergence of eigenvector expansions of solutions is accounted for only in part by this property of the Fourier coefficients. In general, Fourier coefficients converge at least as fast as $1/k$. However, in these second-order examples a stronger influence on the convergence of the solutions is exerted by the eigenvalues, with $1/\lambda_k$ converging approximately at $1/k^2$. The "output expansion" (or solution) consists in a modification of the eigenfunction expansion of the input, wherein high-order eigenfunction components (or normal modes) of the input are attenuated more than are the low-order components. In analogy to a dynamic system with initial conditions, we can think of the systems of Examples 4 and 5 as "low-pass" systems; the systems emphasize (or pass) the low-order eigenfunctions.

*See Birkhoff and Rota [5.3].

Example 6. Solution of a Partial Differential Equation by Eigenfunction Expansion. The following partial differential equation is a model for problems in electrostatics or heat flow:

$$\nabla^2 \mathbf{f} = \mathbf{u}$$

Let the two-dimensional region Ω of the (s,t) plane on which $\mathbf{f}$ and $\mathbf{u}$ are defined be the rectangle $0 \leqslant s \leqslant a$, $0 \leqslant t \leqslant b$. Let $\mathbf{f}(s,t) = 0$ on the boundary Γ of this rectangle. In Example 11 of Section 5.4, we determined that ∇^2 is formally self-adjoint with respect to the standard inner product (5.64). Furthermore the boundary condition is also self-adjoint. Thus we expect to find a basis for $\mathfrak{L}_2(\Omega)$ consisting in orthonormal eigenfunctions of the differential system. The eigendata for the system are given in (4.62) and (4.63); we express the eigenfunctions in normalized form:

$$\lambda_{mn} = -\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 = -\frac{\pi^2}{a^2 b^2}(n^2a^2 + m^2b^2)$$

$$\mathbf{f}_{mn}(s,t) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi s}{a}\right) \sin\left(\frac{n\pi t}{b}\right)$$

It can be shown that these eigenfunctions are complete in $\mathfrak{L}_2(\Omega)$ [5.22, p. 1931. By (5.83), the solution to the differential system, expressed as an eigenfunction expansion, is

$$\mathbf{f}(s,t) = -\frac{4ab}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\int_0^b \int_0^a \mathbf{u}(\nu,\rho) \sin(m\pi\nu/a) \sin(n\pi\rho/b)\, d\nu\, d\rho}{n^2a^2 + m^2b^2} \sin\left(\frac{m\pi s}{a}\right) \sin\left(\frac{n\pi t}{b}\right)$$

We have no closed form solution with which to compare this result. Rather, techniques for finding Green's functions for partial differential operators are usually based upon eigenfunction expansions similar to the one used here. See (5.86).

Further Spectral Concepts

We have developed two different approaches to the solution of an invertible differential system, the inverse (3.35) and the eigenfunction expansion (5.83). We would be surprised if the two techniques were not closely related. Let the differential operator $\mathbf{L}$ act on a space of functions defined on $[a,b]$. Suppose $\mathbf{L}$ together with homogeneous boundary conditions has eigendata $\{\lambda_k, \mathbf{f}_k\}$. Further assume that the eigenfunctions form a basis for the function space which is orthonormal with respect to the standard inner product. We wish to explore the equation $\mathbf{Lf} = \mathbf{u}$ together with the boundary conditions. The solution $\mathbf{f}$ can be expressed in terms of the Green's function $k(t,s)$ as $\mathbf{f}(t) = \int_a^b k(t,s)\mathbf{u}(s)\,ds$. As discussed in Chapter 3,

the Green's function is the solution corresponding to the input $\mathbf{u}(t) = \delta(t-s)$. Using (5.83) we express this solution in terms of the eigendata:

$$k(t,s) = \sum_{k=1}^{\infty} \frac{\int_a^b \delta(\tau - s)\mathbf{f}_k(\tau)\,d\tau}{\lambda_k}\mathbf{f}_k(t)$$

$$= \sum_{k=1}^{\infty} \frac{\mathbf{f}_k(s)\mathbf{f}_k(t)}{\lambda_k} \tag{5.86}$$

Equation (5.86) is known as the *bilinear expansion of the Green's function* for $\mathbf{L}$ in terms of the eigenfunctions of $\mathbf{L}$. The Green's function for the Laplacian operator is in fact derivable from Example 6 using an extension of (5.86) to a space of two-dimensional functions.

Exercise 6. Find the bilinear expansion of the Green's function for Example 4 and compare the first term to the exact Green's function as expressed in (3.42).

The spectral theorem (5.74) can be used to define functions of linear operators analogous to the functions of matrix operators discussed in Section 4.6. Assume the linear operator $\mathbf{T}$ in the Hilbert space $\mathcal{V}$ is self-adjoint and completely continuous. Then, applying $\mathbf{T}$ to the expansion (5.72) of a general vector $\mathbf{x}$ in terms of an orthonormal set $\{\mathbf{x}_k\}$ of eigenvectors for $\mathbf{T}$, we find

$$\mathbf{T}\mathbf{x} = \sum_{k=1}^{\infty} \lambda_k \langle \mathbf{x}, \mathbf{x}_k \rangle \mathbf{x}_k \tag{5.87}$$

where we have used the continuity of $\mathbf{T}$ in order to take $\mathbf{T}$ inside the infinite sum. By combining all terms of (5.87) which are associated with identical eigenvalues, we reexpress (5.87) as

$$\mathbf{T} = \sum_{j=1}^{\infty} \lambda_j \mathbf{P}_j \tag{5.88}$$

where $\mathbf{P}_j$ is the orthogonal projector onto nullspace$(\mathbf{T} - \lambda_j \mathbf{I})$. Thus the effect of $\mathbf{P}_j$ on a general vector $\mathbf{x}$ in $\mathcal{V}$ can be expressed in terms of the orthonormal eigenvectors of $\mathbf{T}$:

$$\mathbf{P}_j \mathbf{x} = \sum_k \langle \mathbf{x}, \mathbf{x}_k \rangle \mathbf{x}_k \tag{5.89}$$

where the summation is over all values of k which correspond to the eigenvalue λ_j. Equation (5.88) is the **spectral decomposition of T;** it can be interpreted as a diagonalization of **T.** If **f** is a real continuous function which is defined at the eigenvalues of **T,** it can be shown that a suitable definition of a function of a transformation is provided by the **fundamental formula for f(T)**:

$$f(\mathbf{T}) = \sum_{j=1}^{\infty} f(\lambda_j)\mathbf{P}_j \tag{5.90}$$

Although we have defined (5.90) only for a self-adjoint, completely continuous **T,** the definition can be extended to any bounded normal linear transformation.* Furthermore, as we know from our examples, it can apply to unbounded differential operators. Equation (5.83), for instance, is essentially an expression of $\mathbf{x} = \mathbf{f}(\mathbf{T})\mathbf{y}$ for the function $f(t) \triangleq t^{-1}$. We applied (5.83) to an unbounded differential operator in Example 4.

Throughout our examination of infinite-dimensional operator equations, we have restricted ourselves to operators for which there is a countable orthonormal set of eigenvectors which form a basis for the space. Self-adjoint, completely continuous transformations are of this type. We have restricted ourselves to these transformations in order to work with only the simplest infinite-dimensional extensions of matrix equations. More generality comes only with considerably increased abstraction. Let **T** be a linear operator on an inner product space $\mathcal{V}$. The eigenvalues and eigenvectors of **T** are determined by the equation $\mathbf{Tx} = \lambda\mathbf{x}$, or alternatively, by the **resolvant operator,** $(\mathbf{T} - \lambda\mathbf{I})^{-1}$; the eigenvalues of **T** are those values of **A** for which the latter inverse does not exist. However the nonexistence of the inverse is only one of the ways in which the resolvant operator can be "irregular." Detailed discussions of the resolvant operator and general spectral concepts can be found in Bachman and Narici [5.2], Stakgold [5.22], Friedman [5.8], and Naylor and Sell [5.17].

Matched Filter Design–An Application of Spectral Decomposition

We wish to recognize the presence or absence of a signal $\mathbf{u}(t)$ of known shape (e.g., a radar return). Our measurement of the signal is corrupted by stationary noise $\mathbf{n}(t)$ whose autocorrelation function, $R(t,s) \triangleq \mathbf{E}[\mathbf{n}(t)\,\mathbf{n}(s)] = R(t-s)$, is known. Because $\mathbf{n}(t)$ is stationary, R is symmetric in t and s, and depends only on the time difference $t - s$; R is also finite and positive. We filter the noisy measurement in order to improve our estimate of the presence or absence of the signal (see Figure 5.11). We select the impulse

* Bachman and Narici [5.2].

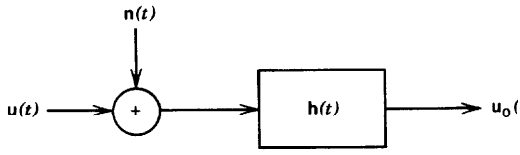


Figure 5.11. A linear filter.

response $\mathbf{h}(t)$ of the linear filter in such a way that the signal-to-noise ratio of the output, $\mathbf{u}_0^2(b)/\mathbf{E}[\mathbf{n}_0^2(b)]$, is maximized at some time $t = b$ units after measurement begins. [A circuit can then be synthesized which has the impulse response $\mathbf{h}(t)$.] The output signal and noise at time $t = b$ are, respectively,*

$$\mathbf{u}_0(b) = \int_0^b \mathbf{h}(s)\mathbf{u}(b-s)\,ds$$

$$\mathbf{n}_0(b) = \int_0^b \mathbf{h}(s)\mathbf{n}(b-s)\,ds$$

Then

$$\mathbf{E}\left[\mathbf{n}_0^2(b)\right] = \mathbf{E}\int_0^b \mathbf{h}(s)\mathbf{n}(b-s)\,ds \int_0^b \mathbf{h}(t)\mathbf{n}(b-t)\,dt$$

$$= \int_0^b \int_0^b \mathbf{h}(s)\mathbf{h}(t)R\,(s-t)\,dt\,ds$$

We use the concepts of P&C 5.30 and Exercise 3, Section 5.1 to interpret $\mathbf{E}[\mathbf{n}_0^2(b)]$ as the square of a norm. Define $\mathbf{T}$ by $(\mathbf{Th})(s) \triangleq \int_0^b \mathbf{h}(t)R\,(s-t)\,dt$. Then, since R is positive and symmetric in its variables, $\mathbf{T}$ is self-adjoint, completely continuous (Hilbert-Schmidt), and positive definite; $\mathbf{T}$ is diagonalizable by means of an orthonormal basis of eigenfunctions $\{\mathbf{f}_k\}$, and the eigenvalues $\{\lambda_k\}$ of $\mathbf{T}$ are positive (P&C 5.28). Therefore, the square roots $\{\sqrt{\lambda_k}\}$ exist, and a unique self-adjoint positive-definite operator $\sqrt{\mathbf{T}}$ is defined by (5.90). Thus

$$\mathbf{E}\left[\mathbf{n}_0^2(b)\right] = \int_0^b \mathbf{h}(s)(\mathbf{Th})(s)\,ds$$

$$= \langle \mathbf{h}, \mathbf{Th} \rangle$$

$$= \|\sqrt{\mathbf{T}}\,\mathbf{h}\|^2$$

* See Appendix 2 for a discussion of convolution and impulse response.

Let $\mathbf{u}_r$ denote the "reverse" of the signal shape $\mathbf{u}$; that is, $\mathbf{u}_r(s) \triangleq \mathbf{u}(b - s)$. Then $\mathbf{u}_0(b) = \langle \mathbf{h}, \mathbf{u}_r \rangle$. Since the eigenvalues $\{ \sqrt{\lambda_k} \}$ of $\sqrt{\mathbf{T}}$ are all positive, $\sqrt{\mathbf{T}}$ is invertible and range$(\sqrt{\mathbf{T}})$ is the whole function space, $\mathcal{L}_2(0, b)$. Therefore, we can assume $\mathbf{u}_r$ is in range$(\sqrt{\mathbf{T}})$; that is, $\mathbf{u}_r = \sqrt{\mathbf{T}}\,\mathbf{g}$ for some function $\mathbf{g}$. Then

$$\mathbf{u}_0^2(b) = |\langle \mathbf{h}, \mathbf{u}_r \rangle|^2 = |\langle \mathbf{h}, \sqrt{\mathbf{T}}\,\mathbf{g} \rangle|^2 = |\langle \sqrt{\mathbf{T}}\,\mathbf{h}, \mathbf{g} \rangle|^2$$

As a consequence, the signal-to-noise ratio satisfies

$$\frac{\mathbf{u}_0^2(b)}{\mathbf{E}\left[\mathbf{n}_0^2(b)\right]} = \frac{|\langle \sqrt{\mathbf{T}}\,\mathbf{h}, \mathbf{g} \rangle|^2}{\|\sqrt{\mathbf{T}}\,\mathbf{h}\|^2} \leqslant \|\mathbf{g}\|^2$$

The latter relationship is the Cauchy-Schwartz inequality (P&C 5.4); equality holds if $\sqrt{\mathbf{T}}\,\mathbf{h} = c\mathbf{g}$ for any constant c, or $\mathbf{T}\mathbf{h} = c\sqrt{\mathbf{T}}\,\mathbf{g} = c\mathbf{u}_r$. We must solve this integral equation for $\mathbf{h}$. It is apparent that $\mathbf{h}$ depends only on the shape of the signal $\mathbf{u}$, but not its magnitude. We can express the solution to the equation in terms of eigendata for $\mathbf{T}$ by means of (5.83):

$$\mathbf{h} = c \sum_{k=1}^{\infty} \frac{\langle \mathbf{u}_r, \mathbf{f}_k \rangle}{\lambda_k} \mathbf{f}_k$$

Suppose the noise is "white"; that is, the autocorrelation function is the limiting case $R(s - t) = N\delta(s - t)$, where N is the noise power and $\delta(s - t)$ is the Dirac delta function. The integral equation becomes

$$(\mathbf{T}\mathbf{h})(s) = N \int_0^b \mathbf{h}(t)\,\delta(s - t)\,dt = N\mathbf{h}(s) = c\mathbf{u}_r(s)$$

or $\mathbf{h}(s)$ is any multiple of $\mathbf{u}_r(s) = \mathbf{u}(b - s)$. The optimum impulse response for this case has the form of the signal running backward in time from the fixed time $t = b$. A filter with this characteristic is called a **matched filter.** We can also use the eigenfunction expansion to determine this solution: The eigendata are determined by

$$(\mathbf{T}\mathbf{h})(s) = N\mathbf{h}(s) = \lambda\mathbf{h}(s)$$

The only eigenvalue is $\lambda = N$. Every function is an eigenfunction. Letting $\{\mathbf{f}_k\}$ be any orthonormal basis for the space, the eigenfunction expansion becomes

$$\mathbf{h} = \frac{c}{N} \sum_{k=1}^{\infty} \langle \mathbf{u}_r, \mathbf{f}_k \rangle \mathbf{f}_k = \frac{c}{N} \mathbf{u}_r$$

The eigenfunction expansion is just a Fourier series expansion of $\mathbf{u}_r$. Although we easily solved for this matched filter, solution of an integral equation and determination of the eigendata of an integral operator are usually difficult problems.

5.6 Problems and Comments

5.1 Let $\mathbf{A}$ be a real symmetric 2×2 matrix with positive eigenvalues.

(*a*) Show that the curve described by the quadratic equation $\langle \mathbf{x}, \mathbf{Ax}\rangle = \mathbf{x}^T\mathbf{Ax} = 1$ is an ellipse in the $\mathbf{x}$ plane. Determine the relationship between the ellipse and the eigendata for $\mathbf{A}$. (Hint: a symmetric matrix has orthogonal eigenvectors. Therefore it can be diagonalized by means of the transformation $\mathbf{A} = \mathbf{S}^T\mathbf{AS}$.)

(*b*) Find the eigendata and sketch the ellipse for

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

5.2 Show that the following definition satisfies the rules for an inner product on $\mathcal{R}^2$:

$$\langle (\xi_1, \xi_2), (\eta_1, \eta_2)\rangle \triangleq 2\xi_1\eta_1 - \xi_1\eta_2 - \xi_2\eta_1 + \xi_2\eta_2$$

5.3 Let $\mathcal{V}$ and $\mathcal{W}$ be inner product spaces over the same scalar field with inner products denoted by $\langle\ ,\ \rangle_{\mathcal{V}}$ and $\langle\ ,\ \rangle_{\mathcal{W}}$, respectively. Let $\mathbf{u}$ and $\mathbf{v}$ be in $\mathcal{V}$; let $\mathbf{w}$ and $\mathbf{z}$ be in $\mathcal{W}$.

(*a*) Show that the following is an inner product on the Cartesian product space $\mathcal{V}\times\mathcal{W}$: $\langle (\mathbf{u},\mathbf{w}), (\mathbf{v},\mathbf{z})\rangle \triangleq \langle \mathbf{u},\mathbf{v}\rangle_{\mathcal{V}} + \langle \mathbf{w},\mathbf{z}\rangle_{\mathcal{W}}$

(*b*) Let $\mathbf{x}$ and $\mathbf{y}$ denote vectors in $\mathcal{C}^2(0, 1) \times \mathcal{C}^1(0, 1)$. Express the elements of $\mathbf{x}$ and $\mathbf{y}$ as 2×1 matrices rather than as 2-tuples. (Then for each t in [0,1], $x(t)$ and $y(t)$ are in the state space, $\mathcal{R}^{2\times 1}$.) Show that the inner product $\langle \mathbf{x}, \mathbf{y}\rangle \triangleq \int_0^1 \mathbf{y}^T(t)\mathbf{x}(t)\,dt$ is essentially a special case of the inner product defined in (*a*).

*5.4 The following useful equalities and inequalities apply to the vectors in any inner product space $\mathcal{V}$:

(*a*) **Pythagorean theorem:** if $\langle \mathbf{x}, \mathbf{y}\rangle = 0$, then

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

(*b*) **Bessel's inequality:** if $\{\mathbf{x}_i\}$ is an orthonormal set in $\mathcal{V}$, then

$$\|\mathbf{x}\|^2 \geq \sum_i |\langle \mathbf{x}, \mathbf{x}_i\rangle|^2$$

(*c*) **Parseval's identity:** equality occurs in (*b*) if and only if $\{x_i\}$ is a basis for $\mathcal{V}$;

(*d*) **Cauchy-Schwartz inequality:** $|\langle \mathbf{x}, \mathbf{y} \rangle| \leqslant \|\mathbf{x}\| \|\mathbf{y}\|$, with equality if and only if **x** and **y** are collinear;

(*e*) **Triangle inequality:** $\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$.

5.5 Equip the vector space $\mathcal{R}^2$ with the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \xi_1\eta_1 - \xi_1\eta_2 - \xi_2\eta_1 + 4\xi_2\eta_2$$

where ξ_i and η_i are the components of **x** and **y,** respectively.

(*a*) Find the matrix $\mathbf{Q}_{\mathcal{E}}$ of the inner product relative to the standard basis for $\mathcal{R}^2$;

(*b*) Find the matrix $\mathbf{Q}_{\mathcal{X}}$ of the inner product relative to the basis $\mathcal{X} \triangleq \{(1,0),(1,1)\}$. Explain the simple form of $\mathbf{Q}_{\mathcal{X}}$.

Let $\langle\ ,\ \rangle$ be an inner product defined on an *n*-dimensional space $\mathcal{V}$. Let $\mathbf{Q}_{\mathcal{X}}$ and $\mathbf{Q}_{\mathcal{Y}}$ be the matrices of this inner product relative to the bases $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let **S** be the coordinate transformation matrix defined by $[\mathbf{x}]_{\mathcal{Y}} = \mathbf{S}[\mathbf{x}]_{\mathcal{X}}$.

(*c*) Determine the relationship between $\mathbf{Q}_{\mathcal{X}}$, $\mathbf{Q}_{\mathcal{Y}}$, and **S**;

(*d*) What special property does **S** possess if $\mathcal{X}$ and $\mathcal{Y}$ are both orthonormal?

5.6 The set $\mathcal{X} \triangleq \{(1,1),(0,-1)\}$ is a basis for $\mathcal{R}^2$. Find an inner product which makes the basis $\mathcal{X}$ orthonormal. Determine the matrix $\mathbf{Q}_{\mathcal{E}}$ of this inner product relative to the standard basis for $\mathcal{R}^2$.

5.7 Let

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The set $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for $\mathcal{M}^{3\times 1}$.

(*a*) Determine an inner product for $\mathcal{M}^{3\times 1}$ with respect to which the basis is orthonormal.

(*b*) Find $\mathbf{Q}_{\mathcal{E}}$, the matrix of the inner product relative to the standard basis for $\mathcal{M}^{3\times 1}$.

5.8 Let $\mathcal{W}_1$ be the subspace of $\mathcal{R}^3$ which is spanned by the pair of vectors (1,0,1) and (0,1,-1). Let $\mathcal{W}_2$ be the subspace of $\mathcal{R}^3$ which is spanned by the vector (1,1,1). Pick an inner product for $\mathcal{R}^3$ which makes every vector in $\mathcal{W}_1$ orthogonal to every vector in $\mathcal{W}_2$.

*5.9 *Positive-definite matrices:* a symmetric $n \times n$ matrix **A** is called **positive definite** if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geqslant 0$ for all real $n \times 1$ vectors **x** and if

equality occurs only for $\mathbf{x} = \boldsymbol{\theta}$. Suppose we pick the kth component of $\mathbf{x}$ equal to zero; it follows that the submatrix of $\mathbf{A}$ obtained by deleting the kth row and kth column of $\mathbf{A}$ must also be positive definite. In fact, any principal submatrix of $\mathbf{A}$ (obtained by deleting a set of rows and the corresponding columns of $\mathbf{A}$) must be positive definite. The determinant of a matrix equals the product of its eigenvalues (P&C 4.6). Furthermore, the eigenvalues of a positive definite matrix are all positive (P&C 5.28). Consequently, if $\mathbf{A}$ is positive definite, the determinant of $\mathbf{A}$ and of each principle submatrix of $\mathbf{A}$ must be positive.

Let $\mathbf{A}_r$ be obtained from $\mathbf{A}$ by deleting all but the first r rows and columns of $\mathbf{A}$; $\det(\mathbf{A}_r)$ is called the rth *leading principle minor* of $\mathbf{A}$. A symmetric $n \times n$ matrix $\mathbf{A}$ is positive definite if and only if the n leading principle minors of $\mathbf{A}$ are positive (see [5.14] and [5.25]). Checking the sign of the leading principle minors is a convenient test for positive definiteness of $\mathbf{A}$.

5.10 Show that the statement $\langle \mathbf{A}, \mathbf{B} \rangle = \text{trace}(\mathbf{B}^T\mathbf{A})$ defines a valid inner product on the real vector space $\mathfrak{M}^{n\times n}$; (the trace of a square matrix is defined to be the sum of the elements on its main diagonal).

5.11 Let $\mathfrak{X} \triangleq \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be an orthonormal basis for a vector space $\mathfrak{V}$. Let $\mathbf{T}$ be a linear operator on $\mathfrak{V}$. Then the element in row i, column j of $[\mathbf{T}]_{\mathfrak{X}\mathfrak{X}}$ is $\langle \mathbf{T}\mathbf{x}_j, \mathbf{x}_i \rangle$ for $i, j = 1, \ldots, n$

5.12 Let $\mathfrak{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ be an orthogonal basis for a real inner product space $\mathfrak{V}$. Approximate a vector $\mathbf{x}$ of $\mathfrak{V}$ by a linear combination, $\mathbf{x}_a = \sum_{k=1}^{n} c_k \mathbf{x}_k$, of the first n vectors of $\mathfrak{X}$ in such a way that $\|\mathbf{x} - \mathbf{x}_a\|^2$ is minimized. Show that the coefficients $\{c_k\}$ are the Fourier coefficients. How are the coefficients affected if we improve the approximation by adding more terms to $\mathbf{x}_a$ (increasing n)?

5.13 Let $\mathbf{A}$ be a 3×3 matrix with eigenvalues $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$ and corresponding linearly independent eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$. Let $\langle \cdot, \cdot \rangle$ denote an inner product for which the above eigenvectors are orthonormal. We wish to solve the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$.

(*a*) Assuming solutions exist, express the general solution $\mathbf{x}$ in terms of the eigendata and the inner product.

(*b*) Determine the conditions that $\mathbf{y}$ must satisfy in order that solutions exist. Express these conditions in terms of the eigendata and the inner product.

5.14 Equip $\mathfrak{M}^{3\times 1}$ with the standard inner product, $\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \mathbf{y}^T\mathbf{x}$. Let

$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$. Obtain an orthonormal basis for span $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ by applying the Gram-Schmidt procedure to $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

5.15 Assign to $\mathfrak{M}^{3\times 1}$ the inner product $\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \mathbf{y}^T\mathbf{Q}\mathbf{x}$, where

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

Let $\mathbf{x}_1 = (1 \quad 0 \quad 1)^T$. Find an orthogonal basis for $\{\mathbf{x}_1\}^\perp$, the orthogonal complement of $\mathbf{x}_1$.

*5.16 *Recurrence formulas for orthogonal polynomials:* let $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots\}$ be a set of polynomials orthogonal with respect to some inner product. Then $\mathbf{p}_n$ can be expressed in terms of $\mathbf{p}_{n-1}$ and $\mathbf{p}_{n-2}$ in the following fashion:

$$\mathbf{p}_n(t) = (c_n t + b_n)\mathbf{p}_{n-1}(t) - a_n\mathbf{p}_{n-2}(t)$$

Once the appropriate coefficients $\{a_n, b_n, c_n\}$ are known, the three-term recurrence formula allows successive determination of the orthogonal polynomials in a manner which is far less cumbersome than the Gram-Schmidt procedure [5.7]. Three-term recurrence formulas exist for other orthogonal sets as well: sine-cosine functions, Bessel functions, and various sets of functions defined on discrete domains. (See [5.24], p. 269 and [5.12]).

Verify for $n = 2$ that the Legendre polynomials of Example 2, Section 5.2 obey the recurrence relation

$$\mathbf{p}_n(t) = \left(\frac{2n-1}{n}\right) t\,\mathbf{p}_{n-1}(t) - \left(\frac{n-1}{n}\right)\mathbf{p}_{n-2}(t)$$

Use this recurrence relation to compute $\mathbf{p}_3$ and verify that it is the next polynomial in the Legendre polynomial set; that is, show that $\mathbf{p}_3$ has the correct norm and is orthogonal to the lower-order polynomials in the set.

5.17 Let $\mathcal{P}^3(-1, 1)$ be the 'space of real polynomial functions of degree less than three with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle \triangleq \int_{-1}^{1} (1 + t^2)\mathbf{f}(t)\mathbf{g}(t)\,dt$$

Let $\mathcal{W}$ be the subspace of $\mathcal{P}^3(-1,1)$ spanned by $\mathbf{f}_0$, where $\mathbf{f}_0(t) \triangleq 1$. Find a basis for $\mathcal{W}^\perp$, the orthogonal complement of $\mathcal{W}$.

5.18 Let $\mathcal{V}$ be the space of complex-valued functions on [0,1] which are bounded, piecewise continuous, and have no more than a finite number of maxima, minima, or discontinuities (these are called the Dirichlet conditions). A typical function in $\mathcal{V}$ is

$$\mathbf{h}(t) = 1 \qquad 0 \leqslant t < \tfrac{1}{2}$$
$$= -1 \qquad \tfrac{1}{2} \leqslant t \leqslant 1$$

Equip $\mathcal{V}$ with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle \triangleq \int_0^1 \mathbf{f}(t)\, \mathbf{g}(t)\, dt$$

Then the set of functions

$$\mathbf{g}_n(t) \triangleq e^{i2\pi nt} \qquad n = 0, \pm 1, \pm 2, \ldots$$

where $i = \sqrt{-1}$, is an orthonormal basis for the space.

(*a*) Determine the coordinates of the function **h** relative to this orthonormal basis; that is, expand **h** in its exponential Fourier series.

(*b*) To what value does the series converge at the discontinuities (t = 0, $\frac{1}{2}$, 1)? (Hint: combine the positive and negative nth order terms of the series.)

*5.19 Let **T** be the linear operator on $\mathcal{M}_c^{n\times 1}$ defined by

$$\mathbf{Tx} \triangleq \mathbf{Ax}$$

where **A** is an $n \times n$ matrix. Let the inner product on $\mathcal{M}_c^{n\times 1}$ be defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \bar{\mathbf{y}}^T \mathbf{Q} \mathbf{x},$$

where **Q** is a hermitian-symmetric, positive-definite matrix. Determine the form of **T***.

5.20 Let **T** be the linear operator on the standard inner product space $\mathcal{L}_2(0, 1)$ defined by

$$(\mathbf{Tf})(t) \triangleq \int_0^t b(s)\mathbf{f}(s)\, ds$$

Determine the form of **T***. Hint: watch the limits of integration.

5.21 Let $\mathbf{T}$: $\mathcal{L}_2(0,1) \times \mathcal{L}_2(0,1) \rightarrow \mathfrak{M}^{2\times 1}$ be defined by

$$\mathbf{Tu} \triangleq \int_0^1 \mathbf{Q}(s)\mathbf{u}(s)\,ds$$

where $\mathbf{Q}(s)$ is a 2×2 matrix and $\mathbf{u}(s)$ is a 2×1 matrix. Find the adjoint $\mathbf{T}^*$ for the inner products

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathfrak{M}^{2\times 1}} \triangleq \mathbf{y}^{\mathrm{T}}\mathbf{x}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{L}_2 \times \mathcal{L}_2} \triangleq \int_0^1 \mathbf{v}^{\mathrm{T}}(s)\mathbf{u}(s)\,ds$$

5.22 Let $\langle \mathbf{x}, \mathbf{z} \rangle_n \triangleq \mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{x}$ and $\langle \mathbf{y}, \mathbf{w} \rangle_m \triangleq \mathbf{w}^{\mathrm{T}}\mathbf{R}\mathbf{y}$ specify the inner products on the real spaces $\mathfrak{M}^{n\times 1}$ and $\mathfrak{M}^{m\times 1}$, respectively, where $\mathbf{Q}$ and $\mathbf{R}$ are symmetric, positive-definite matrices. Define $\mathbf{T}$: $\mathfrak{M}^{n\times 1} \rightarrow \mathfrak{M}^{m\times 1}$ by $\mathbf{Tx} \triangleq \mathbf{Ax}$.

(*a*) Find $\mathbf{T}^*$.

(*b*) Determine the properties which must be satisfied by $\mathbf{A}$, $\mathbf{Q}$, and $\mathbf{R}$ in order that $\mathbf{T}$ be self-adjoint.

5.23 Define $\mathbf{T}$ on $\mathfrak{M}^{2\times 1}$ by $\mathbf{Tx} \triangleq \mathbf{Ax}$, where $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Define the inner product by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^{\mathrm{T}}\mathbf{Q}\mathbf{x}$. Pick $\mathbf{Q}$ such that $\mathbf{T}$ is self-adjoint.

5.24 Let $\mathbf{L} \triangleq \mathbf{D}^2 + \mathbf{D}$ act on those functions $\mathbf{f}$ in $\mathcal{C}^2(a,b)$ which satisfy the boundary conditions $\mathbf{f}(a) = \mathbf{f}'(b) = 0$. Assuming the standard inner product for $\mathcal{C}^2(a,b)$, find the formal adjoint $\mathbf{L}^*$ and the adjoint boundary conditions.

5.25 Define the differential operator $\mathbf{L}$ on $\mathcal{C}^2(a,b)$ by $\mathbf{Lf} \triangleq \mathbf{f}'' - \mathbf{f}'$. Associate with $\mathbf{L}$ the boundary conditions $\mathbf{f}(a) + \mathbf{f}'(a) = \mathbf{f}(b) + \mathbf{f}'(b) = 0$. Find the formal adjoint $\mathbf{L}^*$ and the adjoint boundary conditions relative to the standard inner product.

5.26 The wave equation is

$$\frac{\partial^2 \mathbf{f}}{\partial s^2} + \frac{\partial^2 \mathbf{f}}{\partial \sigma^2} - \frac{\partial^2 \mathbf{f}}{\partial t^2} = 0$$

where s and σ are space variables and t represents time. This equation can be represented in operator notation as

$$(\boldsymbol{\nabla}^2 - \mathbf{D}^2)\mathbf{f} = \boldsymbol{\theta}$$

where the Laplacian operator $\boldsymbol{\nabla}^2$ acts only with respect to the space variables and the ordinary differential operator $\mathbf{D}^2$ acts only with respect to the time variable. Assume $\boldsymbol{\nabla}^2 - \mathbf{D}^2$ acts on the space

$\mathcal{L}_2(\Omega) \times \mathcal{L}_2(0, b)$ with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^b \int_\Omega \mathbf{f}(\mathbf{p}, t)\mathbf{g}(\mathbf{p}, t)\, d\mathbf{p}\, dt$$

where $\mathbf{p} = (s, \sigma)$, Ω is the spatial domain (with boundary Γ), and t is in $[0,b]$. Show that the "wave operator" $\nabla^2 - \mathbf{D}^2$ is formally self-adjoint. Hint: use Examples 10 and 11 of Section 5.4.

*5.27 Let $\mathcal{V}$ be an inner product space (perhaps infinite dimensional) with a basis $\mathcal{X}$. Let $\mathbf{T}$ be a linear operator on $\mathcal{V}$. Show that if $\mathcal{X}$ is orthonormal, $[\mathbf{T}^*]_{\mathcal{X}\mathcal{X}} = \overline{[\mathbf{T}]}^{\mathrm{T}}_{\mathcal{X}\mathcal{X}}$. Hint: express the inner product in terms of coordinates relative to the orthonormal basis.

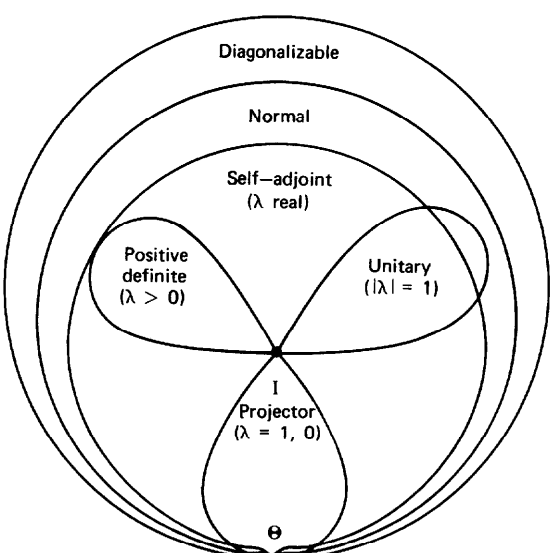
*5.28 Let $\mathbf{T}$ be a linear operator on a complex inner product space $\mathcal{V}$.

(*a*) (1) If $\mathbf{T}^*\mathbf{T} = \mathbf{T}\mathbf{T}^*$, = we call $\mathbf{T}$ a **normal** operator.

(2) If $\mathbf{T}^*\mathbf{T} = = \mathbf{T}\mathbf{T}^* = \mathbf{I}$ (i.e., $\mathbf{T}^* = \mathbf{T}^{-1}$), we call $\mathbf{T}$ a **unitary** operator.

(3) We call $\mathbf{T}$ **non-negative** if $\langle \mathbf{T}\mathbf{x}, \mathbf{x} \rangle \geqslant 0$ for all complex $\mathbf{x}$ in $\mathcal{V}$. If, in addition, $\langle \mathbf{T}\mathbf{x}, \mathbf{x} \rangle = 0$ only for $x = \boldsymbol{\theta}$, we say $\mathbf{T}$ is **positive definite.**

(*b*) If $\mathbf{T}$ is (1) self-adjoint, (2) non-negative, (3) positive definite, (4) unitary, or (5) a projector, then the eigenvalues of $\mathbf{T}$ are, respectively, (1') real, (2') non-negative, (3') positive, (4') of absolute value 1, or (5') equal to 1 or 0. If $\mathbf{T}$ is normal and $\mathcal{V}$ is finite dimensional, then (1')-(5') also imply (1)-(5). The inclusions among these classes of linear operators are illustrated by the following diagram.



*5.29 *Norms of linear transformations:* define $\mathbf{T}$: $\mathcal{M}_c^{n \times 1} \rightarrow \mathcal{M}_c^{m \times 1}$ by $\mathbf{Tx} \triangleq \mathbf{Ax}$, where $\mathbf{A}$ is an $m \times n$ matrix. Assume the standard inner products. Then

$$\|\mathbf{T}\|^2 = \|\mathbf{A}\|^2 = \max_{\mathbf{x}^T\mathbf{x}=1} \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = \lambda_L$$

where λ_L is the eigenvalue of $\mathbf{A}^T\mathbf{A}$ which is of largest magnitude.

(a) Find $\|\mathbf{A}\|$ for the following matrix by carrying out the maximization indicated above:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$$

(b) Find $\|\mathbf{A}\|$ for the matrix $\mathbf{A}$ of *(a)* by determining the eigenvalue λ_L.

(c) A coarse, but easily computed, upper bound on $\|\mathbf{A}\|$ is the *Euclidean norm* of $\mathbf{A}$ defined by

$$\|\mathbf{A}\|_E^2 \triangleq \sum_{i,j} |a_{ij}|^2 = \text{trace}(\mathbf{A}^T\mathbf{A}) = \text{trace}(\mathbf{A}\mathbf{A}^T) = \sum_i |\lambda_i|^2$$

(The numbers $\{\lambda_i\}$ are the eigenvalues of $\mathbf{A}$.) Find $\|\mathbf{A}\|_E$ for the matrix $\mathbf{A}$ of *(a)*. (The last of the equalities applies only for square $\mathbf{A}$.)

(d) If $\mathbf{T}$ is a *bounded normal* operator on a *complex* Hilbert space $\mathcal{V}$, then

$$\|\mathbf{T}\| = \max_{\|\mathbf{x}\|=1} |\langle \mathbf{Tx}, \mathbf{x} \rangle| = \max_i |\lambda_i|$$

where the numbers $\{\lambda_i\}$ are the eigenvalues of $\mathbf{T}$ [5.2, p. 382]. Use this relationship to find $\|\mathbf{T}^{-1}\|$ for $\mathbf{T}$ equal to the differential system of Example 2, Section 4.3. Note that this relationship between $\|\mathbf{T}\|$ and the largest eigenvalue of $\mathbf{T}$ can be used to determine $\|\mathbf{A}\|$ for any *symmetric* matrix $\mathbf{A}$; it cannot be used for the matrix $\mathbf{A}$ of *(a)*.

*5.30 Let $\mathbf{T}$ be a bounded linear operator on a Hilbert space $\mathcal{V}$.

(a) Show that $\langle \mathbf{x}, \mathbf{y} \rangle_T \triangleq \langle \mathbf{x}, \mathbf{Ty} \rangle$ is an inner product on $\mathcal{V}$ if and only if $\mathbf{T}$ is self-adjoint and positive definite.

(b) The operator $\mathbf{T}$ is self-adjoint and positive definite if and only if $\mathbf{T}$ can be decomposed as $\mathbf{T} = \mathbf{U}^2$ where $\mathbf{U}$ is a self-adjoint positive-definite linear operator on $\mathcal{V}$.

(c) Let $\mathcal{V} = \mathcal{M}^{2\times 1}$ with the standard inner product. Let $\mathbf{Tx} \triangleq \mathbf{Qx}$

where

$$\mathbf{Q} = \begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix}$$

Find a self-adjoint, positive-definite operator $\mathbf{U}$ on $\mathfrak{M}^{2\times 1}$ such that $\mathbf{T} = \mathbf{U}^2$.

*5.31 *Reciprocal bases:* let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for $\mathfrak{M}^{n\times 1}$ which is composed of eigenvectors for the invertible $n \times n$ matrix $\mathbf{A}$. Assume the standard inner product for $\mathfrak{M}^{n\times 1}$. The **reciprocal basis** $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ (reciprocal to $\{\mathbf{x}_i\}$) is defined by $\langle \mathbf{x}_i, \mathbf{y}_j \rangle = \mathbf{y}_j^T \mathbf{x}_i = \delta_{ij}$.

(*a*) The vectors in the reciprocal basis are eigenvectors of $\mathbf{A}^T$ (left-hand eigenvectors of $\mathbf{A}$).

(*b*) Every vector $\mathbf{x}$ in $\mathfrak{M}^{n\times 1}$ can be expressed as a biorthogonal expansion, $\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{y}_i \rangle \mathbf{x}_i$. Use this fact to show that the solution to $\mathbf{Ax} = \mathbf{y}$ can be expanded as

$$\mathbf{x} = \sum_{i=1}^{n} \frac{\langle \mathbf{y}, \mathbf{y}_i \rangle}{\lambda_i} \mathbf{x}_i$$

Hint: follow the derivation of (5.24).

(*c*) The **outer product** of two vectors in $\mathfrak{M}^{n\times 1}$ is defined by $\mathbf{x} > < \mathbf{y} \triangleq \mathbf{x}\mathbf{y}^T$. Such a "backwards inner product" is sometimes referred to as a **dyad**. Use the dyad notation to convert the expansion in (*b*) to an explicit matrix multiplication of $\mathbf{y}$. Compare the resulting matrix to the fundamental formula for $\mathbf{A}^{-1}$,

$$\mathbf{A}^{-1} = \frac{1}{\lambda_1} \mathbf{E}_{10}^{\wedge} + \cdots + \frac{1}{\lambda_p} \mathbf{E}_{p0}^{\wedge}$$

where p is the number of *distinct* eigenvalues of $\mathbf{A}$. How are the constituent matrices $\{\mathbf{E}_{i0}^{\wedge}\}$ related to the pair of biorthogonal bases $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_i\}$?

(*d*) Let $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -2 & -1 \end{pmatrix}$. Find a basis for $\mathfrak{M}^{2\times 1}$ consisting in eigenvectors for $\mathbf{A}$; find the reciprocal basis; use the pair of biorthogonal bases to find the constituents of $\mathbf{A}$; use the constituents to compute $\mathbf{A}^{-1}$.

5.32 Let $\mathbf{A}$ be an $n \times n$ matrix. Then $\mathbf{Tx} \triangleq \mathbf{Ax}$ defines a linear operator $\mathbf{T}$ on $\mathfrak{M}^{n\times 1}$. Let $\mathbf{Q}$ be an $n \times n$ symmetric positive-definite matrix.

(*a*) Show that if $\mathbf{T}$ is self-adjoint with respect to the inner product

$\langle \mathbf{x}, \mathbf{y} \rangle_Q \triangleq \mathbf{y}^T\mathbf{Q}\mathbf{x}$, then the operator $\mathbf{U}$ defined by $\mathbf{U}\mathbf{x} \triangleq \mathbf{Q}\mathbf{A}\mathbf{x}$ is self-adjoint with respect to the standard inner product. The matrix equation $\mathbf{A}\mathbf{x} = \mathbf{y}$ can be replaced by an equivalent equation, $\mathbf{Q}\mathbf{A}\mathbf{x} = \mathbf{Q}\mathbf{y}$; the latter equation can be analyzed in terms of a set of eigenvectors (of $\mathbf{Q}\mathbf{A}$) which is orthonormal with respect to the standard inner product.

Let $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$. Find a matrix $\mathbf{Q}$ such that $\mathbf{T}$ is self-adjoint with respect to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle_Q = \mathbf{y}^T\mathbf{Q}\mathbf{x}$. Hint: a test for positive definiteness is given in P&C 5.9.

5.33 By Example 2, Section 5.5, the differential operator $\mathbf{L} \triangleq -\mathbf{D}^2$ and the boundary conditions $\mathbf{f}(0) = \mathbf{f}(b) = 0$ are self-adjoint with respect to the standard inner product on the interval $[0,b]$.

(*a*) The eigendata for $\mathbf{L}$ with the given boundary conditions are

$$\lambda_k = \left(\frac{k\pi}{b}\right)^2, \qquad \mathbf{f}_k(t) = \sin\left(\frac{k\pi t}{b}\right), \qquad k = 1,2,3,...$$

Show that the eigenfunctions form an orthogonal set.

(*b*) Express the solution to the differential system $-\mathbf{f}'' = \mathbf{u}$, $\mathbf{f}(0) = \mathbf{f}(b) = 0$ as an eigenfunction expansion.

(*c*) Compare the first term of the eigenfunction expansion in (*b*) to the exact solution for the specific input function $\mathbf{u}(t) = 1$ and $b = 1$.

5.34 The (nonharmonic) eigendata for the differential operator $-\mathbf{D}^2$ with the boundary conditions $\mathbf{f}(0) = \mathbf{f}(b) + \mathbf{f}'(b) = 0$ are derived in Example 1, Section 4.3.

(*a*) Show that the eigenfunctions are orthogonal with respect to the standard inner product on the interval $[0,b]$ (and, consequently, that $\mathbf{L}$ and the boundary conditions are self-adjoint).

(*b*) Express the solution to the differential system $-\mathbf{f}'' = \mathbf{u}$, $\mathbf{f}(0) = \mathbf{f}(b) + \mathbf{f}'(b) = 0$ as an eigenfunction expansion.

(*c*) Compare the first term of the eigenfunction expansion in (*b*) to the exact solution for the specific input $\mathbf{u}(t) = 1$ and $b = 1$. Hint: $\tan(2.0288) \approx -2.0288$. (The exact solution for the differential system is given in P&C 3.13. Note that the symmetry of the Green's function again implies the self-adjointness of the system with respect to the standard inner product .)

5.35 A (nonsinusoidal) periodic voltage e of frequency ω (period $2\pi/\omega$)

is applied to the terminals of the *R-L* circuit of Figure 5.4. The steady-state current $\mathbf{i}_1$ satisfies the differential equation $L\mathbf{i}_1' + R\mathbf{i}_1 = \mathbf{e}$ with the periodic boundary condition $\mathbf{i}_1(2\pi/\omega) = \mathbf{i}_1(0)$.

(*a*) Find the eigendata for the differential operator $L\mathbf{D} + R\mathbf{I}$ with periodic boundary conditions.

(*b*) Show that the eigenfunctions are orthogonal with respect to the standard *complex* inner product on the interval $[0, 2\pi/\omega]$.

(*c*) Determine the eigenfunction expansion of the steady-state current for an arbitrary periodic voltage. Verify the result by applying the voltage $\mathbf{e}(t) = \sin \omega t$.

5.36 Define $\nabla^2\mathbf{f}(s,t) \triangleq (\partial^2\mathbf{f}/\partial s^2) + (\partial^2\mathbf{f}/\partial t^2)$ on the rectangle $0 \leqslant s \leqslant a$, $0 \leqslant t \leqslant b$. Let $\mathbf{f}$ satisfy the boundary conditions

$$\frac{\partial \mathbf{f}}{\partial s}(0,t) = \frac{\partial \mathbf{f}}{\partial s}(a,t) = \frac{\partial \mathbf{f}}{\partial t}(s,0) = \frac{\partial \mathbf{f}}{\partial t}(s,b) = 0$$

(*a*) The eigendata for ∇^2 with the given boundary conditions are displayed in P&C 4.15. Show that the eigenfunctions are orthogonal with respect to the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle \triangleq \int_0^b \int_0^a \mathbf{f}(s,t)\mathbf{g}(s,t)\, ds\, dt$$

(*b*) Note that one of the eigenvalues is zero. The range of the operator was derived in Example 12, Section 5.4. Express as an eigenfunction expansion the general solution to the partial differential system $\nabla^2\mathbf{f} = \mathbf{u}$ with the given boundary conditions.

*5.37 *A Hilbert space of random variables*: let $\mathcal{V}$ be a vector space of real-valued random variables defined on a particular experiment (Example 11, Section 2.1). An inner product can be defined on $\mathcal{V}$ in terms of the expected value operation (P&C 2.23):

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \mathbf{E}(\mathbf{xy}) = \int \mathbf{x}(\sigma)\mathbf{y}(\sigma)\, \omega(\sigma)\, d\sigma$$

(*a*) Show that $\mathbf{E}(\mathbf{xy})$ is a valid inner product on $\mathcal{V}$.

(*b*) We refer to $\mathbf{E}(\mathbf{x})$ as the *mean* of the random variable $\mathbf{x}$. The *variance* of $\mathbf{x}$ is defined by

$$\mathrm{var}(\mathbf{x}) \triangleq \|\mathbf{x} - \mathbf{E}(\mathbf{x})\|^2 = \mathbf{E}(\mathbf{x}^2) - \mathbf{E}^2(\mathbf{x})$$

The *covariance* between $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathrm{cov}(\mathbf{x}, \mathbf{y}) \triangleq \langle \mathbf{x} - \mathbf{E}(\mathbf{x}), \mathbf{y} - \mathbf{E}(\mathbf{y}) \rangle = \mathbf{E}(\mathbf{xy}) - \mathbf{E}(\mathbf{x})\mathbf{E}(\mathbf{y})$$

The random variables $\mathbf{x}$ and $\mathbf{y}$ are said to be *uncorrelated* if $\text{cov}(\mathbf{x},\mathbf{y}) = 0$. Show that if $\mathbf{x}$ and $\mathbf{y}$ are uncorrelated, then $\text{var}(\mathbf{x} + \mathbf{y}) = \text{var}(\mathbf{x}) + \text{var}(\mathbf{y})$. Show that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, then $\mathbf{E}((\mathbf{x} + \mathbf{y})^2) = \mathbf{E}(\mathbf{x}^2) + \mathbf{E}(\mathbf{y}^2)$ (Pythagorean theorem). If either $\mathbf{x}$ or $\mathbf{y}$ has zero mean, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if and only if they are uncorrelated.

(c) The vector space $\mathcal{H}$ which consists in all random variables (defined on the experiment) of finite norm is a Hilbert space. Show that $\mathcal{H}$ consists in precisely those random variables which have finite mean and finite variance.

5.38 *Karhunen-Loêve Expansion*: let $\mathbf{x} \triangleq (\mathrm{x}(1)\ldots\mathrm{x}(n))^{\mathrm{T}}$ be a discrete finite random process with zero mean; that is, $\mathbf{x}$ consists in a sequence of n random variables $\{\mathbf{x}(i)\}$, all defined on a single experiment,* and $\mathbf{E}(\mathbf{x}(i)) = 0$, $i = 1, \ldots, n$. (For notational convenience we treat the n elements of $\mathbf{x}$ as an $n \times 1$ column vector.) A particular running of the underlying experiment yields a sample function $\bar{\mathbf{x}}$, a specific column of n numbers. The sample function $\bar{\mathbf{x}}$ has a unique Fourier series expansion $\bar{\mathbf{x}} = \sum_{j=1}^{n} \langle \bar{\mathbf{x}}, \mathbf{y}_j \rangle \mathbf{y}_j$ corresponding to each orthonormal basis $\mathcal{Y} = \{\mathbf{y}_j\}$ for the standard inner product space $\mathcal{M}^{n \times 1}$. We can also expand the random process itself in a Fourier series, $\mathbf{x} = \sum_{j=1}^{n} c_j \mathbf{y}_j$, where $c_j = \langle \mathbf{x}, \mathbf{y}_j \rangle = \sum_{p=1}^{n} \mathbf{x}(p)\, \mathbf{y}_j(p)$, and $\mathbf{y}_j(p)$ is the pth element of $\mathbf{y}_j$. However, since the elements $\{\mathbf{x}(p)\}$ of $\mathbf{x}$ are random variables, the Fourier coefficients c_j are also random variables. We wish to pick the basis $\mathcal{Y}$ for $\mathcal{M}^{n \times 1}$ in such a way that the random variables $\{c_j\}$ are statistically orthogonal ($\mathbf{E}(c_j c_k) = 0$), The resulting Fourier series expansion is known as the *Karhunen-Loêve expansion* of the random process.† The sequence of random variables $\{\mathbf{x}(i)\}$ can be represented by the sequence of random variables $\{c_i\}$; the latter are uncorrelated.

(a) If we substitute into $\mathbf{E}(\mathbf{x}(i)c_k)$ the Fourier series expansion $\mathbf{x} = \sum_{j=1}^{n} c_j \mathbf{y}_j$, we find $\mathbf{E}(\mathbf{x}(i)c_k) = \sum_{j=1}^{n} \mathbf{E}(c_j c_k)\mathbf{y}_j(i)$. On the other hand, if we substitute the Fourier coefficient expansion $c_k = \sum_{p=1}^{n} \mathbf{x}(p)\mathbf{y}_k(p)$, we obtain $\mathbf{E}(\mathbf{x}(i)c_k) = \sum_{p=1}^{n} \mathbf{E}(\mathbf{x}(i)\mathbf{x}(p))\, \mathbf{y}_k(p)$. By equating these two expansions, show that the random variables $\{c_k\}$ are orthogonal if and only if the basis functions $\{\mathbf{y}_k\}$ satisfy

$$\mathbf{R}\mathbf{y}_k = \mathbf{E}(c_k^2)\mathbf{y}_k, \quad k = 1, \ldots, n$$

where $\mathbf{R}$ is the autocorrelation matrix for the random process;

* See Example 11, Section 2.1.

† See Papoulis [5.19] for a discussion of the Karhunen-Loêve expansion for continuous random processes.

R is defined by

$$\mathbf{R}(i,p) = \mathbf{E}(\mathbf{x}(i)\mathbf{x}(p)), \qquad i,p = 1,\ldots,n$$

(*b*) Let the autocorrelation matrix of a two-element random process **x** be

$$\mathbf{R} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Find an orthonormal basis $\{\mathbf{y}_1, \mathbf{y}_2\}$ for the standard inner product space $\mathfrak{M}^{2\times 1}$ relative to which the coordinates of **x** are statistically orthogonal. Verify your results by computing the coordinates, c_1 and c_2.

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