3

Linear Differential Operators

Differential equations seem to be well suited as models for systems. Thus an understanding of differential equations is at least as important as an understanding of matrix equations. In Section 1.5 we inverted matrices and solved matrix equations. In this chapter we explore the analogous inversion and solution process for linear differential equations.

Because of the presence of boundary conditions, the process of inverting a differential operator is somewhat more complex than the analogous matrix inversion. The notation ordinarily used for the study of differential equations is designed for easy handling of boundary conditions rather than for understanding of differential operators. As a consequence, the concept of the inverse of a differential operator is not widely understood among engineers. The approach we use in this chapter is one that draws a strong analogy between linear differential equations and matrix equations, thereby placing both these types of models in the same conceptual framework. The key concept is the Green's function. It plays the same role for a linear differential equation as does the inverse matrix for a matrix equation.

There are both practical and theoretical reasons for examining the process of inverting differential operators. The inverse (or integral form) of a differential equation displays explicitly the input-output relationship of the system. Furthermore, integral operators are computationally and theoretically less troublesome than differential operators; for example, differentiation emphasizes data errors, whereas integration averages them. Consequently, the theoretical justification for applying many of the computational procedures of later chapters to differential systems is based on the inverse (or integral) description of the system. Finally, the application of the optimization techniques of Chapters 6-8 to differential systems often depends upon the prior determination of the integral forms of the systems.

One of the reasons that matrix equations are widely used is that we have a practical, automatable scheme, Gaussian elimination, for inverting a matrix or solving a matrix equation. It is also possible to invert certain types of differential equations by computer automation. The greatest progress in understanding and automation has been made for linear,

constant-coefficient differential equations with initial conditions. These equations are good models for many dynamic systems (systems which evolve with time). In Section 3.4 we examine these linear constant-coefficient models in state-space form and also in the form of *n*th-order differential equations. The inversion concept can be extended to partial differential equations.

3.1 A Differential Operator and Its Inverse

Within the process of inverting a differential operator there is an analogue of the elimination technique for matrix inversion. However, the analogy between the matrix equation and the differential equation is clouded by the presence of the boundary conditions. As an example of a linear differential equation and its associated boundary conditions, we use

$$-\mathbf{f}'' = \mathbf{u} \qquad \text{with } \mathbf{f}(0) = \alpha_1 \text{ and } \mathbf{f}(b) = \alpha_2 \tag{3.1}$$

Equation (3.1) can be viewed as a description of the relationship between the steady-state temperature distribution and the sources of heat in an insulated bar of length b. The temperature distribution $\mathbf{f}$ varies only as a function of position t along the bar. The temperature distribution is controlled partly by $\mathbf{u}$, the heat generated (say, by induction heating) throughout the bar, and partly by constant temperature baths (of temperatures α_1 and α_2, respectively) at the two ends $t = 0$ and $t = b$. Thus both the distributed input $\mathbf{u}$ and the boundary inputs $\{\alpha_i\}$ have practical significance. The concepts of *distributed and boundary inputs* extend to other ordinary and partial differential equations.

A Discrete Approximation of the Differential System

In order to obtain a more transparent analogy to matrix equations and thereby clarify the role of the boundary conditions, we temporarily approximate the differential equation by a set of difference equations.* Let $b = 4$, substitute into (3.1) the finite-difference approximation

$$-\frac{d^2\mathbf{f}(t)}{dt^2} \approx -\frac{[\mathbf{f}(t+1)-\mathbf{f}(t)]/1-[\mathbf{f}(t)-\mathbf{f}(t-1)]/1}{1}$$

$$= -\mathbf{f}(t-1)+2\mathbf{f}(t)-\mathbf{f}(t+1)$$

*The approximation of derivatives by finite differences is a practical numerical approach to the solution of ordinary and partial differential equations. The error owing to the finite-difference approximation can be made as small as desired by using a sufficiently fine approximation to the derivatives. (See Forsythe and Wasow [3.3].) Special techniques are usually used to solve the resulting algebraic equations. See P&C 3.3 and Varga [3.12].

and evaluate the equation at $t = 1$, 2, and 3:

$$\begin{aligned} -\mathbf{f}(0)+2\mathbf{f}(1)\ -\mathbf{f}(2) &= \mathbf{u}(1) \\ -\mathbf{f}(1)+2\mathbf{f}(2)\ -\mathbf{f}(3) &= \mathbf{u}(2) \\ -\mathbf{f}(2)+2\mathbf{f}(3)-\mathbf{f}(4) &= \mathbf{u}(3) \\ \mathbf{f}(0) &= \alpha_1 \\ \mathbf{f}(4) &= \alpha_2 \end{aligned} \tag{3.4}$$

It is obvious that this set of algebraic equations would not be invertible without the boundary conditions. We can view the boundary conditions either as an increase in the number of equations or as a decrease in the number of unknowns. The left side of (3.2), including the boundary conditions, is a matrix multiplication of the general vector $(\mathbf{f}(0)\ \mathbf{f}(1)\cdots\mathbf{f}(4))^T$ in the space $\mathcal{M}^{5\times 1}$. The corresponding right-hand side of (3.2) is $(\mathbf{u}(1)\ \mathbf{u}(2)\ \mathbf{u}(3)\ \alpha_1\ \alpha_2)^T$; the boundary values increase the dimension of the range of definition by two. On the other hand, if we use the boundary conditions to eliminate two variables, we reduce the dimension of the domain of the matrix operator by two, $\mathbf{f}(0)$ and $\mathbf{f}(4)$ become part of the right-hand side, and the reduced matrix operates on the general vector $(\mathbf{f}(1)\ \mathbf{f}(2)\ \mathbf{f}(3))^T$ in $\mathcal{M}^{3\times 1}$. By either the "expanded" or the "reduced" view, the transformation with its boundary conditions is invertible. In the next section we explore the differential equation and its boundary conditions along the same lines as we have used for this discrete approximation.

The Role of the Boundary Conditions

A differential operator without boundary conditions is like a matrix with fewer rows than columns: it leads to an underdetermined differential equation. In the same manner as in the discrete approximation (3.2), appropriate boundary conditions make a linear differential operator invertible. In order that we be able to denote the inverse of (3.1) in a simple manner as we do for matrix equations, we must combine the differential operator $-\mathbf{D}^2$ and the two boundary conditions into a single operator on a vector space. We can do so using the "increased equations" view of the boundary conditions. Let $\mathbf{f}$ be a function in the space $\mathcal{C}^2(0,b)$ of twice continuously differentiable functions; then $-\mathbf{f}''$ will be in $\mathcal{C}(0,b)$, the space of continuous functions. Define the differential system operator $\mathbf{T}$: $\mathcal{C}^2(0,b)\rightarrow\mathcal{C}(0,b)\times\mathcal{R}^2$ by

$$\mathbf{Tf} \triangleq (-\mathbf{f}'', \mathbf{f}(0), \mathbf{f}(b)) \tag{3.3}$$

The system equations become

$$\mathbf{Tf} = (\mathbf{u}, \alpha_1, \alpha_2) \tag{3.4}$$

We are seeking an explicit expression of $\mathbf{T}^{-1}$ such that $\mathbf{f} = \mathbf{T}^{-1}(\mathbf{u}, \alpha_1, \alpha_2)$. Because of the abstractness of $\mathbf{T}$, an operation which produces a mixture of a distributed quantity u and discrete quantities $\{\alpha_i\}$, it is not clear how to proceed to determine $\mathbf{T}^{-1}$.

Standard techniques for solution of differential equations are more consistent with the "decreased unknowns" interpretation of the boundary conditions. Ordinarily, we solve the differential equation, $-\mathbf{f}'' = \mathbf{u}$, ignoring the boundary conditions. Then we apply the boundary conditions to eliminate the arbitrary constants in the solution. If we think of the operator $-\mathbf{D}^2$ as being restricted through the whole solution process to act only on functions which satisfy the boundary conditions, then the "arbitrary" constants in the solution to the differential equation are not arbitrary; rather, they are specific (but unknown) functions of the boundary values, $\{\alpha_i\}$. We develop this interpretation of the inversion process into an explicit expression for the inverse of the operator $\mathbf{T}$ of (3.3).

How do we express the "restriction" of $-\mathbf{D}^2$ in terms of an operator on a *vector space?* The set of functions which satisfy the boundary conditions is not a subspace of $\mathcal{C}^2(0,b)$; it does not include the zero function (unless $\alpha_1 = \alpha_2 = 0$). The analogue of this set of functions in the three-dimensional arrow space is a plane which does not pass through the origin. We frame the problem in terms of vector space concepts by separating the effects of the distributed and boundary inputs. In point of fact, it is the difference in the nature of these two types of inputs that has prevented the differential equation and the boundary conditions from being expressed as a single equation.* Decompose the differential system (3.1) into two parts, one involving only the distributed input, the other only the boundary inputs:

$$-\mathbf{f}_d'' = \mathbf{u} \qquad \text{with} \quad \mathbf{f}_d(0) = \mathbf{f}_d(b) = 0 \tag{3.5}$$

$$-\mathbf{f}_b'' = \boldsymbol{\theta} \qquad \text{with} \quad \mathbf{f}_b(0) = \alpha_1, \quad \mathbf{f}_b(b) = \alpha_2 \tag{3.6}$$

Equations (3.5) and (3.6) possess unique solutions. By superposition, these solutions combine to yield the unique solution $\mathbf{f}$ to (3.1); that is, $\mathbf{f} = \mathbf{f}_d + \mathbf{f}_b$. Each of these differential systems can be expressed as a single operator on a vector space. We invert the two systems separately.

We work first with (3.5). The operator $-\mathbf{D}^2$ is onto $\mathcal{C}(0, b)$; that is, we can obtain any continuous function by twice differentiating some function in $\mathcal{C}^2(0, b)$. However, $-\mathbf{D}^2$ is singular; the general vector in nullspace $(-\mathbf{D}^2)$ is of the form $\mathbf{f}(t) = c_1 + c_2 t$. We modify the definition of the operator $-\mathbf{D}^2$ by reducing its domain. Let $\mathcal{V}$ be the subspace of functions in $\mathcal{C}^2(0,b)$ which satisfy the homogeneous boundary conditions of (3.5),

*Friedman [3.4] does include the boundary conditions in the differential equation by treating the boundary conditions as delta functions superimposed on the distributed input.

$\mathbf{f}(0) = \mathbf{f}(b) = 0$. Define the modified differential operator $\mathbf{T}_d$: $\mathcal{V} \to \mathcal{C}(0, b)$ by $\mathbf{T}_d\mathbf{f} \triangleq -\mathbf{D}^2\mathbf{f}$ for all $\mathbf{f}$ in $\mathcal{V}$. The "distributed input" differential system (3.5) becomes

$$\mathbf{T}_d\mathbf{f}_d = \mathbf{u} \tag{3.7}$$

The boundary conditions are now included in the definition of the operator; in effect, we have "reduced" the operator $-\mathbf{D}^2$ to the operator $\mathbf{T}_d$ by using the two boundary conditions to eliminate two "variables" or two degrees of freedom from the domain of the operator $-\mathbf{D}^2$. The operator $\mathbf{T}_d$ is nonsingular; the equation $-\mathbf{f}''(t) = 0$ has no nonzero solutions in $\mathcal{V}$. Furthermore, $\mathbf{T}_d$ is onto; eliminating from the domain of $-\mathbf{D}^2$ those functions which do not satisfy the zero boundary conditions of (3.5) does not eliminate any functions from the range of $-\mathbf{D}^2$. Suppose $\mathbf{g}$ is in $\mathcal{C}^2(0, b)$, and that $\mathbf{g}(0)$ and $\mathbf{g}(b)$ are not zero. Define the related function $\mathbf{f}$ in $\mathcal{V}$ by $\mathbf{f}(t) \triangleq \mathbf{g}(t) - [\mathbf{g}(0) + t(\mathbf{g}(b) - \mathbf{g}(0))/b]$. We have simply subtracted a "straight line" to remove the nonzero end points from g; as a result, $\mathbf{f}(0) = \mathbf{f}(b) = 0$. But $-\mathbf{D}^2\mathbf{f} = -\mathbf{D}^2\mathbf{g}$. Both $\mathbf{f}$ and $\mathbf{g}$ lead to the same function in $\mathcal{C}(0, b)$. Every vector in $\mathcal{C}(0, b)$ comes (via $-\mathbf{D}^2$) from some function in $\mathcal{V}$. Thus $\mathbf{T}_d$ is onto and invertible.

The differential system (3.6) can also be expressed as a single invertible operator. The nonzero boundary conditions of (3.6) describe a transformation $\mathbf{U}$: $\mathcal{C}^2(0, b) \to \mathcal{R}^2$, where

$$\mathbf{U}\mathbf{f} \triangleq (\mathbf{f}(0), \mathbf{f}(b))$$

Since $\mathcal{C}^2(0, b)$ is infinite dimensional but $\mathcal{R}^2$ is not, $\mathbf{U}$ must be singular. We modify the operator $\mathbf{U}$ by reducing its domain. Let $\mathcal{W}$ be the subspace of functions in $\mathcal{C}^2(0, b)$ which satisfy the homogeneous differential equation of (3.6), $-\mathbf{f}_b''(t) = 0$; $\mathcal{W}$ is the two-dimensional space $\mathcal{P}^2$ consisting in functions of the form $\mathbf{f}(t) = c_1 + c_2 t$. We define the modified operator $\mathbf{T}_b$: $\mathcal{P}^2 \to \mathcal{R}^2$ by $\mathbf{T}_b\mathbf{f} \triangleq (\mathbf{f}(0), \mathbf{f}(b))$ for all $\mathbf{f}$ in $\mathcal{P}^2$. The "boundary input" differential system (3.6) can be expressed as the two-dimensional equation

$$\mathbf{T}_b\mathbf{f}_b = (\alpha_1, \alpha_2) \tag{3.8}$$

The differential equation and boundary conditions of (3.6) have been combined into the single operator, $\mathbf{T}_b$. It is apparent that $\mathbf{T}_b$ is invertible-the operator equation is easily solved for its unique solution.

The Inverse Operator

We have rephrased (3.5) and (3.6) in terms of the invertible operators $\mathbf{T}_d$ and $\mathbf{T}_b$, respectively. Because (3.5) and (3.6) constitute a restructuring of

(3.1), we can express the solution to (3.1) as

$$\begin{aligned}\mathbf{f} &= \mathbf{f}_d + \mathbf{f}_b \\ &= \mathbf{T}_d^{-1}\mathbf{u} + \mathbf{T}_b^{-1}(\alpha_1, \alpha_2) \\ &= \mathbf{T}^{-1}(\mathbf{u}, \alpha_1, \alpha_2)\end{aligned} \tag{3.9}$$

where $\mathbf{T}$ is the operator of (3.3).

Since $\mathbf{T}_d$ is a differential operator, we expect $\mathbf{T}_d^{-1} : \mathcal{C}(0, b) \to \mathcal{V}$ to be an integral operator. We express it explicitly in the general form (2.34):

$$\mathbf{f}_d(t) = (\mathbf{T}_d^{-1}\mathbf{u})(t) = \int_0^b k(t,s)\mathbf{u}(s)\,ds \tag{3.10}$$

The kernel function k is commonly referred to as the **Green's function** for the differential system (3.1). In order that (3.10) correctly express the inverse of $\mathbf{T}_d$, $\mathbf{f}_d(t)$ must satisfy the differential system (3.5) from which $\mathbf{T}_d$ is derived. Substituting (3.10) into (3.5) yields

$$\begin{aligned}-\mathbf{f}_d''(t) &= -\frac{d^2}{dt^2}\int_0^b k(t,s)\mathbf{u}(s)\,ds \\ &= \int_0^b -\frac{d^2k(t,s)}{dt^2}\mathbf{u}(s)\,ds = \mathbf{u}(t)\end{aligned}$$

with

$$\mathbf{f}_d(0) = \int_0^b k(0,s)\mathbf{u}(s)\,ds = 0$$

$$\mathbf{f}_d(b) = \int_0^b k(b,s)\mathbf{u}(s)\,ds = 0$$

for all $\mathbf{u}$ in $\mathcal{C}(0, b)$. These equations are satisfied for all continuous $\mathbf{u}$ if and only if

$$-\frac{d^2k(t,s)}{dt^2} = \delta(t-s) \qquad \text{with} \quad k(0,s) = k(b,s) = 0 \tag{3.11}$$

That is, the Green's function k, *as a function of its first variable* t, must satisfy the differential equation and boundary conditions (3.5) for $\mathbf{u}(t) = \delta(t - s)$, where $\delta(t-s)$ is a unit impulse (or Dirac delta function) applied at the point $t = s$.* We can use (3.11) to determine the Green's function.

*See Appendix 2 for a discussion of delta functions. We use some license in interchanging the order of differentiation and integration when delta functions are present. The interchange can be justified, however, through the theory of distributions (Schwartz [3.10]).

For practical purposes we can think of $\delta(t-s)$ as a narrow continuous pulse of unit area, centered at $t = s$. [In terms of the steady-state heat-flow problem (3.1), the function $\delta(t-s)$ in (3.11) represents the generation of a unit quantity of heat per unit time in the cross section of the bar at $t = s$.] However, $\delta(t-s)$ is not a function in the usual sense; its value is not defined at $t = s$. It is not in $\mathcal{C}^2(0, b)$. Therefore, the solution k to (3.11) cannot be in $\mathcal{C}^2(0, b)$. We simply note that the domain $\mathcal{C}^2(0, b)$ and range of definition $\mathcal{C}(0, b)$ of the operator $-\mathbf{D}^2$ were defined somewhat arbitrarily. We can allow a "few" discontinuities or delta functions in $-\mathbf{D}^2\mathbf{f}$ if we also add to $\mathcal{C}^2(0, b)$ those functions whose second derivatives contain a "few" discontinuities or delta functions.

The operator $\mathbf{T}_b^{-1}: \mathcal{R}^2 \rightarrow \mathcal{P}^2$ can also be expressed explicitly. Since $\mathbf{T}_b^{-1}$ acts linearly on the vector (α_1, α_2) in $\mathcal{R}^2$ to yield a polynomial in $\mathcal{P}^2$, we express $\mathbf{T}_b^{-1}$ as

$$\mathbf{f}_b = \mathbf{T}_b^{-1}(\alpha_1, \alpha_2) = \alpha_1\rho_1 + \alpha_2\rho_2 \tag{3.12}$$

where ρ_1 and ρ_2 are functions in $\mathcal{P}^2$. We refer to the function $\rho_j(t)$ as the **boundary kernel** for the differential system (3.1). Just as the Green's function is a function of two variables, t and s, so the boundary kernel is a function of both the continuous variable t and the discrete variable j. Because of the simplicity of the differential operator of this example, the introduction of the boundary kernel seems unnecessary and artificial. For more complicated differential operators, however, the boundary kernel provides a straightforward approach to determination of the full inverse operator. In order that (3.12) correctly describe $\mathbf{T}_b^{-1}$, $\mathbf{f}_b$ must satisfy the differential system (3.6):

$$-\mathbf{f}_b'' = -\alpha_1\rho_1'' - \alpha_2\rho_2'' = 0$$
$$\mathbf{f}_b(0) = \alpha_1\rho_1(0) + \alpha_2\rho_2(0) = \alpha_1$$
$$\mathbf{f}_b(b) = \alpha_1\rho_1(b) + \alpha_2\rho_2(b) = \alpha_2$$

for all α_1 and α_2. Thus the boundary kernel ρ must obey

$$\begin{aligned} -\rho_1''(t) &= 0 \qquad \text{with} \quad \rho_1(0) = 1,\ \rho_1(b) = 0 \\ -\rho_2''(t) &= 0 \qquad \text{with} \quad \rho_2(0) = 0,\ \rho_2(b) = 1 \end{aligned} \tag{3.13}$$

We can use (3.13) to determine the boundary kernel.

We have defined carefully the differential system operator $\mathbf{T}$, the "distributed input" system operator $\mathbf{T}_d$, and the "boundary input" system operator $\mathbf{T}_b$ in order to be precise about the vector space concepts involved with inversion of differential equations. However, to continue use of this

precise notation would require an awkward transition back and forth between the vector space notation and the notation standard to the field of differential equations. We rely primarily on the standard notation. We use the term **differential system** to refer to the differential operator with its boundary conditions (denoted {- $\mathbf{D^2}$, $\mathbf{f(0)}$, f(b)} in this example) and also to the differential equation with its boundary conditions [denoted as in (3.1)]. We refer to both the inverse of the operator and the inverse of the equation as the *inverse of the differential system.* Where we refer to the purely differential part of the system separately, we usually denote it explicitly, for example, as $-\mathbf{D^2}$ or as $-\mathbf{f}'' = \mathrm{u}$.

A Green's Function and Boundary Kernel

We solve for the Green's function k of the system (3.1) by direct integration of (3.11). The successive integration steps are depicted graphically in Figure 3.1. It is clear from the figure that the integral of $-d^2k/dt^2$ is constant for $t < s$ and $t > s$, and contains a jump of size 1 at $t = s$. We permit the value of the constant c to depend upon the point s at which the unit impulse is applied.

$$\begin{aligned} -\frac{dk(t,s)}{dt} &= c(s), \qquad t < s \\ &= c(s) + 1, \quad t > s \end{aligned}$$

Integration of $-dk/dt$ yields continuity of $-k$ at s:

$$\begin{aligned} -k(t,s) &= c(s)t + d(s), && t \leqslant s \\ &= c(s)s + d(s) + (c(s)+1)(t-s), && t \geqslant s \end{aligned}$$

Applying the boundary conditions we find

$$\begin{aligned} -k(0,s) &= c(s)(0) + d(s) = 0 && \Rightarrow d(s) = 0 \\ -k(b,s) &= c(s)s + (c(s)+1)(b-s) = 0 && \Rightarrow c(s) = \frac{s-b}{b} \end{aligned}$$

Thus

$$\begin{aligned} k(t,s) &= \frac{(b-s)t}{b}, \quad t \leqslant s \\ &= \frac{(b-t)s}{b}, \quad t \geqslant s \end{aligned} \tag{3.14}$$

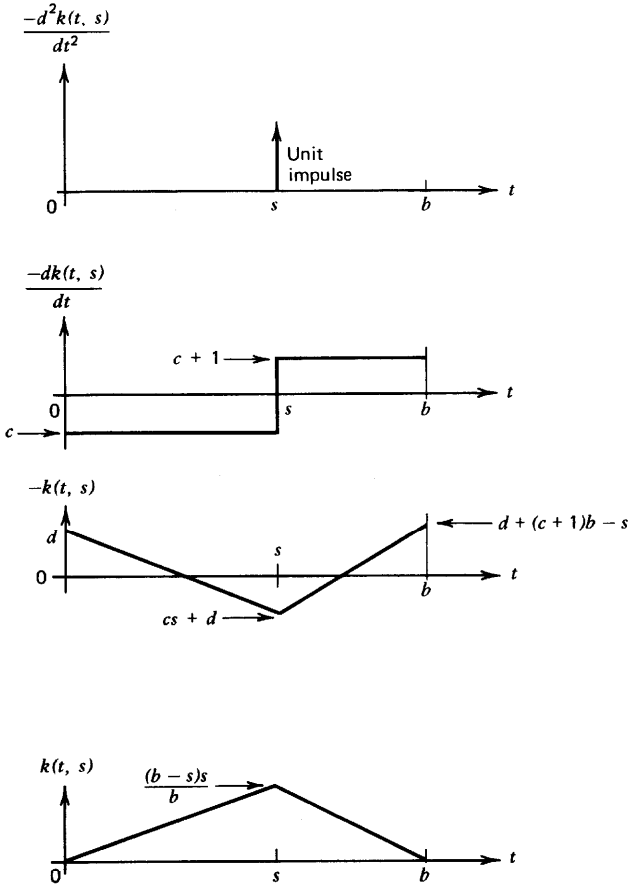

Figure 3.1. Graphical integration of (3.11).

where both t and s lie in the interval $[0, b]$.

By integration of (3.13) we determine the boundary kernel ρ associated with (3.1). The general solution to the jth differential equation is $\rho_j(t) = c_{j1} + c_{j2}t$. Using the boundary conditions we find

$$\rho_1(t) = \frac{b-t}{b}$$
$$\rho_2(t) = \frac{t}{b} \tag{3.15}$$

Having found k and ρ, we insert them into (3.10) and (3.12) to obtain

$\mathbf{T}_d^{-1}$ and $\mathbf{T}_b^{-1}$. Combining the two inverses as in (3.9) produces

$$\mathbf{f}(t) = \int_0^b k(t,s)\mathbf{u}(s)\,ds + \alpha_1\rho_1(t) + \alpha_2\rho_2(t)$$

$$= \int_0^t \frac{(b-t)s}{b}\mathbf{u}(s)\,ds + \int_t^b \frac{(b-s)t}{b}\mathbf{u}(s)\,ds + \alpha_1\frac{b-t}{b} + \alpha_2\frac{t}{b} \quad (3.16)$$

Equation (3.16) is an explicit description of the inverse of the linear differential system (3.1).

A Matrix Analogy

A differential equation with an appropriate set of boundary conditions is analogous to a square matrix equation. We explore this analogy in order to remove some of the abstractness and mystery from differential operators and their inverses. An example of a matrix equation and its corresponding inverse is

$$\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}3 & -2\\ -1 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$$

Any such pair of equations can be expressed as $\mathbf{Ax} = \mathbf{y}$ and $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$, respectively, for some square matrix $\mathbf{A}$. The inverse matrix equation is more clearly analogous to an inverse differential equation (or integral equation) if we express the matrix multiplication in the form of a summation. Denote the elements of $\mathbf{x}$ and $\mathbf{y}$ by ξ_i and η_i, respectively. Then the equation $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$ becomes*

$$\xi_i = \sum_{j=1}^{n} (\mathbf{A}^{-1})_{ij}\eta_j, \qquad i = 1,\ldots,n \quad (3.17)$$

The symbol $(\mathbf{A}^{-1})_{ij}$ represents the element in row i and column j of the $n \times n$ matrix $\mathbf{A}^{-1}$. Thus the inverse matrix, a function of the two integer variables i and j, is the kernel of a summation operator. In the form (3.17), the inverse matrix equation $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$ is obviously a discrete analogue of the integral equation $\mathbf{f}_d(t) = \int_0^b k(t,s)\mathbf{u}(s)\,ds$ of (3.10). The Green's function $k(t,s)$ is the analogue of the inverse matrix $\mathbf{A}^{-1}$. If we compare the inverse matrix equation (3.17) to (3.16), the full inverse of the differential system (3.1), the analogy is clouded somewhat by the presence of the boundary terms. The true analogue of $\mathbf{A}^{-1}$ is the *pair* of kernel functions, k and ρ.

*See (2.35).

Because $k(t,s)$ and $\rho_j(t)$ appear as "weights" in an integral or summation, the inverse form of the differential system is somewhat more useful to the intuition than is the differential system itself.

We can also draw an analogy between the process of inverting the matrix $\mathbf{A}$ and the process of solving for k and ρ. The solution to the equation $\mathbf{Ax} = \boldsymbol{\varepsilon}_i$, where $\boldsymbol{\varepsilon}_i$ is the ith standard basis vector for $\mathfrak{M}^{n\times 1}$, is the ith column of $\mathbf{A}^{-1}$. The solution process is analogous to solving (3.11) for $k(t,s)$ with s fixed; it is also analogous to solving (3.13) for ρ_j with j fixed. The row reduction $(\mathbf{A} \vdots \mathbf{I}) \rightarrow (\mathbf{I} \vdots \mathbf{A}^{-1})$ produces all columns of the inverse matrix simultaneously. Thus the inversion of $\mathbf{A}$ by row reduction is analogous to the determination of k and ρ by solving (3.11) and (3.13), respectively. In general, the process of computing k and ρ requires more effort than does the direct solution of (3.1) for specific inputs $\mathbf{u}$ and $\{\alpha_i\}$. However, the resulting inverse equation (3.16) contains information about the solution for any set of inputs.

3.2 Properties of *n*th-Order Systems and Green's Functions

In Section 3.1 we introduced the concepts of a differential operator and its inverse by means of a simple second-order example, (3.1). We now explore these concepts in detail for more general linear differential systems. Included in this section is an examination of noninvertible differential systems and a development of conditions for invertibility. Techniques for explicit determination of the Green's function and boundary kernel are treated in Section 3.3.

We define a **regular** nth-order linear differential operator $\mathbf{L} : \mathcal{C}^n(a,b) \rightarrow \mathcal{C}(a,b)$ by

$$(\mathbf{Lf})(t) \triangleq g_0(t)\mathbf{f}^{(n)}(t) + g_1(t)\mathbf{f}^{(n-1)}(t) + \cdots + g_n(t)\mathbf{f}(t) \tag{3.18}$$

where the coefficients $\{g_i\}$ are continuous and $g_0(t) \neq 0$ on $[a, b]$.* The corresponding nth-order differential equation is $\mathbf{Lf} = \mathbf{u}$, where the distributed input function $\mathbf{u}$ is continuous on $[a, b]$. It is well known that $\mathbf{L}$ is onto $\mathcal{C}(a,b)$; the nth-order differential equation without boundary conditions always has solutions (Ince [3.6]). The **homogeneous differential equation** is defined as the equation $\mathbf{Lf} = 0$, without boundary conditions (the input u is zero). The homogeneous differential equation for the operator

*If the interval $[a,b]$ were infinite, if g_0 were zero at some point, or if one of the coefficient functions were discontinuous, we would refer to (3.18) as a *singular* differential operator. In Section 5.5 we refer to the regular second-order linear differential operator as a regular Sturm-Liouville operator.

(3.18) always has n linearly independent solutions[†]; we call a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of independent solutions a **fundamental set of solutions** for L. We sometimes express such a set as the **complementary function** for L:

$$\mathbf{f}_c \triangleq c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \tag{3.19}$$

where $c_1, \ldots, c_n$ are unspecified constants. Both the complementary function $\mathbf{f}_c$ and the fundamental set of solutions $\{\mathbf{v}_i\}$ are, in reality, descriptions of the n-dimensional nullspace of $\mathbf{L}$.

In order that $\mathbf{L}$ of (3.18) be invertible, we must add n appropriate boundary conditions to eliminate the n arbitrary constants in the complementary function. We denote the ith boundary condition for (3.18) by $\beta_i(\mathbf{f}) = \alpha_i$, where α_i is a scalar and β_i is a linear functional on $\mathcal{C}^n(a, b)$.[‡] A typical boundary condition is some linear combination of $\mathbf{f}$ and its first $n - 1$ derivatives evaluated at the end points of the interval of definition. For example,

$$\beta_1(\mathbf{f}) \triangleq \gamma_1\mathbf{f}(a) + \gamma_2\mathbf{f}'(a) + \gamma_3\mathbf{f}(b) + \gamma_4\mathbf{f}'(b) = \alpha_1 \tag{3.20}$$

(where the $\{\gamma_i\}$ are scalars) is as general a boundary condition as we would normally expect to encounter for a second-order differential operator acting on functions defined over $[a,b]$. The second boundary condition for the second-order differential equation, $\beta_2(\mathbf{f}) = \alpha_2$, would be of the same form, although the particular linear combination of derivatives which constitutes β_2 would have to be linearly independent of that specified by the coefficients $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ in β_1. There is, of course, no reason why the boundary conditions could not involve evaluations of $\mathbf{f}$ and its derivatives at interior points of the interval of definition. We refer to the boundary condition $\beta_i(\mathbf{f}) = 0$, where the boundary input α_i is zero, as a **homogeneous boundary condition.**

Consider the following nth-order differential system:

$$\begin{aligned} \mathbf{Lf} &= \mathbf{u} \\ \beta_i(\mathbf{f}) &= \alpha_i, \qquad i = 1, \ldots, m \end{aligned} \tag{3.21}$$

where $\mathbf{L}$ is defined in (3.18) and β_i is an nth-order version of (3.20); m is typically but not necessarily equal to n. We call a solution $\mathbf{f}_p$ to (3.21) a

[†]See P&C 3.4.

[‡]Of course, it is possible for the boundary conditions associated with a physical system to be nonlinear functions of $\mathbf{f}$. We consider here only linear differential equations and linear boundary conditions.

particular solution for the differential system. A **completely homogeneous solution** $\mathbf{f}_h$ for the differential system is a solution to the homogeneous differential equation with homogeneous boundary conditions (the homogeneous differential system):

$$\begin{aligned} \mathbf{Lf} &= \boldsymbol{\theta} \\ \boldsymbol{\beta}_i(\mathbf{f}) &= 0, \qquad i = 1, \ldots, m \end{aligned} \tag{3.22}$$

Thus a completely homogeneous solution for the differential system is a solution with all inputs zero. Any solution $\mathbf{f}$ to (3.21) can be written as $\mathbf{f} = \mathbf{f}_p + \mathbf{f}_h$, where $\mathbf{f}_p$ is any particular solution and $\mathbf{f}_h$ is some homogeneous solution. The set of completely homogeneous solutions constitutes the nullspace of the differential system (or the nullspace of the underlying differential operator).* A system with a nonzero nullspace is not invertible.

Exercise 1. Suppose

$$(\mathbf{Lf})(t) \triangleq \mathbf{f}''(t) = \mathbf{u}(t) \tag{3.23}$$

with the boundary conditions

$$\boldsymbol{\beta}_1(\mathbf{f}) \triangleq \mathbf{f}'(0) = \alpha_1 \qquad \boldsymbol{\beta}_2(\mathbf{f}) \triangleq \mathbf{f}'(1) = \alpha_2 \tag{3.24}$$

What is the completely homogeneous solution to (3.23)-(3.24)? Show that the general solution to (3.23)-(3.24) is

$$\mathbf{f}(t) = \int_0^t \int_0^\sigma \mathbf{u}(\tau)\, d\tau\, d\sigma + \alpha_1 t + \mathbf{f}(0) \tag{3.25}$$

where

$$\int_0^1 \mathbf{u}(\tau)\, d\tau = \alpha_1 - \alpha_2 \tag{3.26}$$

Note that the differential system (3.23)-(3.24) is not invertible. No solution exists unless the inputs $\mathbf{u}$ and $\{\alpha_i\}$ satisfy (3.26).

The Role of the Homogeneous Differential System

The matrix analogue of the *n*th-order differential system (3.21) is the matrix equation $\mathbf{Ax} = \mathbf{y}$ (where $\mathbf{A}$ is not necessarily square). Row reduction of $\mathbf{A}$ determines the nullspace of $\mathbf{A}$ (the solution to $\mathbf{Ax} = \boldsymbol{\theta}$); it also shows

*See (3.4) and (3.9).

the dependencies in the rows of $\mathbf{A}$ and the degree of degeneracy of the equation-the degree to which the range of the matrix transformation fails to fill the range of definition. To actually find the range of the matrix transformation (specific conditions on $\mathbf{y}$ for which the equation is solvable), we can follow either of two approaches: (a) row reduce $\mathbf{A}^T$ (the rows of $\mathbf{A}^T$ span the range of $\mathbf{A}$)*; or (b) row reduce $(\mathbf{A} \vdots \mathbf{I})$. If $\mathbf{A}$ is square and invertible, approach (b) amounts to inversion of $\mathbf{A}$.

For the differential system (3.21), the analogue of row reduction of $\mathbf{A}$ is the analysis of the completely homogeneous system (3.22). We focus first on this analysis, thereby determining the extent to which (3.21) is underdetermined or overdetermined. Then assuming the system (3.21) is invertible, we perform the analogue of row reduction of $(\mathbf{A} \vdots \mathbf{I})$—inversion of the differential operator.

The solutions to the homogeneous differential equation, $\mathbf{Lf} = \boldsymbol{\theta}$, are expressed as the complementary function $\mathbf{f}_c$ of (3.19). We apply the m homogeneous boundary conditions to $\mathbf{f}_c$, thereby eliminating some of the arbitrary constants in $\mathbf{f}_c$:

$$\begin{aligned} \beta_1(\mathbf{f}_c) &= c_1\beta_1(\mathbf{v}_1) + \cdots + c_n\beta_1(\mathbf{v}_n) = 0 \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ \beta_m(\mathbf{f}_c) &= c_1\beta_m(\mathbf{v}_1) + \cdots + c_n\beta_m(\mathbf{v}_n) = 0 \end{aligned}$$

or

$$\mathbf{B}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \triangleq \begin{pmatrix} \beta_1(\mathbf{v}_1)\cdots\beta_1(\mathbf{v}_n) \\ \vdots \qquad\quad \vdots \\ \beta_m(\mathbf{v}_1)\cdots\beta_m(\mathbf{v}_n) \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.27}$$

The nullspace of the differential system (3.21) consists in the functions $\mathbf{f}_c = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, where some of the arbitrary constants $\{c_i\}$ are eliminated by (3.27).

The key to the differential system (3.21) lies in $\mathbf{B}$, a **boundary condition matrix** (or **compatibility matrix)** for the system. In point of fact, $\mathbf{B}$ completely characterizes (3.22). It describes not just the boundary conditions, but rather the effect of the boundary conditions on a set of fundamental solutions for $\mathbf{L}$. In general, the m boundary conditions, in concert with the

*see P&C 2.19. In Section 5.4 we introduce the adjoint operator, the analogue of $\mathbf{A}^T$. The orthogonal decomposition theorem (5.67) is the basis of a method for determining the range of an operator from the nullspace of its adjoint; this method is the analogue row reduction of $\mathbf{A}^T$.

*n*th-order differential equation $\mathbf{Lf} = \mathbf{u}$, can specify either an underdetermined or overdetermined set of equations. Exercise 1 exhibits symptoms of both the underdetermined and overdetermined cases. Of course, B is not unique since it can be based on any fundamental set of solutions. Yet the rank of B is unique; $\mathbf{rank(B)}$ tells much about the solutions to (3.21)*:

1. If $\mathbf{rank(B)} = m = n$, then (3.27) precisely eliminates the completely homogeneous solution, and (3.21) is then the analogue of an invertible square matrix equation; the system is invertible.

2. If $\mathbf{rank(B)} = p < n$, then $(n - p)$ of the constants $\{c_i\}$ in the characteristic function remain arbitrary and the nullspace of the system has dimension $(n - p)$. There are $(n - p)$ degrees of freedom in the solutions to (3.21); the system is singular.

3. If $\mathbf{rank(B)} = p < m$, then $(m - p)$ rows of $\mathbf{B}$ are dependent on the rest. As demonstrated by Exercise 1, these dependencies in the rows of $\mathbf{B}$ must be matched by $(m - p)$ scalar-valued relations among the boundary values $\{\alpha_i\}$ and the distributed input $\mathbf{u}$, or there can be no solutions to (3.21) (P&C 3.5). The system is not onto $\mathcal{C}(a, b)$.

The following example demonstrates the relationship between $\mathbf{rank(B)}$ and the properties of the differential system.

Example 1. The Rank of the Boundary Condition Matrix. Let

$$(\mathbf{Lf})(t) \triangleq \mathbf{f}''(t) = \mathbf{u}(t)$$

for t in [0, 1]. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1(t) = 1$ and $\mathbf{v}_2(t) = t$, is a fundamental set of solutions for t. We apply several different sets of boundary conditions, demonstrating the three cases mentioned above.

1. $\boldsymbol{\beta}_1(\mathbf{f}) \triangleq \mathbf{f}(0) = \alpha_1$, $\boldsymbol{\beta}_2(\mathbf{f}) \triangleq \mathbf{f}(1) = \alpha_2$. In this case,

$$\mathbf{B} = \begin{pmatrix} \mathbf{v}_1(0) & \mathbf{v}_2(0) \\ \mathbf{v}_1(1) & \mathbf{v}_2(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Since $\mathbf{rank(B)} = 2 = m = n$, the system is invertible. We find the unique solution by direct integration:

$$\mathbf{f}(t) = \int_0^t \int_0^s \mathbf{u}(\tau)\, d\tau\, ds + \left[\alpha_2 - \alpha_1 - \int_0^1 \int_0^s \mathbf{u}(\tau)\, d\tau\, ds \right] t + \alpha_1$$

2. $\boldsymbol{\beta}_1(\mathbf{f}) \triangleq \mathbf{f}(0) = \alpha_1$. For this single boundary condition,

$$\mathbf{B} = \begin{pmatrix} \mathbf{v}_1(0) & \mathbf{v}_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

*Ince [3.6].

and $\mathbf{rank(B)} = 1$. Since $n = 2$, we should expect one degree of freedom in the solution. Since $m = \mathbf{rank(B)}$, we should expect a solution to exist for all scalars α_1 and all continuous functions $\mathbf{u}$. By direct integration, the solution is

$$\mathbf{f}(t) = \int_0^t \int_0^s \mathbf{u}(\tau)\, d\tau\, ds + d_1 t + \alpha_1$$

where d_1 is an arbitrary constant.

3. $\beta_1(\mathbf{f}) \triangleq \mathbf{f}(0) = \alpha_1$, $\beta_2(\mathbf{f}) \triangleq \mathbf{f}(1) = \alpha_2$, $\beta_3(\mathbf{f}) \triangleq \mathbf{f}'(0) = \alpha_3$. Then,

$$\mathbf{B} = \begin{pmatrix} \mathbf{v}_1(0) & \mathbf{v}_2(0) \\ \mathbf{v}_1(1) & \mathbf{v}_2(1) \\ \mathbf{v}_1'(0) & \mathbf{v}_2'(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Because $\mathbf{rank(B)} = 2$ and $m = 3$, one scalar-valued function of $\mathbf{u}$, α_1, α_2, and α_3 must be satisfied in order that a solution exist. Since $\mathbf{rank(B)} = n$, if a solution exists for a given set of inputs $(\mathbf{u}, \alpha_1, \alpha_2)$, that solution is unique. We find the solution by direct integration and application of the three boundary conditions:

$$\mathbf{f}(t) = \int_0^t \int_0^s \mathbf{u}(\tau)\, d\tau\, ds + \alpha_3 t + \alpha_1$$

where $\mathbf{u}$, α_1, α_2, and α_3 must satisfy

$$\alpha_2 - \alpha_1 - \alpha_3 - \int_0^1 \int_0^s \mathbf{u}(\tau)\, d\tau\, ds = 0$$

4. $\beta_1(\mathbf{f}) \triangleq \mathbf{f}'(0) = \alpha_1$, $\beta_2(\mathbf{f}) \triangleq \mathbf{f}'(1) = \alpha_2$. This case is presented in Exercise 1.

$$\mathbf{B} = \begin{pmatrix} \mathbf{v}_1'(0) & \mathbf{v}_2'(0) \\ \mathbf{v}_1'(1) & \mathbf{v}_2'(1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$\mathbf{Rank(B)} = 1$, but $m = n = 2$. We expect one scalar-valued condition on the inputs, and one degree of freedom in the solutions. The general solution and the restriction on the inputs are given in (3.25) and (3.26), respectively.

It is apparent from Example 1 that if $m < n$, the system is underdetermined; there are at least $(n - m)$ degrees of freedom in the solutions. On the other hand, if $m > n$, the system is usually overdetermined; since $\mathbf{rank(B)} < n$ for an nth-order differential system, the input data must satisfy at least $(m - n)$ different scalar-valued restrictions in order that the differential equation and boundary conditions be solvable.

Ordinarily, $m = n$; that is, the differential equation which represents a physical system usually has associated with it n independent boundary

conditions $\{\beta_i\}$. These n boundary conditions are independent in the sense that they represent independent linear combinations of $\mathbf{f}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(n-1)}$ evaluated at one or more points of $[a,b]$. However, we see from the fourth case of Example 1 that a boundary condition matrix $\mathbf{B}$ can be degenerate even if the boundary conditions are independent. Thus for a "square" differential operator, the condition for invertibility (or compatibility) is

$$\det(\mathbf{B}) \neq 0 \tag{3.28}$$

where $\mathbf{B}$ is a boundary condition matrix as defined in (3.27).

It can be shown that (3.28) is satisfied for any differential operator for which $m = n$ and for which the boundary conditions are linearly independent and are all at one point (P&C 3.4). Only for multipoint boundary value problems can the test (3.28) fail. Exercise 1 is such a case.

For the rest of this chapter we assume (3.28) is satisfied, and proceed to determine the inverse of the differential system (3.21). In Section 4.3, where we determine eigenvalues and eigenfunctions of differential operators, we seek conditions under which (3.28) is **not** satisfied. These conditions occur, of course, only with multipoint boundary value problems.

The Green's Function and the Boundary Kernel

Our procedure for inverting the system (3.21) parallels the procedure used with the second-order example (3.1). Of course, the compatibility condition (3.28) must be satisfied. Assume $m = n$. We begin by splitting (3.21) into two parts, one involving only the distributed input, the other only the boundary inputs:

$$\begin{aligned} &\mathbf{Lf} = \mathbf{u} \\ &\beta_i(\mathbf{f}) = 0 \quad i = 1, \ldots, n \end{aligned} \tag{3.29}$$

$$\begin{aligned} &\mathbf{Lf} = \boldsymbol{\theta} \\ &\beta_i(\mathbf{f}) = \alpha_i \quad i = 1, \ldots, n \end{aligned} \tag{3.30}$$

where $\mathbf{L}$ is given in (3.18). The completely homogeneous equation (3.22) is a special case of both (3.29) and (3.30). Thus both are characterized by any boundary condition matrix $\mathbf{B}$ derived from (3.22). If (3.28) is satisfied, both (3.29) and (3.30) are invertible. The inverse of (3.29) is an integral operator with a distributed kernel. The inverse of (3.30) is a summation operator involving a boundary kernel. These two kernels describe explicitly the dependence of $\mathbf{f}(t)$ on the input data $\mathbf{u}(t)$ and $\{\alpha_j\}$.

Assume the inverse of (3.29) is representable in the integral form

$$\mathbf{f}(t) = \int_a^b k(t,s)\mathbf{u}(s)\,ds \tag{3.31}$$

for all t in $\{a,b\}$. The kernel k is known as the Green's function for the system (3.21). If (3.31) is the correct inverse for (3.29), $\mathbf{f}$ must satisfy (3.29):

$$
\begin{aligned}
(\mathbf{L}\mathbf{f})(t) &= \mathbf{L}\int_a^b k(t,s)\mathbf{u}(s)\,ds \\
&= \int_a^b \mathbf{L}k(t,s)\mathbf{u}(s)\,ds = \mathbf{u}(t) \\
(\boldsymbol{\beta}_i\mathbf{f})(t) &= \boldsymbol{\beta}_i\int_a^b k(t,s)\mathbf{u}(s)\,ds \\
&= \int_a^b \boldsymbol{\beta}_i k(t,s)\mathbf{u}(s)\,ds = 0 \qquad i=1,\ldots,n
\end{aligned}
$$

for all $\mathbf{u}$ in $\mathcal{C}(a,b)$. Both $\mathbf{L}$ and $\boldsymbol{\beta}_i$ treat the variable s as a constant, *acting on $k(t,s)$ only as a function of t.* Each operator acts on the whole "t" function $k(\cdot,s)$. It is evident that $\mathbf{L}k(t,s)$ exhibits the "sifting" property of a delta function (see Appendix 2). On the other hand, $\boldsymbol{\beta}_i k(t,s)$ acts like the zero function. Consequently, the Green's function k must satisfy

$$
\begin{aligned}
&\mathbf{L}k(t,s) \triangleq g_0(t)\frac{d^n k(t,s)}{dt^n} + \cdots + g_n(t)k(t,s) = \delta(t-s) \\
&\boldsymbol{\beta}_i k(t,s) = 0 \qquad j=1,\ldots,n
\end{aligned}
\tag{3.32}
$$

for all t and s in $[a,b]$. Because the delta function appears in (3.32), we cannot rigorously interchange the order of the differential operator $\mathbf{L}$ and the integration without resorting to the theory of generalized functions (Appendix 2). However, we can justify the formal interchange for each specific problem by showing that the Green's function k derived from (3.32) does indeed lead to the solution of (3.29) for every continuous function $\mathbf{u}$.

Assume the inverse of (3.30) is representable as a summation operator of the form:

$$
\mathbf{f}(t) = \sum_{j=1}^{n} \rho_j(t)\alpha_j \tag{3.33}
$$

We can think of ρ as a kernel function of the two variables j and t. We call ρ the boundary kernel for (3.21). To find the equations which determine ρ,

we substitute (3.33) into (3.30):

$$\mathbf{Lf} = \mathbf{L}\sum_{j=1}^{n} \rho_j \alpha_j$$

$$= \sum_{j=1}^{\mathrm{ll}} \alpha_j (\mathbf{L}\rho_j) = \boldsymbol{\theta}$$

$$\boldsymbol{\beta}_i(\mathbf{f}) = \boldsymbol{\beta}_i \sum_{j=1}^{n} \rho_j \alpha_j$$

$$= \sum_{j=1}^{n} \alpha_j \boldsymbol{\beta}_i(\rho_j) = \alpha_i \quad i = 1, \ldots, n$$

for all $\{\alpha_i\}$. Suppose we let $\alpha_k = 1$ and $\alpha_j = 0$ for $j \neq k$. It follows that for $k = 1, \ldots, n$, $\mathbf{L}\rho_k = \boldsymbol{\theta}$, $\boldsymbol{\beta}_i\rho_k = 1$ for $k = i$, and $\boldsymbol{\beta}_i\rho_k = 0$ for $k \neq i$. Thus the boundary kernel ρ must satisfy

$$(\mathbf{L}\rho_j)(t) \triangleq g_0(t)\frac{d^n\rho_j(t)}{dt^n} + \cdots + g_n(t)\rho_j(t) = 0 \qquad j = 1, \ldots, n \tag{3.34}$$

$$\boldsymbol{\beta}_i(\rho_j) = \delta_{ij} \qquad i = 1, \ldots, n; \quad j = 1, \ldots, n$$

for all t in $[a, b]$, where δ_{ij} is the Kronecker delta (see A2.11 of Appendix 2). According to (3.34), the n components $\{\rho_j\}$ of the boundary kernel constitute a fundamental set of solutions for the operator $\mathbf{L}$; furthermore, $\{\rho_j\}$ is a fundamental set for which the boundary condition matrix $\mathbf{B}$ of (3.27) is the $n \times n$ identity matrix.

By solving (3.32) and (3.34), we can invert any regular nth-order differential system which has a nonsingular boundary condition matrix. The inverse of the differential system (3.21) (with $m = n$) consists in the sum of the inverses of (3.29) and (3.30), namely,

$$\mathbf{f}(t) = \int_a^b k(t,s)\mathbf{u}(s)\,ds + \sum_{j=1}^{n} \alpha_j \rho_j(t) \tag{3.35}$$

where k and ρ are determined by (3.32) and (3.34), respectively.

Theoretically, we can invert any linear differential operator, ordinary or partial, which has appropriate boundary conditions. That is, we can convert any invertible linear differential equation to an integral equation analogous to (3.35). As a model for a system, the integral equation is more

desirable than the differential equation from two standpoints. First, the boundary conditions are included automatically. Second, integral operators tend to "smooth" functions whereas differential operators introduce discontinuities and delta functions.* It is well known that numerical differentiation amplifies errors in empirical data, but numerical integration does not (Ralston [3.9, p. 791). The rest of this chapter is devoted to techniques for determining the inverse (or integral) model for various types of ordinary differential operators. Techniques and examples which apply to partial differential operators can be found in Friedman [3.4], Stakgold [3.11], Morse and Feshbach [3.8], and Bergman and Schiffer [3.1].

3.3 Inversion of *n*th-Order Differential Systems

In Section 3.1 we determined the Green's function and boundary kernel for a simple second-order system, (3.1). The Green's function and boundary kernel for the general *n*th-order differential systems of Section 3.2 cannot be determined by the direct integration technique used for that simple system, In this section we describe general procedures for solving (3.32) and (3.34) to obtain k and ρ for the nth-order differential system (3.21) with n independent boundary conditions. The procedures are demonstrated in detail for regular second-order variable-coefficient differential systems.

Obtaining a Complementary Function

Most techniques for determining particular solutions to differential systems are based on the complementary function (3.19). Techniques for determining the Green's function k and the boundary kernel ρ also depend heavily on the complementary function (or the equivalent, a fundamental set of solutions). In point of fact, the individual segments or components of k and ρ are of the form of the complementary function.

It is well known that the complementary function for a **constant-coefficient** differential operator consists in sums of exponentials. Let $\mathbf{L}$ of (3.18) be the constant-coefficient operator

$$\mathbf{L} \triangleq \mathbf{D}^n + a_1\mathbf{D}^{n-1} + \cdots + a_n\mathbf{I} \tag{3.36}$$

To find which exponentials are contained in the complementary function for $\mathbf{L}$, we insert a particular exponential $\mathbf{v}(t) = e^{\mu t}$ into the equation $\mathbf{Lf} = \boldsymbol{\theta}$

*Integral operators are continuous, whereas differential operators are not. See the discussion of continuous operators in Section 5.4.

and solve for μ. The result is

$$\mu^n + a_1 \mu^{n-1} + \cdots + a_n = 0 \tag{3.37}$$

This equation, known as the **characteristic equation for L**, has n roots $\mu_1, \mu_2, \ldots, \mu_n$. If the n roots are distinct, the complementary function is

$$\mathbf{f}_c(t) = c_1 \exp(\mu_1 t) + \cdots + c_n \exp(\mu_n t) \tag{3.38}$$

Equation (3.38) can be verified by substituting $\mathbf{f}_c$ into $\mathbf{Lf} = \boldsymbol{\theta}$. If two roots are equal, say, $\mu_1 = \mu_2$, then the corresponding fundamental solutions in (3.38) must be replaced by $c_1 \exp(\mu_1 t) + c_2 t \exp(\mu_1 t)$. This equal root case is discussed further in Section 4.4.

We are unable to deal with the variable-coefficient operator (3.18) with much generality. An approach that can be used to *seek* the complementary function for the variable-coefficient operator is the **power series method** (the method of Frobenius). The method consists in assuming a power series form for the complementary function, substituting the series into the homogeneous differential equation, equating the coefficient on each power of t to zero, and solving for the coefficients of the power series. The sum of the series, where it converges, is at least part of the complementary function. The sum will not, in general, consist of elementary functions. For example, Bessel functions arise as fundamental solutions to Bessel's equation (a second-order variable-coefficient differential equation); the power series method provides an expression for one of the two fundamental solutions to Bessel's equation. In the event that the power series method does not provide a full set of fundamental solutions for the differential equation, other methods must be used to complete the complementary function. See Ince [3.6] or Wiley [3.13, p. 255].

Example 1. Power Series Method—Variable Coefficients Suppose

$$(\mathbf{Lf})(t) \triangleq \mathbf{f}'(t) + t\mathbf{f}(t) \tag{3.39}$$

We find the complementary function for (3.39) by assuming a power series of the general form

$$\mathbf{f}_c(t) = t^a (c_0 + c_1 t + c_2 t^2 + \cdots)$$

where the constant a allows for noninteger powers of t. We first insert $\mathbf{f}_c$ into the homogeneous equation and regroup terms:

$$\begin{aligned} \mathbf{f}'(t) + t\mathbf{f}(t) &= ac_0 t^{a-1} + (a+1)c_1 t^a + [(a+2)c_2 + c_0]t^{a+1} \\ &\quad + [(a+3)c_3 + c_1]t^{a+2} + [(a+4)c_4 + c_2]t^{a+3} + \cdots \\ &= 0 \end{aligned}$$

Equating each coefficient to zero, we obtain

$$\begin{aligned} ac_0 &= 0 \\ (a+1)c_1 &= 0 \\ (a+2)c_2 + c_0 &= 0 \\ (a+3)c_3 + c_1 &= 0 \\ (a+4)c_4 + c_2 &= 0 \\ &\vdots \end{aligned}$$

We assume, without loss of generality, that $c_0 \neq 0$. It follows that $a = 0$ and c_0 is arbitrary; then

$$c_1 = c_3 = c_5 = \cdots = 0,$$

$$c_2 = -\frac{c_0}{2}, \qquad c_4 = \frac{c_0}{4(2)}, \qquad c_6 = -\frac{c_0}{6(4)(2)},$$

and

$$\begin{aligned} \mathbf{f}_c(t) &= c_0\left[1 - \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 - \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \cdots\right] \\ &= c_0 \exp\left(\frac{-t^2}{2}\right) \end{aligned}$$

Determination of the Green's Function and Boundary Kernel—An Example

We solved for the kernel functions k and ρ associated with (3.1) by direct integration of the differential equation. Unfortunately, that simple approach does not apply to most differential equations. In the following example we introduce a general technique for finding k and ρ.

The model for a particular armature-controlled dc motor and load is the differential equation

$$\ddot{\phi}(t) + \dot{\phi}(t) = \mathbf{u}(t) \tag{3.40}$$

where $\mathbf{u}(t)$ is the armature voltage at time t and $\phi(t)$ is the angular position of the motor shaft relative to some reference position. Let the boundary conditions be

$$\phi(0) = \alpha_1 \quad \text{and} \quad \phi(b) = \alpha_2 \tag{3.41}$$

That is, we seek the "trajectory" (or angular position versus time), of the

shaft in order that it be in position α_1 at time 0 and pass through position α_2 at time b. Comparing this problem to that of (3.21), we note that $\mathbf{L}=\mathbf{D}^2+\mathbf{D}$, $\boldsymbol{\beta}_1(\boldsymbol{\phi})=\boldsymbol{\phi}(0)$, and $\boldsymbol{\beta}_2(\boldsymbol{\phi})=\boldsymbol{\phi}(b)$. The symbol $\boldsymbol{\phi}$ replaces the symbol $\mathbf{f}$ used earlier.

Finding the Green's function for the differential system (3.40)-(3.41) is equivalent to exploring the trajectory $\boldsymbol{\phi}$ of the motor shaft for all possible applied voltages $\mathbf{u}(t)$, but for $\alpha_1 = \alpha_2 = 0$. The Green's function must satisfy (3.32):

$$\frac{d^2k(t,s)}{dt^2}+\frac{dk(t,s)}{dt}=\delta(t-s)$$

$$k(0,s)=k(b,s)=0$$

Clearly $k(t,s)$ satisfies the homogeneous differential equation in each of the regions $[0,s)$ and $(s,b]$; that is, in the regions where $\delta(t-s)$ is zero. We let $k(t,s)=\mathbf{f}_c(t)$ for each of the two regions $[0,s)$ and $(s,b]$:

$$k(t,s)=c_1+c_2e^{-t}, \qquad t \text{ in } [0,s)$$

$$=d_1+d_2e^{-t}, \qquad t \text{ in } (s,b]$$

Since $k(t,s)$ is a function of s, the arbitrary constants must depend on s. We eliminate half of the arbitrary constants by applying the boundary conditions

$$k(0,s)=c_1+c_2=0 \qquad \Rightarrow c_2=-c_1$$

$$k(b,s)=d_1+d_2e^{-b}=0 \Rightarrow d_2=-e^bd_1$$

It is the second (or highest) derivative of k that introduces the delta function in (3.32); for if the first derivative included a delta function, the second derivative would introduce the derivative of the delta function.* Since d^2k/dt^2 includes a unit impulse at $t = s$, dk/dt must include a unit step at $t = s$, and k itself must be continuous at $t = s$. We express these facts by the two "discontinuity" conditions:

$$k(s^+,s)=k(s^-,s) \qquad \text{(continuity of } k \text{ at } t=s)$$

$$\frac{dk(s^+,s)}{dt}-\frac{dk(s^-,s)}{dt}=1 \qquad \left(\text{unit step in } \frac{dk}{dt} \text{ at } t=s\right)$$

*See Appendix 2 for a discussion of unit steps, delta functions, and derivatives of delta functions.

Applying these conditions to $k(t,s)$, we find

$$d_1 + d_2 e^{-s} = c_1 + c_2 e^{-s}$$

$$-d_2 e^{-s} - (-c_2 e^{-s}) = 1$$

A messy elimination procedure among the boundary condition equations and discontinuity condition equations yields

$$c_1(s) = \frac{e^s - e^b}{e^b - 1} \text{ and } d_1(s) = \frac{e^s - 1}{e^b - 1}$$

It follows that

$$k(t,s) = \frac{(1-e^{-t})(e^s - e^b)}{e^b - 1} \qquad t \leqslant s$$

$$= \frac{(1-e^b e^{-t})(e^s - 1)}{e^b - 1} \qquad t \geqslant s \tag{3.42}$$

To get a feel for the nature of this system (for which $\phi(0) = \phi(b) = 0$), we use k to determine the shaft trajectory ϕ and velocity profile $\dot{\phi}$ for a specific input $\mathbf{u}(t) = 1$:

$$\phi(t) = \int_0^b k(t,s)\mathbf{u}(s)\,ds$$

$$= \frac{1-e^b e^{-t}}{e^b - 1}\int_0^t (e^s - 1)\,ds + \frac{1-e^{-t}}{e^b - 1}\int_t^b (e^s - e^b)\,ds$$

$$= t - \left(\frac{be^b}{e^b - 1}\right)(1 - e^{-t})$$

$$\dot{\phi}(t) = 1 - \left(\frac{be^b}{e^b - 1}\right)e^{-t}$$

The trajectory ϕ and the velocity profile $\dot{\phi}$ are plotted in Figure 3.2 for $b = 1$. Observe that, in general, the motor shaft cannot be at rest at $t = 0$ and at $t = b$ if the shaft positions are specified; it is precisely the freedom in the initial and terminal velocities which allows us to choose both the end points, $\phi(0)$ and $\phi(b)$, and an arbitrary continuous input voltage $\mathbf{u}$.

The boundary kernel ρ for the system (3.40)-(3.41) describes the trajectory $\phi(t)$ as a function of the boundary conditions $\phi(0) = \alpha_1$ and

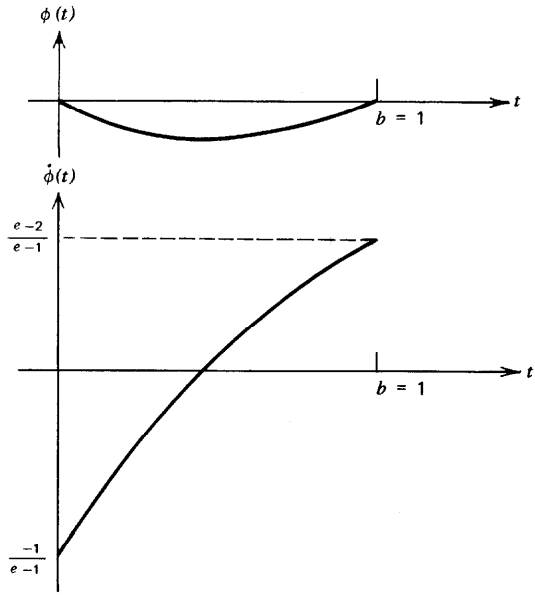

Figure 3.2. Shaft position and velocity for $\phi(0) = \phi(1) = 0$ and $u(t) = 1$.

$\phi(b) = \alpha_2$ with no voltage applied to the motor; that is,

$$\phi(t) = \rho_1(t)\alpha_1 + \rho_2(t)\alpha_2$$

Perhaps the most direct approach to the determination of $\rho_j(t)$ is to let $\phi(t) = c_1 v_1(t) + c_2 v_2(t)$, a linear combination of the fundamental solutions for (3.40), then apply the boundary conditions (3.41) to obtain the coefficients c_i as a function of α_1 and α_2. Rather than use this approach, we attack the defining equations for $\rho_j(t)$ in a more formal manner which parallels the determination of the Green's function. The two approaches are equivalent in the amount of computation they require. The boundary kernel satisfies (3.34):

$$\begin{aligned} &\ddot{\rho}_1(t) + \dot{\rho}_1(t) = 0 \qquad \ddot{\rho}_2(t) + \dot{\rho}_2(t) = 0 \\ &\beta_1(\rho_1) = \rho_1(0) = 1 \quad \beta_1(\rho_2) = \rho_2(0) = 0 \\ &\beta_2(\rho_1) = \rho_1(b) = 0 \quad \beta_2(\rho_2) = \rho_2(b) = 1 \end{aligned}$$

The boundary condition statements are reminiscent of the boundary condition matrix (3.27). In point of fact, ρ_1 and ρ_2 each consist in a linear combination of the fundamental solutions $v_1(t) = 1$ and $v_2(t) = e^{-t}$. Apply-

ing the boundary conditions to $\rho_1(t) = c_1 + c_2 e^{-t}$, we get

$$\beta_1(\rho_1) = \rho_1(0) = c_1 + c_2 e^{-0} = 1$$

$$\beta_2(\rho_1) = \rho_1(b) = c_1 + c_2 e^{-b} = 0$$

or

$$\mathbf{B}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & e^{-b} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $\mathbf{B}$ is, indeed, the boundary condition matrix of (3.27). Similarly, using $\rho_2(t) = d_1 + d_2 e^{-t}$, we find

$$\mathbf{B}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can combine the two coefficient equations into the single matrix equation

$$\mathbf{B}\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which has the solution

$$\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \mathbf{B}^{-1} = \left(\frac{1}{e^b - 1}\right)\begin{pmatrix} -1 & e^b \\ e^b & -e^b \end{pmatrix}$$

The function ρ_j is a specific linear combination of the two fundamental solutions specified above; the jth column of $\mathbf{B}^{-1}$ specifies the linear combination. Thus

$$\rho_1(t) = \frac{-1}{e^b - 1} + \frac{e^b}{e^b - 1} e^{-t}$$

$$\rho_2(t) = \frac{e^b}{e^b - 1} + \frac{-e^b}{e^b - 1} e^{-t} \tag{3.43}$$

The shaft position and velocity, as functions of the boundary conditions, are

$$\phi(t) = \frac{e^b e^{-t} - 1}{e^b - 1}\alpha_1 + \frac{e^b(1 - e^{-t})}{e^b - 1}\alpha_2$$

$$\dot{\phi}(t) = \frac{e^b e^{-t}}{e^b - 1}(\alpha_2 - \alpha_1)$$

Figure 3.3 shows the position and velocity of the motor shaft for $\alpha_1 = 0$ and $\alpha_2 = 1$. The shaft is already in motion at $t = 0$, and exhibits an "undriven" decay in velocity until it reaches the position $\phi(b) = 1$ rad. If the boundary conditions were $\alpha_1 = \alpha_2 = 1$, the shaft would sit at rest in the position $\phi(t) = 1$ rad; again an undriven trajectory.

The inverse of the system (3.40)-(3.41) is the sum of the separate solutions for the distributed and boundary inputs. That is,

$$\mathbf{f}(t) = \int_0^b k(t,s)\mathbf{u}(s)\,ds + \rho_1(t)\alpha_1 + \rho_2(t)\alpha_2$$

where k and ρ are given in (3.42) and (3.43), respectively. The nature of the system (3.40)-(3.41) does not seem in keeping with the nature of dynamic [real-time) systems. The motor must anticipate the input $\mathbf{u}(t)$ (or the Impulse $\delta(t-s)$) and appropriately select its velocity at $t = 0$ in order to be able to meet the requirement on its position at $t = b$. We are more likely to meet such a two-point boundary value problem when the independent variable t represents not time, but rather a space variable. Yet a two-point

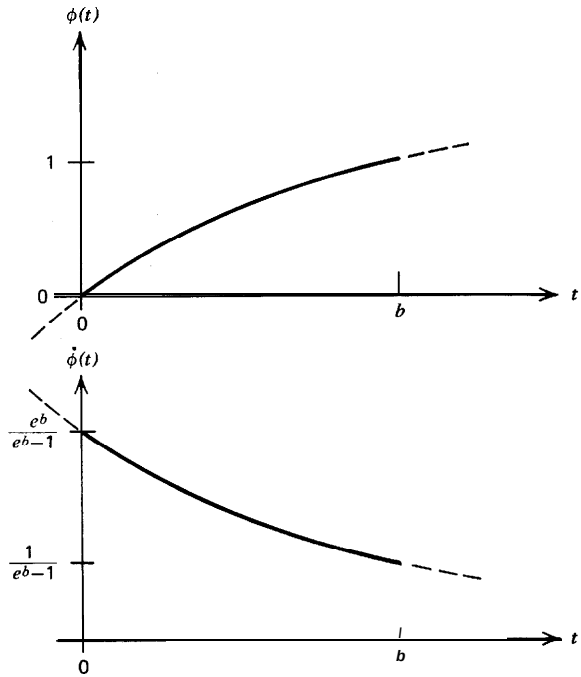

Figure 3.3. Undriven shaft position and velocity for $\alpha_1 = 0$ and $\alpha_2 = 1$.

boundary value problem can arise in a dynamic system if we impose requirements on the future behavior of the system as we did in (3.41).

Summary of the Technique

The technique demonstrated above for determining the Green's function and the boundary kernel depends upon knowledge of the complementary function. We can apply the technique to the regular *n*th-order system (3.21) if the corresponding complementary function can be determined. Assume $\mathbf{L}$ of (3.21) has the complementary function $\mathbf{f} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$. Further assume that the system is invertible (i.e., we have n independent boundary conditions for which (3.28) is satisfied). We obtain the Green's function k and the boundary kernel ρ for the system (3.21) by following the technique used for the system (3.40)-(3.41).

Equation (3.32) determines the Green's function k. The unit impulse $\delta(t-s)$ is zero for all $t \neq s$. Therefore, $k(t,s)$ satisfies the homogeneous differential equation for $t \neq s$; $k(t,s)$ is equal to the complementary function (3.19) in each of the two regions $[a,s)$ and $(s,b]$. Because the complementary function $\mathbf{f}_c$ is used in two separate regions, we must determine two sets of n arbitrary constants:

$$\begin{aligned} k(t,s) &= b_1\mathbf{v}_1(t) + \cdots + b_n\mathbf{v}_n(t), \quad t \text{ in } [a,s) \\ &= d_1\mathbf{v}_1(t) + \cdots + d_n\mathbf{v}_n(t), \quad t \text{ in } (s,b] \end{aligned} \tag{3.44}$$

Half of the $2n$ constants can be eliminated by the *homogeneous* boundary conditions of (3.21): $\beta_i k(t,s) = 0$, $i = 1, \ldots, n$. The rest are determined by appropriate "discontinuity" conditions at $t = s$. Only the highest derivative term, $g_0(t)d^n k(t,s)/dt^n$, can introduce the delta function into (3.32) (otherwise derivatives of delta functions would appear); therefore, we match the two halves of $k(t,s)$ at $t = s$ in such a way that we satisfy the following n conditions:

$$\begin{aligned} &k, \frac{dk}{dt}, \ldots, \frac{d^{n-2}k}{dt^{n-2}} \quad \text{are continuous at } t = s \\ &\frac{d^{n-1}k(s^+,s)}{dt^{n-1}} - \frac{d^{n-1}k(s^-,s)}{dt^{n-1}} = \frac{1}{g_0(s)} \end{aligned} \tag{3.45}$$

That is, $d^{n-1}k(t,s)/dt^{n-1}$ must contain a step of size $1/g_0(s)$ at $t = s$. Then $g_0(t)d^n k(t,s)/dt^n$ will include the term $\delta(t-s)$.*

*See Appendix 2 for a discussion of steps, delta functions, and derivatives of delta functions.

The boundary kernel $\boldsymbol{\rho}$ is specified by (3.34). Each component of $\boldsymbol{\rho}$ is a linear combination of the fundamental solutions for $\mathbf{L}$:

$$\rho_j = c_{1j}\mathbf{v}_1 + \cdots + c_{nj}\mathbf{v}_n \qquad j = 1, \ldots, n \tag{3.46}$$

Applying the n boundary conditions of (3.21) as required by (3.34), we find

$$\begin{pmatrix} \beta_1(\rho_j) \\ \vdots \\ \beta_n(\rho_j) \end{pmatrix} = \begin{pmatrix} \beta_1(\mathbf{v}_1) \cdots \beta_1(\mathbf{v}_n) \\ \vdots \quad\quad \vdots \\ \beta_n(\mathbf{v}_1) \cdots \beta_n(\mathbf{v}_n) \end{pmatrix} \begin{pmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1_j \\ \vdots \\ 0 \end{pmatrix}, \qquad j = 1, \ldots, n$$

These n sets of equations can be expressed as

$$\begin{pmatrix} \beta_1(\mathbf{v}_1) \cdots \beta_1(\mathbf{v}_n) \\ \vdots \quad\quad \vdots \\ \beta_n(\mathbf{v}_1) \cdots \beta_n(\mathbf{v}_n) \end{pmatrix} \begin{pmatrix} c_{11} \cdots c_{1n} \\ \vdots \quad\quad \vdots \\ c_{n1} \cdots c_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3.47}$$

It follows that the coefficients for $\boldsymbol{\rho}_j$ in (3.46) are the elements in the jth column of $\mathbf{B}^{-1}$, where $\mathbf{B}$ is the boundary condition matrix defined in (3.27). Specifically, c_{ij} is the element in row i and column j of $\mathbf{B}^{-1}$.

Exercise 1. Let $\mathbf{f}'(t) + t\mathbf{f}(t) = \mathbf{u}(t)$ with $\mathbf{f}(0) = \alpha_1$. (The complementary function for this differential equation was determined in Example 1.) Show that the inverse of this differential system is

$$\mathbf{f}(t) = \exp(-t^2/2) \int_0^t \exp(s^2/2)\mathbf{u}(s)\, ds + \alpha_1 \exp(-t^2/2) \tag{3.48}$$

Second-Order Differential Systems

Many of the ordinary and partial differential equations that arise in the modeling of physical systems are second order. Some of the second-order partial differential equations can be reduced, by a substitution of variables or by integral transforms, to second-order ordinary differential equations.* Furthermore, use of the "separation of variables" technique in solving second-order partial differential equations produces sets of second-order

*See Kaplan [3.7].

ordinary differential equations. Thus the general second-order ordinary differential equation with variable coefficients is of considerable practical importance. We present explicit expressions for the Green's function and the boundary kernel for an arbitrary regular second-order differential system; these expressions are obtained in terms of a fundamental set of solutions for the differential operator.

The regular second-order differential system is†

$$\begin{gathered} g_0(t)\mathbf{f}''(t) + g_1(t)\mathbf{f}'(t) + g_2(t)\mathbf{f}(t) = \mathbf{u}(t) \\ \boldsymbol{\beta}_1(\mathbf{f}) = \alpha_1 \quad \text{and} \quad \boldsymbol{\beta}_2(\mathbf{f}) = \alpha_2 \end{gathered} \tag{3.49}$$

where g_i is continuous and $g_0(t) \neq 0$ in the region of interest. Assume $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent solutions to the homogeneous differential equation. By (3.44), the Green's function is of the form

$$\begin{aligned} k(t,s) &= b_1\mathbf{v}_1(t) + b_2\mathbf{v}_2(t), \quad t < s \\ &= d_1\mathbf{v}_1(t) + d_2\mathbf{v}_2(t), \quad t > s \end{aligned}$$

The discontinuity conditions (3.45) become

$$d_1\mathbf{v}_1(s) + d_2\mathbf{v}_2(s) = b_1\mathbf{v}_1(s) + b_2\mathbf{v}_2(s) \text{ (continuity of } k)$$

$$d_1\mathbf{v}_1'(s) + d_2\mathbf{v}_2'(s) - b_1\mathbf{v}_1'(s) - b_2\mathbf{v}_2'(s) = \frac{1}{g_0(s)} \left(\text{step of size } \frac{1}{g_0(s)} \text{ in } \frac{dk}{dt} \right)$$

Since dk/dt has a step of size $1/g_0(s)$, then $g_0(t)d^2k/dt^2$ includes a *unit* impulse. These two discontinuity equations can be put in the matrix form

$$\begin{pmatrix} \mathbf{v}_1(s) & \mathbf{v}_2(s) \\ \mathbf{v}_1'(s) & \mathbf{v}_2'(s) \end{pmatrix} \begin{pmatrix} d_1 - b_1 \\ d_2 - b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/g_0(s) \end{pmatrix}$$

The solution is

$$d_1 - b_1 = -\frac{\mathbf{v}_2(s)}{w(s)\,g_0(s)}, \qquad d_2 - b_2 = \frac{\mathbf{v}_1(s)}{w(s)\,g_0(s)}$$

where $w(s)$ is the *Wronskian determinant*:*

$$w(s) \triangleq \begin{vmatrix} \mathbf{v}_1(s) & \mathbf{v}_2(s) \\ \mathbf{v}_1'(s) & \mathbf{v}_2'(s) \end{vmatrix} \tag{3.50}$$

† In Section 5.5 we refer to the differential operator of (3.49) as a regular Sturm-Liouville operator.

*Note that the solution is undefined for $w(s) = 0$. It can be shown that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent solutions to the homogeneous differential equation, then $w(s) \neq 0$ for all s in the interval of interest. See P&C 3.7.

The boundary conditions $\beta_1 k(t, s) = \beta_2 k(t, s) = 0$ provide two more linear algebraic equations which, together with the above pair of equations, determine the constants b_1, b_2, d_1, and d_2, and therefore, $k(t,s)$. However, without specific information about the nature of the boundary conditions, we can carry the solution no further. The solution for a dynamic system (initial conditions) is given in Exercise 2. Two-point boundary conditions are treated in Exercise 3.

Exercise 2. Let the boundary conditions of (3.49) be

$$\beta_1(\mathbf{f}) \triangleq \mathbf{f}(a) \quad \text{and} \quad \beta_2(\mathbf{f}) \triangleq \mathbf{f}'(a) \tag{3.5 1}$$

Show that the corresponding Green's function is

$$\begin{aligned} k(t,s) &= 0, && t \text{ in } [a,s) \\ &= \frac{\Delta(s,t)}{g_0(s)w(s)}, && t \text{ in } (s,\infty) \end{aligned} \tag{3.52}$$

where w is given by (3.50), and

$$\Delta(s,t) \triangleq \begin{vmatrix} \mathbf{v}_1(s) & \mathbf{v}_2(s) \\ \mathbf{v}_1(t) & \mathbf{v}_2(t) \end{vmatrix}$$

Show also that the corresponding boundary kernel is

$$\begin{aligned} \rho_1(t) &= \frac{\mathbf{v}_2'(a)\mathbf{v}_1(t) - \mathbf{v}_1'(a)\mathbf{v}_2(t)}{w(a)} \\ \rho_2(t) &= \frac{\mathbf{v}_1(a)\mathbf{v}_2(t) - \mathbf{v}_2(a)\mathbf{v}_1(t)}{w(a)} \end{aligned} \tag{3.53}$$

Exercise 3. Let the boundary conditions of (3.49) be $\beta_1(\mathbf{f}) \triangleq \mathbf{f}(a)$ and $\beta_2(\mathbf{f}) \triangleq \mathbf{f}(b)$. Show that for this two-point boundary value problem

$$\begin{aligned} k(t,s) &= \frac{1}{g_0(s)w(s)} \begin{cases} \dfrac{\Delta(b,s)\Delta(a,t)}{\Delta(a,b)}, & a \leqslant t \leqslant s \\ \dfrac{\Delta(b,s)\Delta(a,t)}{\Delta(a,b)} + \Delta(s,t), & s \leqslant t \leqslant b \end{cases} \\ \rho_1(t) &= -\frac{\Delta(b,t)}{\Delta(a,b)} \quad \text{and} \quad \rho_2(t) = \frac{\Delta(a,t)}{\Delta(a,b)} \end{aligned} \tag{3.54}$$

where $\Delta(s, t)$ is given beneath (3.52) and $w(s)$ is defined in (3.50).

Exercise 4. Use (3.54) to find k and ρ for the dc motor system (3.40)-(3.41). Compare the result with (3.42) and (3.43).

It is apparent that we could derive an explicit expression for the inverse of a regular nth-order linear differential system [assuming the boundary conditions satisfy the invertibility condition (3.28)]. The inverse would involve n independent fundamental solutions and the nth-order Wronskian determinant of these n solutions. Of course, as indicated by Exercise 3, the manipulation can be complicated. The determination of the Green's function for an nth-order two-point boundary value problem requires the solution of $2n$ simultaneous algebraic equations with coefficients which are functions of s. In contrast, the Green's function for the initial condition problem (or one-point boundary value problem) requires the solution of only n simultaneous equations because $k(t,s) = 0$ for $t < s$. Of particular interest is the constant-coefficient initial condition problem, for which determination of the Green's function reduces to inversion of an $n \times n$ matrix of constants.

3.4 Time-Invariant Dynamic Systems

The initial value problem is at the heart of dynamic systems-systems for which the variable t represents time. The linear time-invariant (or constant-coefficient) dynamic system merits special attention if only because its inversion is easily automated using standard computer programs for solving matrix equations. Furthermore, many dynamic systems are adequately represented as linear time-invariant systems. We examine these systems in detail in this section.

The Inverse of the nth-Order System

The general nth-order constant-coefficient differential equation with initial conditions is

$$\begin{aligned} &\mathbf{f}^{(n)}(t) + a_1\mathbf{f}^{(n-1)}(t) + \cdots + a_n\mathbf{f}(t) = \mathbf{u}(t) \\ &\beta_i(\mathbf{f}) \triangleq \mathbf{f}^{(i-1)}(0) = \alpha_i \qquad i = 1,\ldots,n \end{aligned} \tag{3.55}$$

for real scalars $\{a_i\}$ and $t \geqslant 0$. The characteristic equation for (3.55) is (3.37); assume it has n distinct roots $\mu_1, \ldots, \mu_n$ (the multiple root case is considered in Section 4.4). Then the fundamental solutions for (3.55) are $\mathbf{v}_i(t) \triangleq \exp(\mu_i t)$, $i = 1,\ldots, n$.

The Green's function, as given by (3.44), is

$$k(t,s) = b_1 \exp(\mu_1 t) + \cdots + b_n \exp(\mu_n t), \quad t \text{ in } [0,s)$$
$$= d_1 \exp(\mu_1 t) + \cdots + d_n \exp(\mu_n t), \quad t \text{ in } (s,\infty)$$

All n boundary conditions apply to the first half of $k(t, s)$, the half involving the unknowns $b_1, \ldots, b_n$. As a result,

$$\begin{pmatrix} k(0,s) \\ \frac{dk}{dt}(0,s) \\ \vdots \\ \frac{d^{n-1}k(0,s)}{dt^{n-1}} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_n \\ \vdots & & \vdots \\ \mu_1^{n-1} & \cdots & \mu_n^{n-1} \end{pmatrix} \begin{pmatrix} b_1 \\ \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \vdots \\ 0 \end{pmatrix}$$

This boundary condition matrix is the **Wronskian matrix** of the functions $\{\exp(\mu_i t)\}$ at $t = 0$. The matrix is also known as the **Vandermond matrix** for the system (3.55). It is invertible if and only if the roots $\mu_1, \ldots, \mu_n$ are distinct as assumed.* Therefore, $b_1 = \ldots = b_n = 0$, and $k(t,s) = 0$ for t in $[0,s)$. The discontinuity conditions (3.45) at $t = s$ are

$$d_1 \exp(\mu_1 s) + \ldots + d_n \exp(\mu_n s) = 0 \qquad (k \text{ continuous})$$
$$d_1 \mu_1 \exp(\mu_1 s) + \cdots + d_n \mu_n \exp(\mu_n s) = 0 \qquad (dk/dt \text{ continuous})$$
$$d_1 \mu_1^{n-2} \exp(\mu_1 s) + \ldots + d_n \mu_n^{n-2} \exp(\mu_n s) = 0 \qquad (d^{n-2}k/dt^{n-2} \text{ continuous})$$
$$d_1 \mu_1^{n-1} \exp(\mu_1 s) + \ldots + d_n \mu_n^{n-1} \exp(\mu_n s) = 1 \qquad (\text{unit step in } d^{n-1}k/dt^{n-1})$$

We substitute the new variables $\hat{d}_i \triangleq d_i \exp(\mu_i s)$, $i = 1, \ldots, n$ into the discontinuity equations to obtain

$$\begin{pmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_n \\ \vdots & & \vdots \\ \mu_1^{n-1} & \cdots & \mu_n^{n-1} \end{pmatrix} \begin{pmatrix} \hat{d}_1 \\ \vdots \\ \hat{d}_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{3.56}$$

*If the roots were not distinct, we would use a different set of fundamental solutions $\{\mathbf{v}_i\}$, and obtain a different boundary condition matrix. The Wronskian matrix is explored in P&C 3.7. The Vandermond matrix is examined in P&C 4.16.

Because the roots $\{\mu_i\}$ are distinct, the Vandermond matrix is invertible, and (3.56) can be solved by means of a standard computer program to obtain $\{d_i\}$. Notice that the new variables $\{d_i\}$ are independent of s. The s dependence of the variables $\{d_i\}$ has been removed by the substitution. In terms of the new variables, the Green's function becomes

$$\begin{aligned} k(t,s) &= 0 && \text{for } 0 \leqslant t \leqslant s \\ &= \hat{d}_1 \exp[\mu_1(t-s)] + \cdots + \hat{d}_n \exp[\mu_n(t-s)] && \text{for } t \geqslant s \end{aligned} \tag{3.57}$$

The boundary kernel for the system (3.55) is found from (3.46) and (3.47). Equation (3.47) is

$$\begin{pmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_n \\ \vdots & & \vdots \\ \mu_1^{n-1} & \cdots & \mu_n^{n-1} \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} = \mathbf{I}$$

Then, by (3.46),

$$\rho_j(t) = c_{1j} \exp(\mu_1 t) + \cdots + c_{nj} \exp(\mu_n t), \quad j = 1, \ldots, n \tag{3.58}$$

where the coefficients for ρ_j are obtained from the jth column of the inverse Vandermond matrix:

$$\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_n \\ \vdots & & \vdots \\ \mu_1^{n-1} & \cdots & \mu_n^{n-1} \end{pmatrix}^{-1} \tag{3.59}$$

The inverse of the differential equation and boundary conditions of (3.55) is

$$\begin{aligned} \mathbf{f}(t) &= \int_0^\infty k(t,s)\mathbf{u}(s)\,ds + \sum_{j=1}^{n} \rho_j(t)\alpha_j \\ &= \int_0^t \{\hat{d}_1 \exp[\mu_1(t-s)] + \cdots + \hat{d}_n \exp[\mu_n(t-s)]\}\mathbf{u}(s)\,ds \\ &\quad + \sum_{j=1}^{n} [c_{1j} \exp(\mu_1 t) + \cdots + c_{nj} \exp(\mu_n t)]\mathbf{f}^{(j-1)}(0) \end{aligned} \tag{3.60}$$

where $\{\hat{d}_i\}$ and $\{c_{ij}\}$ are specified by (3.56) and (3.59), respectively. The computer program which produces (3.59) will simultaneously solve (3.56). Section 4.4 explores the computational difficulties which arise when the characteristic equation of the system has nearly equal roots.

The shape of the Green's function for a time-invariant (i.e., constant-coefficient) dynamic system depends only on $t-s$, the delay between the time s that an impulse is applied at the system input and the time t that the output $k(t,s)$ is observed. That is, $k(t,s)=k(t-s,0)$. Therefore, actual measurement of the response of the physical system to an approximate impulse is a suitable method for determining the Green's function. The response of such a system, initially at rest, to an impulse input $\mathbf{u}(t)=\delta(t)$ is commonly referred to as the **impulse response** of the system. We denote the impulse response by g, where $g(t) \triangleq k(t,0)$. Then the integral term in (3.60) can be rewritten as a convolution of $\mathbf{u}$ and g.*

$$\int_0^t k(t,s)\mathbf{u}(s)\,ds=\int_0^t g(t-s)\mathbf{u}(s)\,ds$$

The components of the boundary kernel also can be measured physically; $\rho_j(t)$ is the response of the system with no distributed input $\mathbf{u}$, and with the initial conditions $\alpha_j = 1$, $\alpha_i = 0$, for $i \neq j$. Furthermore, we see from (3.56)-(3.59) that $\hat{d}_i = c_{in}$ for $i=1,\ldots,n$. Therefore, the impulse response is equal to one of the initial condition responses; specifically,

$$\rho_n(t)=k(t,0)=g(t) \tag{3.61}$$

Applying a unit impulse $\delta(t)$ is equivalent to instantaneously applying to the system (at rest) the unit initial condition $\mathbf{f}^{(n-1)}(0) = 1$ (all other initial conditions remaining zero); if we can apply this initial condition some other way, we do not need an approximate impulse in order to measure the impulse response of the system.

Exercise 1. The differential equation (3.40) for an armature-controlled dc motor is

$$\ddot{\phi}(t)+\dot{\phi}(t)=\mathbf{u}(t)$$

Show that for given initial conditions, $\phi(0)$ and $\dot{\phi}(0)$, the Green's function,

*See Appendix 2 for a discussion of convolution.

boundary kernel, and inverse equation are

$$\begin{aligned} k(t,s) &= 0, && 0 \leqslant t \leqslant s \\ &= 1 - e^{-(t-s)}, && t \geqslant s \end{aligned}$$

$$\rho_1(t) = 1$$

$$\rho_2(t) = 1 - e^{-t} \tag{3.62}$$

$$\phi(t) = \int_0^t [1 - e^{-(t-s)}]\mathbf{u}(s)\,ds + \phi(0) + (1 - e^{-t})\dot{\phi}(0)$$

Compare (3.62) with (3.52) and (3.53).

The State-Space Model

The *n*th-order constant-coefficient differential equation with initial conditions, (3.59, can be expressed as a first-order vector differential equation by redefining the variables. If $\mathbf{u}(t) = 0$, the quantities $\mathbf{f}(0), \mathbf{f}^{(1)}(0), \ldots, \mathbf{f}^{(n-1)}(0)$ determine the trajectory $\mathbf{f}(t)$ for all t; these n quantities together form a more complete description of the state (or condition) of the system at $t = 0$ than does $\mathbf{f}(0)$ alone. Let $\mathbf{f}_1 \triangleq \mathrm{f}$, $\mathbf{f}_2 \triangleq \mathbf{f}^{(1)}, \ldots, \mathbf{f}_n \triangleq \mathbf{f}^{(n-1)}$. Then (3.55) can be expressed as the following set of n first-order differential equations.

$$\begin{aligned} \dot{\mathbf{f}}_1(t) &= \mathbf{f}_2(t) \\ \dot{\mathbf{f}}_2(t) &= \mathbf{f}_3(t) \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{f}}_{n-1}(t) &= \mathbf{f}_n(t) \\ \dot{\mathbf{f}}_n(t) &= -a_n\mathbf{f}_1(t) - \cdots - a_1\mathbf{f}_n(t) + \mathbf{u}(t) \end{aligned}$$

By defining $\mathbf{x} \triangleq (\mathbf{f}_1 \ldots \mathbf{f}_n)^{\mathrm{T}}$ and $\dot{\mathbf{x}} \triangleq (\dot{\mathbf{f}}_1 \cdots \dot{\mathbf{f}}_n)^{\mathrm{T}}$, we write the n individual equations as

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \mathbf{u}(t) \tag{3.63}$$

The square matrix of (3.63) is known as the **companion matrix** for the nth

order differential operator (3.55). The initial conditions of (3.55) become

$$\mathbf{x}(0)=\begin{pmatrix}\mathbf{f}(0)\\ \mathbf{f}^{(1)}(0)\\ \vdots\\ \mathbf{f}^{(n-1)}(0)\end{pmatrix}=\begin{pmatrix}\alpha_1\\ \\ \vdots\\ \alpha_n\end{pmatrix} \tag{3.64}$$

We call $\mathbf{x}(t)$ the *state vector* of the system at time t. Since the differential system (3.55) has a unique solution, the state at time t can be determined from the state at any time previous to t. The state provides precisely enough information concerning the condition of the system to determine the future behavior of the system for a given input. The vector $\mathbf{x}(t)$ is in $\mathcal{R}^{n \times 1}$. Therefore, we call $\mathcal{R}^{n \times 1}$ the *state space* of the system. The variables $\{\mathbf{f}_i(t)\}$ are known as *state variables.*

Equations (3.63) and (3.64) are of the general form

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) \text{ given} \tag{3.65}$$

However, the notation of (3.65) is more general than that of (3.63) and (3.64). The input $\mathbf{u}$ can include more than one function. A meaningful equation is defined by any $n \times n$ matrix $\mathbf{A}$ and $n \times m$ matrix $\mathbf{B}$; the resulting vector equation describes the evolution in time of a system with m inputs and n outputs. A general set of coupled linear time-invariant differential equations can be expressed in this state-space form (P&C 3.18). We refer to (3.65) as a **state equation.** We call $\mathbf{x}(t)$ the **state vector** and its elements the **state variables; A** and **B** are the **system matrix** and the **input matrix,** respectively.*

We should note that the description of a dynamic system by a state-space model is not unique. If we multiply both sides of (3.65) by an arbitrary invertible $n \times n$ matrix $\mathbf{S}$, we obtain

$$\mathbf{S}\dot{\mathbf{x}}(t)=\mathbf{S}\mathbf{A}\mathbf{x}(t)+\mathbf{S}\mathbf{B}\mathbf{u}(t)$$

Defining $\mathbf{y} = \mathbf{S}\mathbf{x}$, we find

$$\begin{aligned}\dot{\mathbf{y}}(t)&=\mathbf{S}\mathbf{A}\mathbf{S}^{-1}\mathbf{y}(t)+\mathbf{S}\mathbf{B}\mathbf{u}(t)\\ &=\hat{\mathbf{A}}\mathbf{y}(t)+\hat{\mathbf{B}}\mathbf{u}(t)\end{aligned}$$

*See Zadeh and Desoer [3.14] or DeRusso, Roy, and Close [3.2] for a more complete discussion of state-space models.

with $\mathbf{y}(0) = \mathbf{S}\mathbf{x}(0)$ given. This second state-space differential equation is equivalent to (3.65) as a representative of the system. The state vector $\mathbf{y}(t)$ is a representation of x(t) in new coordinates. Thus the state variables and system matrix which describe a given system are not unique. In Section 4.2 we explore the essential characteristics of a matrix, its eigenvalues. We find that the similarity transformation $\mathbf{SAS}^{-1}$ does not affect the eigenvalues. Consequently, all system matrices which represent the same system have the same essential characteristics. State space models of dynamic systems are analyzed in terms of their eigenvalues in Sections 4.3 and 4.5.

Example 1. A State Equation. The differential equation for the armature-controlled dc motor of Exercise 1 is

$$\ddot{\phi}(t) + \dot{\phi}(t) = \mathbf{u}(t), \qquad \phi(0) = \alpha_1, \qquad \dot{\phi}(0) = \alpha_2$$

Defining the state variables $\mathbf{f}_1(t) \stackrel{\Delta}{=} \phi(t)$ and $\mathbf{f}_2(t) \stackrel{\Delta}{=} \dot{\phi}(t)$, we obtain the following state equation

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\mathbf{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbf{u}(t), \qquad \mathbf{x}(0) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

The system matrix is the companion matrix for the second-order differential equation.

Let us find an integral equation which is the inverse (or explicit solution) of the first-order vector-valued differential system (3.65). Although we work directly with the system in the specific form (3.65), we note that the equation can be expressed in terms of a general differential operator $\mathbf{L}$ acting on a vector-valued function space. Let $\mathbf{f}$ be in $\mathcal{C}^n(0, \infty)$; then $\mathbf{f}^{(k)}$ is in $\mathcal{C}^{n-k}(0, \infty)$ and $\mathbf{x}$ is in the Cartesian product space:

$$\mathcal{V} = \mathcal{C}^n(0, \infty) \times \mathcal{C}^{n-1}(0, 00) \times \cdots \times \mathcal{C}^1(0, \infty)$$

The system (3.65) is equivalent to the following operator equation on $\mathcal{V}$:

$$\mathbf{L}\mathbf{x} \stackrel{\Delta}{=} \dot{\mathbf{x}} - \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{u} \quad \text{with } \mathbf{x}(0) \text{ given} \tag{3.66}$$

We express $\mathbf{x}$ as an integral operation on the whole vector-valued function $\mathbf{Bu}$.

Inversion of the State Equation

The state equation for an *n*th-order time-invariant dynamic system is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.67}$$

where $\mathbf{A}$ and $\mathbf{B}$ are arbitrary $n \times n$ and $n \times m$ matrices, respectively. The state vector $\mathbf{x}(t)$ and the input vector $\mathbf{u}(t)$ are in $\mathfrak{M}^{n\times 1}$ and $\mathfrak{M}^{m\times 1}$, respectively. We invert (3.67) by the same approach we used for the nth-order differential system; we invert separately the two component equations

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \boldsymbol{\theta} \tag{3.68}$$

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \boldsymbol{\theta}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.69}$$

Assume the inverse of the "boundary input" system (3.69) is of the form

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t)\mathbf{x}(0) \tag{3.70}$$

where the boundary kernel $\boldsymbol{\Phi}(t)$ is a $n \times n$ matrix commonly referred to as the **state transition** matrix. (It describes the "undriven" transition from the state at "0" to the state at t.) In order that (3.70) be the correct inverse, $\mathbf{x}(t)$ must satisfy (3.69),

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \frac{d\boldsymbol{\Phi}(t)}{dt}\mathbf{x}(0) - \mathbf{A}\boldsymbol{\Phi}(t)\mathbf{x}(0) = \boldsymbol{\theta}, \qquad \mathbf{x}(0) = \boldsymbol{\Phi}(0)\mathbf{x}(0)$$

for any initial condition vector $\mathbf{x}(0)$. Therefore, the state transition matrix must satisfy

$$\frac{d\boldsymbol{\Phi}(t)}{dt} - \mathbf{A}\boldsymbol{\Phi}(t) = \boldsymbol{\Theta}, \qquad \boldsymbol{\Phi}(0) = \mathbf{I} \tag{3.7 1}$$

Rather than treat the system (3.71) one element at a time, we work with the whole $n \times n$ matrix-valued system. We use the power series method to find the complementary function for the system. Assume

$$\boldsymbol{\Phi}(t) = \mathbf{C}_0 + \mathbf{C}_1 t + \mathbf{C}_2 t^2 + \cdots$$

where each $\mathbf{C}_i$ is a constant $n \times n$ matrix. We substitute $\boldsymbol{\Phi}(t)$ into the differential equation of (3.71) and equate the coefficient on each power of t to the zero matrix $\boldsymbol{\Theta}$ to find

$$\mathbf{C}_1 = \mathbf{A}\mathbf{C}_0, \qquad \mathbf{C}_2 = \left(\frac{1}{2!}\right)\mathbf{A}^2\mathbf{C}_0, \qquad \mathbf{C}_3 = \left(\frac{1}{3!}\right)\mathbf{A}^3\mathbf{C}_0, \qquad \cdots$$

It follows that $\mathbf{C}_0$ is arbitrary and

$$\boldsymbol{\Phi}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots\right)\mathbf{C}_0 \triangleq e^{\mathbf{A}t}\mathbf{C}_0 \tag{3.72}$$

We have used the symbol $e^{\mathbf{A}t}$ to represent the sum of the "exponential-looking" matrix series of (3.72):

$$e^{\mathbf{A}t} \triangleq \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} \cdots$$

We call $e^{\mathbf{A}t}$ a **fundamental matrix** for the state equation of (3.67); the matrix is analogous to a fundamental set of solutions for an nth-order differential equation. Applying the boundary conditions of (3.71) to (3.72), we find $\mathbf{\Phi}(0) = e^{\mathbf{A}0}\mathbf{C}_0 = \mathbf{I}$. It is clear from the definition of $e^{\mathbf{A}t}$ that $e^{\mathbf{A}0} = \mathbf{I}$; therefore, $\mathbf{C}_0 = \mathbf{I}$ and the state transition matrix (or boundary kernel) for the state-space system (3.67) is

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} \tag{3.73}$$

Example 2. A State Transition Matrix. In Example 1 we found the system matrix $\mathbf{A}$ for the differential equation $\ddot{\phi}(t) + \dot{\phi}(t) = u(t)$:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

To find the fundamental matrix for this system, we sum the defining infinite series:

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}t + \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\frac{t^2}{2!} + \cdots$$

$$= \begin{pmatrix} 1 & \left(t - \frac{t^2}{2!} + \frac{t^3}{3!} - \cdots\right) \\ 0 & \left(1 - t + \frac{t^2}{2!} - \cdots\right) \end{pmatrix} = \begin{pmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

If the matrix $\mathbf{A}$ of Example 2 were not simple, it would be difficult to sum the infinite series for $e^{\mathbf{A}t}$ by the method of that example. It would not be easy to recognize the function to which each scalar series converges. Arbitrary functions of matrices are examined in detail in Section 4.6, and practical techniques for computing functions of matrices are developed. These techniques can be used to compute $e^{\mathbf{A}t}$ for an arbitrary square matrix $\mathbf{A}$.

Exercise 2. Show that for the fundamental matrix $e^{\mathbf{A}t}$ of Example 2,

$$\begin{aligned} e^{\mathbf{A}t}e^{\mathbf{A}\tau} &= e^{\mathbf{A}(t+\tau)} \\ (e^{\mathbf{A}t})^{-1} &= e^{-\mathbf{A}t} \end{aligned} \tag{3.74}$$

Properties (3.74) apply to all time-invariant systems, that is, all systems which have a constant system matrix (P&C 3.19).

We can view $\mathbf{Bu}$ as a vector-valued distributed input to (3.67). Therefore, we assume the inverse of the distributed-input state equation (3.68) is an integral equation of the form

$$\mathbf{x}(t) = \int_0^\infty \mathbf{K}(t,s)\mathbf{B}\mathbf{u}(s)\,ds \tag{3.75}$$

where the $n \times n$ matrix $\mathbf{K}(t,s)$ is called the **matrix Green's function** for the system (3.67). (By the integral of a matrix we mean the matrix of integrals.) We substitute (3.75) into (3.68) to determine the equations which describe $\mathbf{K}$:

$$\begin{aligned} \dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) &= \int_0^\infty \left[\frac{d}{dt}\mathbf{K}(t,s) - \mathbf{A}\mathbf{K}(t,s) \right] \mathbf{B}\mathbf{u}(s)\,ds = \mathbf{B}\mathbf{u}(t) \\ \mathbf{x}(0) &= \int_0^\infty \mathbf{K}(0,s)\mathbf{B}\mathbf{u}(s)\,ds = \boldsymbol{\theta} \end{aligned} \tag{3.76}$$

for all vectors $\mathbf{u}$ with elements which are continuous functions. To see more clearly the conditions on $\mathbf{K}(t,s)$ which follow from (3.76), note that

$$\int_0^\infty \delta(t-s)\mathbf{I} \begin{pmatrix} g_{11}(s) & \cdots & g_{1m}(s) \\ \vdots & & \vdots \\ g_{n1}(s) & \cdots & g_{nm}(s) \end{pmatrix} ds = \begin{pmatrix} g_{11}(t) & \cdots & g_{1m}(t) \\ \vdots & & \vdots \\ g_{n1}(t) & \cdots & g_{nm}(t) \end{pmatrix}$$

In other words, if we let $\mathbf{G}(s)$ denote the matrix with elements $g_{ij}(s)$, then the equation $\int_0^\infty \mathbf{K}(t,s)\mathbf{G}(s)\,ds = \mathbf{G}(t)$ is satisfied by $\mathbf{K}(t,s) = \delta(t-s)\mathbf{I}$. Thus in order to satisfy (3.76), it is sufficient that $\mathbf{K}$ meet the following requirements:

$$\begin{aligned} \frac{d\mathbf{K}(t,s)}{dt} - \mathbf{A}\mathbf{K}(t,s) &= \delta(t-s)\mathbf{I} \\ \mathbf{K}(0,s) &= \boldsymbol{\Theta} \end{aligned} \tag{3.77}$$

The approach we use to solve (3.77) for $\mathbf{K}$ is essentially the same as that used for the nth-order scalar system (3.55). For $t \neq s$, $\mathbf{K}(t,s)$ satisfies the same $n \times n$ differential equation, (3.71), as does the state transition matrix. Thus, using the general solution to (3.71) found earlier,

$$\begin{aligned} \mathbf{K}(t,s) &= e^{\mathbf{A}t}\mathbf{B}_0, \quad t \text{ in } [0,s) \\ &= e^{\mathbf{A}t}\mathbf{D}_0, \quad t \text{ in } (s,\infty) \end{aligned}$$

where $\mathbf{B}_0$ and $\mathbf{D}_0$ are $n \times n$ constant matrices. The boundary conditions of (3.76) require $\mathbf{K}(0,s) = e^{\mathbf{A}0}\mathbf{B}_0 = \mathbf{\Theta}$; since $e^{\mathbf{A}0} = \mathbf{I}$, $\mathbf{B}_0 = \mathbf{\Theta}$. From (3.77), we also note that $\mathbf{K}$ must satisfy a discontinuity condition at $t = s$. The delta functions on the right-hand side of (3.77) must be introduced by the highest derivative, $d\mathbf{K}/dt$; otherwise derivatives of delta functions would appear. Consequently, the diagonal elements of $\mathbf{K}$ contain a unit step at $t = s$, whereas off-diagonal elements are continuous:

$$\mathbf{K}(s^+,s) - \mathbf{K}(s^-,s) = e^{\mathbf{A}s}\mathbf{D}_0 - \mathbf{\Theta} = \mathbf{I}$$

Then, using (3.74), $\mathbf{D}_0 = (e^{\mathbf{A}s})^{-1} = e^{-\mathbf{A}s}$, and

$$\begin{aligned} \mathbf{K}(t,s) &= \mathbf{\Theta}, \quad t < s \\ &= e^{\mathbf{A}(t-s)}, \quad t > s \end{aligned} \tag{3.78}$$

The- inverse of the state-space system (3.67) is the sum of (3.75) and (3.70); $\mathbf{\Phi}$ and $\mathbf{K}$ are given by (3.73) and (3.78), respectively:

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{u}(s)\,ds + e^{\mathbf{A}t}\mathbf{x}(0) \tag{3.79}$$

The inverse system is fully determined by the state transition matrix $e^{\mathbf{A}t}$ and the input matrix $\mathbf{B}$. In Section 4.6 we determine how to evaluate $e^{\mathbf{A}t}$ by methods other than summing of the series (3.72).

At the heart of the solution (3.60) for the nth-order dynamic system (3.55) is the Vandermond matrix for the system. If the state equation is derived from the nth-order differential equation as in (3.63), we would expect the Vandermond matrix to be involved in the solution (3.79) of the state equation. We find in P&C 4.16 and (4.98) that if the system matrix $\mathbf{A}$ is the companion matrix for an nth-order dynamic system, the Vandermond matrix is intimately related to both $\mathbf{A}$ and $e^{\mathbf{A}t}$.

Exercise 3. Show that for the system of Examples 1 and 2,

$$\mathbf{x}(t) = \begin{pmatrix} \phi(t) \\ \dot{\phi}(t) \end{pmatrix} = \int_0^t \begin{pmatrix} 1 - e^{-(t-s)} \\ e^{-(t-s)} \end{pmatrix} \mathbf{u}(s)\,ds + \begin{pmatrix} \phi(0) + \dot{\phi}(0) - \dot{\phi}(0)e^{-t} \\ \dot{\phi}(0)e^{-t} \end{pmatrix} \tag{3.80}$$

Equation (3.80) should be compared with its second-order scalar equivalent (3.62). The state-space solution usually contains more information than its scalar counterpart-information about derivatives of the solution is stated explicitly.

Exercise 4. Use the solution (3.79) at $t = a$ to determine the form of the solution to the state-space system (3.67) if the initial conditions are given at $t = a$ instead of $t = 0$; that is, show that

$$\mathbf{f}(t) = \int_a^t e^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{u}(s)\,ds + e^{\mathbf{A}(t-a)}\mathbf{x}(a)$$

The discussion beneath the nth-order scalar solution (3.60) extends to the more general state-space solution (3.79). We can interpret $\mathbf{K}(t, 0) = e^{\mathbf{A}t}$ as the matrix impulse response of the state-space system. Since the matrix $\mathbf{A}$ is constant, it is appropriate to measure physically the state transition matrix $e^{\mathbf{A}t}$. By (3.70), the jth column of $\mathbf{\Phi}(t)$ (or $e^{\mathbf{A}t}$) consists in the "undriven" decay of $\mathbf{x}(t)$ from the initial condition $\mathbf{x}(0) = \boldsymbol{\varepsilon}_j$, the jth standard basis vector for $\mathcal{M}^{n \times 1}$. From measurements of the n columns of $e^{\mathbf{A}t}$ we can determine the full inverse equation (3.79) without explicit determination of the system matrix $\mathbf{A}$ (P&C 3.20).

The techniques used to invert the first-order state-space system (3.67) are applied to a *second-order* vector differential system in P&C 4.32. As with the state-space system, the Green's function for this system can be obtained from the boundary kernel; the latter can be measured physically. The inverse for this second-order vector system involves several functions of matrices. We discuss methods for evaluating general functions of matrices in Section 4.6.

3.5 Problems and Comments

3.1 *Forward integration:* the differential system $\mathbf{f}''(t) + \frac{1}{4}\mathbf{f}(t) + (1/400)\mathbf{f}^3(t) = 0$, $\mathbf{f}(0) = 10$, $\mathbf{f}'(0) = 0$ describes the unforced oscillations of a mass hanging on a spring. The spring has a nonlinear force-elongation characteristic; $\mathbf{f}(t)$ denotes the position of the mass at time t. There are many numerical integration techniques for obtaining an approximate solution to such a nonlinear differential equation with initial conditions (see [3.9]). The following technique is one of the simplest. We concern ourselves only with integer values of t, and replace the derivatives by the finite-difference approximations $\mathbf{f}'(n) \approx \mathbf{f}(n+1) - \mathbf{f}(n)$ and $\mathbf{f}''(n) \approx \mathbf{f}(n+1) - 2\mathbf{f}(n) + \mathbf{f}(n-1)$. Use these finite-difference approximations and the differential system

to express $\mathbf{f}(n+1)$ in terms of $\mathbf{f}(n)$ and $\mathbf{f}(n-1)$. Compute $\mathbf{f}(1), \mathbf{f}(2), \ldots, \mathbf{f}(8)$. How might the above finite-difference approximation be modified to obtain a more accurate solution to the differential equation?

3.2 *Backward integration:* a (nonlinear) differential equation with final end-point conditions (rather than initial conditions) can be solved by backward numerical integration. Backward integration can be carried out be means of any forward integration routine. Suppose the differential system is of the form $\mathbf{f}^{(n)}(t) + \mathbf{F}(\mathbf{f}(t), \mathbf{f}'(t), \ldots, \mathbf{f}^{(n-1)}(t), t) = 0$ with $\mathbf{f}(t_f), \mathbf{f}'(t_f), \ldots, \mathbf{f}^{(k-1)}(t_f)$ specified. Show that the change of variables $\mathbf{f}(t) = \mathbf{f}(t_f - s) = \mathbf{g}(s)$ converts the final conditions on $\mathbf{f}$ to initial conditions on $\mathbf{g}$ and produces a differential equation in $\mathbf{g}$ which differs from the differential equation in $\mathbf{f}$ in the sign on the odd-order derivatives.

3.3 *Relaxation:* the finite-difference approximation to a two-point boundary value problem can be solved by a simple iterative technique known as relaxation [3.3]. Suppose $\mathbf{f}''(s) = 1$ with $\mathbf{f}(0) = \mathbf{f}(5) = 0$. Consider the values of $\mathbf{f}$ only at integer values of s. Replace the second derivative by the approximation $\mathbf{f}''(n) \approx \mathbf{f}(n+1) - 2\mathbf{f}(n) + \mathbf{f}(n-1)$, and express $\mathbf{f}(n)$ in terms of $\mathbf{f}(n-1)$ and $\mathbf{f}(n+1)$. Let the initial values of $\mathbf{f}(1), \ldots, \mathbf{f}(4)$ be zero. A single step in the iteration consists in solving successively for each of the values $\mathbf{f}(1), \ldots, \mathbf{f}(4)$ in terms of current values of $\mathbf{f}$ at the two neighboring points. Repetitive improvement of the set of values $\{\mathbf{f}(k)\}$ results in convergence of this set of values to the solution of the set of difference equations, regardless of numerical errors, and regardless of the order in which the values are improved during each iteration.

(*a*) Carry out six iterations for the above problem.

(*b*) Find the exact solution to the set of difference equations by solving the equations simultaneously. Compare the results of the iteration of (*a*) with the exact solution for the differential system.

*3.4 An intuitive understanding of the following properties of differential systems can be gained by examining a finite-difference approximation to the second-order case. See [3.6] for a rigorous discussion of these statements.

(*a*) A regular *n*th-order linear differential equation has *n* independent solutions.

(*b*) A boundary condition consisting in a linear combination of values of $\mathbf{f}, \mathbf{f}', \ldots, \mathbf{f}^{(n-1)}$ need not be independent of the regular *n*th-order differential equation; consider, for example, $\mathbf{f}''(s) = 0$ with $\mathbf{f}'(0) - \mathbf{f}'(1) = 0$.

(*c*) If the boundary conditions associated with a regular nth-order differential equation consist in n independent linear combinations of the values $\mathbf{f}(a), \mathbf{f}'(a), \ldots, \mathbf{f}^{(n-1)}(a)$, at a single point a in the domain of $\mathbf{f}$, then the differential system has a unique solution.

3.5 The following differential system is degenerate:

$$\boldsymbol{\phi}'' + \boldsymbol{\phi}' = \mathbf{u} \quad \text{with} \begin{cases} \boldsymbol{\phi}(0) - \boldsymbol{\phi}(1) = \alpha_1 \\ 2\boldsymbol{\phi}(1) - 2\boldsymbol{\phi}(0) + \boldsymbol{\phi}'(0) - \boldsymbol{\phi}'(1) = \alpha_2 \\ \boldsymbol{\phi}'(1) - \boldsymbol{\phi}'(0) = \alpha_3 \end{cases}$$

Find the solutions to the differential system in terms of the inputs $\mathbf{u}$, α_1, α_2, and α_3. Also find the relations among the inputs that must be satisfied in order that solutions exist. (Hint: the solution to the differential equation is expressed in terms of $\boldsymbol{\phi}(0)$ and $\boldsymbol{\phi}'(0)$ in (3.80).) What relationship exists between the number of dependent rows in a boundary condition matrix for a system and the number of different relations which must be satisfied by the inputs to that system?

3.6 Let $\mathbf{L}$ be a regular nth-order differential operator and $\{\boldsymbol{\beta}_i(\mathbf{f}) = 0, i = 1, \ldots, m\}$ a set of homogeneous boundary conditions. Let $\mathcal{V}$ be the space of functions in $\mathcal{C}^n(a, b)$ which satisfy the homogeneous differential equation $\mathbf{Lf} = \boldsymbol{\theta}$. Let $\mathcal{F} \triangleq \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a fundamental set of solutions for $\mathbf{L}$; $\mathcal{F}$ is a basis for $\mathcal{V}$. Define $\mathbf{T}: \mathcal{V} \to \mathcal{R}^n$ by $\mathbf{Tf} \triangleq (\boldsymbol{\beta}_1(\mathbf{f}), \ldots, \boldsymbol{\beta}_m(\mathbf{f}))$ for all $\mathbf{f}$ in $\mathcal{V}$. Let $\mathcal{E}$ be the standard basis for $\mathcal{R}^n$. Show that the matrix $[\mathbf{T}]_{\mathcal{F}\mathcal{E}}$ is a boundary condition matrix for the differential system $\{\mathbf{L}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m\}$.

*3.7 The Wronskian: let $\mathbf{f}_1, \ldots, \mathbf{f}_n$ be in $\mathcal{C}^n(a, b)$. The Wronskian matrix of $\mathbf{f}_1, \ldots, \mathbf{f}_n$ at t is defined by

$$\mathbf{W}(t) \triangleq \begin{pmatrix} \mathbf{f}_1(t) & \cdots & \mathbf{f}_n(t) \\ \mathbf{f}_1^{(1)}(t) & \cdots & \mathbf{f}_n^{(1)}(t) \\ \vdots & & \vdots \\ \mathbf{f}_1^{(n-1)}(t) & \cdots & \mathbf{f}_n^{(n-1)}(t) \end{pmatrix}$$

The Wronskian determinant is $w(t) \triangleq \det(\mathbf{W}(t))$.

(*a*) Show that $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ cannot be linearly dependent unless $w(t) = 0$ for all t in $[a, b]$.

(*b*) The fact that $w(t) = 0$ for *some* t does not ordinarily imply that the set $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is dependent; try, for example, $\mathbf{f}_1(t) = t^2$ and $\mathbf{f}_2(t) = t^3$ at $t = 0$. Suppose, however, that $\mathbf{f}_1, \ldots, \mathbf{f}_n$ are solutions to an nth-order homogeneous differential equation defined on $[a, b]$. Then if $w(t) = 0$ for *any* t in $[a, b]$, $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is a linearly dependent set.

3.8 *Difference equations:* an arbitrary linear constant-coefficient difference equation can be expressed in the form

$$a_0(\mathbf{E}^n\mathbf{f})(k) + a_1(\mathbf{E}^{n-1}\mathbf{f})(k) + \cdots + a_n\mathbf{f}(k) = \mathbf{u}(k), \qquad k = 0, 1, 2, \ldots$$

where $\mathbf{E}$ is the shift operator defined by $(\mathbf{Ef})(k) \triangleq \mathbf{f}(k+1)$; we concern ourselves only with integer values of the argument of $\mathbf{f}$. The order of the difference equation is the number of boundary conditions needed to specify a unique solution to the equation; that is, the order is $n - p$, where p is the lowest power of $\mathbf{E}$ to appear in the equation. (See [3.2].)

(*a*) The solutions to the homogeneous difference equation (the equation with $\mathbf{u}(k) = 0$) usually consist of combinations of geometric sequences. Substitution of the sequence $\mathbf{f}(k) = r^k$, $k = 0, 1, 2, \ldots$, into the homogeneous equation shows that non-trivial sequences must satisfy the following characteristic equation: $a_0r^n + a_1r^{n-1} + \ldots + a_n = 0$. Find a basis for the nullspace of the difference operator $\mathbf{T}$ defined by

$$\begin{aligned}(\mathbf{Tf})(k) &\triangleq 2(\mathbf{E}^2\mathbf{f})(k) - 3(\mathbf{Ef})(k) + \mathbf{f}(k)\\ &= 2\mathbf{f}(k+2) - 3\mathbf{f}(k+1) + \mathbf{f}(k)\end{aligned}$$

What is the dimension of the nullspace of an nth-order difference operator?

(*b*) Let $\mathbf{f}_1, \ldots, \mathbf{f}_n$ be infinite sequences of the form $\mathbf{f}_i(k)$, $k = 0, 1, 2, \ldots$. The *Casorati matrix* of $\mathbf{f}_1, \ldots, \mathbf{f}_n$ is defined by

$$\mathbf{C}(k) \triangleq \begin{pmatrix} \mathbf{f}_1(k) & \cdots & \mathbf{f}_n(k) \\ \mathbf{Ef}_1(k) & \cdots & \mathbf{Ef}_n(k) \\ \vdots & & \vdots \\ \mathbf{E}^{n-1}\mathbf{f}_1(k) & \cdots & \mathbf{E}^{n-1}\mathbf{f}_n(k) \end{pmatrix}$$

The infinite sequences $\mathbf{f}_1, \ldots, \mathbf{f}_n$ are linearly independent if and

only if $c(k) \neq 0$ for $k = 0, 1, 2, \ldots$, where $c(k)$ is the *Casorati determinant,* $\det(\mathbf{C}(k))$. Use the Casorati determinant to show the independence of the basis vectors found in (a).

3.9 Use the power series method to find the complementary function for the differential operator $(\mathbf{D}-1)^2$.

3.10 Define $\mathbf{L}: \mathcal{C}^1(0,1) \to \mathcal{C}(0,1)$ by $\mathbf{L} \triangleq -\mathbf{D} - a\mathbf{I}$. Find the Green's function k and the inverse equation for the differential system $\mathbf{Lf} = \mathbf{u}$, $\mathbf{f}(0) = \mathbf{f}(1)$.

3.11 Define $\mathbf{L}: \mathcal{C}^2(0,b) \to \mathcal{C}(0,b)$ by $\mathbf{L} \triangleq \mathbf{D}^2 - 3\mathbf{D} + 2\mathbf{I}$. Find the Green's function k, the boundary kernel ρ, and the inverse equation for the differential system $\mathbf{Lf} = \mathbf{u}$, $\mathbf{f}(0) = \alpha_1$, $\mathbf{f}(b) = \alpha_2$.

3.12 Find the inverse equation for each of the following differential systems:

(a) $\mathbf{f}'' + 6\mathbf{f}' + 5\mathbf{f} = \mathbf{u}$, $\mathbf{f}(0) = \alpha_1$, $\mathbf{f}'(0) = \alpha_2$

(b) $\mathbf{f}'' + 2\mathbf{f}' + 2\mathbf{f} = \mathbf{u}$, $\mathbf{f}(0) = \alpha_1$, $\mathbf{f}'(0) = \alpha_2$

(c) $\mathbf{f}''' + 6\mathbf{f}'' + 5\mathbf{f}' = \mathbf{u}$, $\mathbf{f}(0) = \alpha_1$, $\mathbf{f}'(0) = \alpha_2$, $\mathbf{f}''(0) = \alpha_3$

3.13 The following differential system describes the steady-state temperature distribution along an insulated bar of length b: $-\mathbf{f}'' = \mathbf{u}$, $\mathbf{f}(0) = \alpha_1$, $\mathbf{f}'(b) + \mathbf{f}(b) = \alpha_2$. (The second boundary condition implies that heat is removed by convection at point b.) Show that the inverse equation for this system is

$$\mathbf{f}(t) = \left(1 - \frac{t}{1+b}\right)\int_0^t s\mathbf{u}(s)\,ds + t\int_t^b \left(1 - \frac{s}{1+b}\right)\mathbf{u}(s)\,ds$$

$$+ \alpha_1\left(1 - \frac{t}{1+b}\right) + \alpha_2\left(\frac{t}{1+b}\right)$$

3.14 For the differential system $t\mathbf{f}'(t) - \mathbf{f}(t) = \mathbf{u}(t)$, $\mathbf{f}'(t_1) = \alpha$, $t_1 > 0$,

(a) Find the complementary function by the power series method;

(b) Find the Green's function $k(t, s)$;

(c) Find the boundary kernel $\rho_j(t)$;

(d) State explicitly the inverse equation.

*3.15 Let μ_1 and μ_2 be the roots of the characteristic equation for the differential system $\mathbf{f}'' + a_1\mathbf{f}' + a_2\mathbf{f} = \mathbf{u}$, $\mathbf{f}(0) = \mathbf{f}'(0) = 0$.

(a) Use (3.56) and (3.57) to find the Green's function k for this system. If $\mu_2 \approx \mu_1$, computed values of $\mu_2 - \mu_1$ and $\exp(\mu_2 t) - \exp(\mu_1 t)$ will be badly in error. What is the effect of near equality of the roots on the numerical computation of $k(t,s)$ and $\int k(t,s)\mathbf{u}(s)\,ds$?

(*b*) If $\mu_2 \approx \mu_1$, the fundamental set $\{\exp(\mu_1 t), \exp(\mu_2 t)\}$ is nearly dependent. A better fundamental set (not nearly dependent) in this circumstance is

$$\mathbf{v}_1(t) = \frac{\exp(\mu_1 t) + \exp(\mu_2 t)}{2} \qquad \mathbf{v}_2(t) = \frac{\exp(\mu_1 t) - \exp(\mu_2 t)}{\mu_1 - \mu_2}$$

Derive a power series expansion of $\mathbf{v}_2$ which can be used to compute values of $\mathbf{v}_2$ without numerical division by the inaccurate quantity $\mu_2 - \mu_1$. Show that as $\mu_2 \to \mu_1$, $\{\mathbf{v}_1(t), \mathbf{v}_2(t)\} \to \{\exp(\mu_1 t), t\exp(\mu_1 t)\}$.

(*c*) Equation (3.52) expresses the Green's function k in terms of the functions $\{\mathbf{v}_i\}$ of (*b*). Evaluate the Wronskian determinant w in this expression in terms of exponentials. Values of $k(t,s)$ and $\int k(t, s)\mathbf{u}(s)\, ds$ can be computed accurately by using this expression for $k(t,s)$ together with computed values of $\mathbf{v}_1, \mathbf{v}_2$, and w. Show that this expression for $k(t,s)$ is a rearrangement of the expression for $k(t, s)$ found in (*a*).

3.16 One method for obtaining the Green's function for a constant-coefficient differential system is to solve (3.32) by means of one-sided Laplace transforms. Use this technique to show that the inverse of the differential equation $\ddot{\mathbf{f}} + \omega^2 \mathbf{f} = \mathbf{u}$, with constant ω and given values of $\mathbf{f}(0)$ and $\dot{\mathbf{f}}(0)$, is

$$\mathbf{f}(t) = \mathbf{f}(0)\cos\omega t + \frac{\dot{\mathbf{f}}(0)}{\omega}\sin\omega t + \frac{1}{\omega}\int_0^t \sin\omega(t-s)\mathbf{u}(s)\, ds$$

3.17 The approximation of derivatives by finite-differences leads to the approximate representation of differential equations by difference equations. For instance, the use of a second-central difference plus a forward difference converts the second-order differential system $\phi'' + \phi' = \mathbf{u}$, $\phi(0) = \alpha_1$, $\phi'(0) = \alpha_2$ to the approximately equivalent second-order difference system $2\phi(i+2) - 3\phi(i+1) + \phi(i) = \mathbf{u}(i+1)$, $\phi(0) = \alpha_1$, $\phi(1) = \alpha_2 + \alpha_1$. A general form for the nth-order constant-coefficient difference system with initial conditions is

$$\mathbf{f}(i+n) + a_1\mathbf{f}(i+n-1) + \cdots + a_n\mathbf{f}(i) = \mathbf{v}(i)$$

$$\mathbf{f}(0) = \gamma_1, \quad \mathbf{f}(1) = \gamma_2 \quad \ldots, \quad \mathbf{f}(n-1) = \gamma_n$$

for $i = 0, 1, 2, \ldots$.

By analogy to the inverse equation for the nth-order differential system, we assume the inverse of the nth-order difference system is

of the form

$$\mathbf{f}(i) = \sum_{j=0}^{\infty} k(i,j)\mathbf{v}(j) + \sum_{m=1}^{n} \rho_m(i)\mathbf{f}(m-1)$$

for $i = 0, 1, 2, \ldots$.

(*a*) Show that the discrete Green's function $k(i,j)$ is specified by the difference system

$$k(i+n,j) + a_1 k(i+n-1,j) + \cdots + a_n k(i,j) = \delta_{ij}$$

$$k(0,j) = k(1,j) = \cdots = k(n-1,j) = 0$$

for $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$.

(*b*) Show that the discrete boundary kernel $\rho_m(i)$ is specified by the difference system

$$\rho_m(i+n) + a_1\rho_m(i+n-1) + \cdots + a_n\rho_m(i) = 0$$

$$\rho_m(p) = \delta_{m,p+1}$$

for $i = 0, 1, 2, \ldots, m = 1, \ldots, n$, and , ..., $n-1$.

(*c*) Find the inverse of the second-order difference system mentioned above by solving the difference systems corresponding to those in (*a*) and (*b*). Hint: solutions to homogeneous constant-coefficient difference equations consist in sums of geometric sequences of the form $\mathbf{f}(i) = r^i$, $i = 0, \pm 1, \pm 2, \ldots$.

3.18 The following pair of coupled differential equations relates a pair of system outputs $\{\mathbf{f}_i(t)\}$ to a pair of inputs $\{\mathbf{u}_i(t)\}$:

$$\mathbf{f}_1'' + 3\mathbf{f}_1' + 2\mathbf{f}_2 = \mathbf{u}_1,$$

$$\mathbf{f}_2'' + \mathbf{f}_1' + \mathbf{f}_2 = \mathbf{u}_2,$$

$\mathbf{f}_1(0), \mathbf{f}_1'(0), \mathbf{f}_2(0), \mathbf{f}_2'(0)$ specified.

(*a*) Find a first-order state equation of the form (3.65) which is equivalent to the set of coupled equations. (Hint: use as state variables the output functions and their first derivatives.) Is the state equation unique?

(*b*) The solution to the state equation is determined by the state transition matrix (3.73). How could this matrix function be computed for the system in (*a*)?

3.19 *Properties of state transition matrices:* the concept of a state transition matrix extends to time-varying dynamic systems [3.14]. sup-

pose a dynamic system satisfies $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, where $\mathbf{x}(t_0)$ is given and $\mathbf{A}(t)$ is an $n \times n$ matrix. We can express the solution in the form $\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0)$. We refer to the $n \times n$ matrix $\mathbf{\Phi}(t_0, t)$ as the **state transition matrix.** The state transition matrix has the following properties:

(*a*) $\dfrac{d}{dt}\mathbf{\Phi}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0)$, $\mathbf{\Phi}(t_0, t_0) = I$;

(*b*) $\mathbf{\Phi}(t_0, t_1)\mathbf{\Phi}(t_1, t_2) = \mathbf{\Phi}(t_0, t_2)$ for all t_0, t_1, and t_2;

(*c*) $\mathbf{\Phi}(t_1, t_0)^{-1} = \mathbf{\Phi}(t_0, t_1)$;

(*d*) If $\mathbf{A}(t)\int_{t_0}^{t}\mathbf{A}(s)\,ds = \int_{t_0}^{t}\mathbf{A}(s)\,ds\,\mathbf{A}(t)$, then $\mathbf{\Phi}(t, t_0) = \exp\int_{t_0}^{t}\mathbf{A}(s)\,ds$ (see P&C 4.29);

(*e*) $\det \mathbf{\Phi}(t, t_0) = \exp\int_{t_0}^{t} \operatorname{trace}[\mathbf{A}(s)]\,ds$, where $\operatorname{trace}[\mathbf{A}(s)]$ is the sum of the diagonal elements of $\mathbf{A}(s)$.

3.20 A certain system can be represented by a differential equation of the form $\ddot{\mathbf{f}} + a_1\dot{\mathbf{f}} + a_2\mathbf{f} = \mathbf{u}$. The values of the coefficients a_1 and a_2 are unknown. However, we have observed the response of the undriven system ($\mathbf{u}(t) = 0$ for $t > 0$) with various initial conditions. In particular, for $\mathbf{f}(0) = 1$ and $\dot{\mathbf{f}}(0) = 0$, we find that $\mathbf{f}(t) = 2e^{-t} - e^{-2t}$ and $\dot{\mathbf{f}}(t) = 2(e^{-2t} - e^{-t})$ for $t \geqslant 0$. Also, for $\mathbf{f}(0) = 0$ and $\dot{\mathbf{f}}(0) = 1$, we find that $\mathbf{f}(t) = e^{-t} - e^{-2t}$ and $\dot{\mathbf{f}}(t) = 2e^{-2t} - e^{-t}$ for $t \geqslant 0$.

(*a*) Determine the state equation in terms of a_1 and a_2.

(*b*) Use the transient measurements to determine the state transition matrix and the precise inverse of the state equation.

3.21 *Discrete-time state equations:* by using finite-difference approximations for derivatives, an arbitrary *n*th-order linear constant-coefficient differential equation with initial conditions can be approximated by an *n*th-order linear constant-coefficient difference equation of the form

$$\mathbf{f}((k+n)\tau) + a_1\mathbf{f}((k+n-1)\tau) + \cdots + a_n\mathbf{f}(k\tau) = \mathbf{u}(k\tau)$$

for $k = 0, 1, 2, \ldots$, with $\mathbf{f}(0), \mathbf{f}(\tau), \ldots, \mathbf{f}((n-1)\tau)$ given. The quantity τ is the time increment used in the finite-difference approximation.

(*a*) Put this *n*th-order difference equation in state-space form; that is, develop an equivalent first-order vector difference equation.

(*b*) Determine the form of the inverse of the discrete-time state equation.

3.6 References

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