converting (2.16) to a matrix equation in the unknowns $\{c_i\}$ is to evaluate the equation at three different values of t. Each such evaluation yields an algebraic equation in $\{c_i\}$. The resulting matrix equation is different from (2.15), but the solution is the same. We now describe a general method, built around a natural basis, for converting (2.13) to a matrix equation. The coordinate matrix of a vector **x** relative to the basis $\mathcal{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is $[\mathbf{x}]_{\mathcal{X}} = (c_1 \cdots c_n)^{\mathrm{T}}$, where the coordinates c_i are obtained by solving the vector equation

$$\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

A general method for obtaining an equivalent matrix equation consists in taking coordinates of the vector equation relative to a natural basis $\mathcal{N}$—a basis relative to which coordinates can be obtained by inspection. The vector equation becomes

$$[\mathbf{x}]_{\mathcal{N}} = \left[\sum_{i=1}^{n} c_i\mathbf{x}_i \right]_{\mathcal{N}}$$

$$= \sum_{i=1}^{n} c_i[\mathbf{x}_i]_{\mathcal{N}}$$

$$= ([\mathbf{x}_1]_{\mathcal{N}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathcal{N}}) [\mathbf{x}]_{\mathcal{X}} \qquad (2.17)$$

We determine $[\mathbf{x}]_{\mathcal{N}}, [\mathbf{x}_1]_{\mathcal{N}}, \dots, [\mathbf{x}_n]_{\mathcal{N}}$ by inspection. Then we solve (2.17) routinely for $[\mathbf{x}]_{\mathcal{X}}$.

Example 9.   Finding Coordinates via a Natural Basis.   Let the set $\mathcal{F} \stackrel{\Delta}{=} \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be a basis for $\mathcal{P}^3$, where $\mathbf{f}_1(t) = 1 + 2t + 3t^2$, $\mathbf{f}_2(t) = 2 + 3t + 2t^2$, and $\mathbf{f}_3(t) = 2 + 5t + 5t^2$. We seek $[\mathbf{f}]_{\mathcal{F}}$ for the vector $\mathbf{f}(t) = 1 + t + t^2$. To convert the defining equation for coordinates into a matrix equation, we use the natural basis $\mathcal{N} \stackrel{\Delta}{=} \{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$, where $\mathbf{g}_k(t) = t^{k-1}$. For this problem, (2.17) becomes

$$[\mathbf{f}]_{\mathcal{N}} = \left([\mathbf{f}_1]_{\mathcal{N}} \vdots [\mathbf{f}_2]_{\mathcal{N}} \vdots [\mathbf{f}_3]_{\mathcal{N}} \right)[\mathbf{f}]_{\mathcal{F}}$$

or

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \vdots & 2 & \vdots & 2 \\ 2 & \vdots & 3 & \vdots & 5 \\ 3 & \vdots & 2 & \vdots & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

The solution to this equation is $[\mathbf{f}]_{\mathcal{F}} = (\tfrac{3}{5} \ \tfrac{3}{5} \ -\tfrac{2}{5})^{\mathrm{T}}$. (Compare with Example 8.)

Typically, the solution of (2.17) requires the elimination procedure

$$([\mathbf{x}_1]_{\mathfrak{R}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathfrak{R}} \vdots [\mathbf{x}]_{\mathfrak{R}}) \rightarrow (\mathbf{I} \vdots [\mathbf{x}]_{\mathfrak{R}}) \qquad (2.18)$$

If we wish to solve for the coordinates of more than one vector, we still perform the elimination indicated in (2.18), but augment the matrix with all the vectors whose coordinates we desire. Thus if we wish the coordinates for $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$, we perform elimination on

$$([\mathbf{x}_1]_{\mathfrak{R}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathfrak{R}} \vdots [\mathbf{z}_1]_{\mathfrak{R}} \vdots [\mathbf{z}_2]_{\mathfrak{R}} \vdots [\mathbf{z}_3]_{\mathfrak{R}})$$

This elimination requires less computation than does the process which goes through inversion of the matrix $([\mathbf{x}_1]_{\mathfrak{R}} \vdots \cdots \vdots [\mathbf{x}_n]_{\mathfrak{R}})$, regardless of the number of vectors whose coordinates we desire (P&C 1.3).

Example 10. ***A Basis and Coordinates for a Subspace.*** Let $\mathfrak{W}$ be the **subspace** of $\mathcal{P}^3$ consisting in all functions $\mathbf{f}$ defined by the rule $\mathbf{f}(t) = \xi_1 + \xi_2 t + (\xi_1 + \xi_2)t^2$ for some ξ_1 and ξ_2. Note that the standard basis functions for $\mathcal{P}^3$ are not contained in $\mathfrak{W}$. The functions defined by $\mathbf{g}_1(t) = 1 + t^2$ and $\mathbf{g}_2(t) = t + t^2$ are clearly independent vectors in $\mathfrak{W}$. Because there are two "degrees of freedom" in $\mathfrak{W}$ (i.e., two parameters ξ_1 and ξ_2 must be given to specify a particular function in $\mathfrak{W}$) we expect the set $\mathcal{G} \overset{\Delta}{=} \{\mathbf{g}_1, \mathbf{g}_2\}$ to span $\mathfrak{W}$ and thus be a basis. We seek the coordinate matrix $[\mathbf{f}]_{\mathcal{G}}$ of an arbitrary vector $\mathbf{f}$ in $\mathfrak{W}$. That is, we seek c_1 and c_2 such that

$$\mathbf{f}(t) = c_1 \mathbf{g}_1(t) + c_2 \mathbf{g}_2(t)$$

The matrix equation (2.17) can be written by inspection using the natural basis $\mathfrak{R}$ of Example 9:

$$[\mathbf{f}]_{\mathfrak{R}} = ([\mathbf{g}_1]_{\mathfrak{R}} \vdots [\mathbf{g}_2]_{\mathfrak{R}})[\mathbf{f}]_{\mathcal{G}}$$

or

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_1 + \xi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Then $c_i = \xi_i$ and

$$[\mathbf{f}]_{\mathcal{G}} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

Because we were able to solve uniquely for the coordinates, we know that $\mathcal{G}$ is indeed a basis for $\mathfrak{W}$. The subspace $\mathfrak{W}$ is equivalent to the subspace of Example 1. Note that the elimination procedure does not agree precisely

with (2.18) because there are only two degrees of freedom among the three coefficients of the arbitrary vector $\mathbf{f}$ in $\mathfrak{W}$.

Dimension

The equivalence between the three vector spaces $\mathfrak{R}^3$, $\mathfrak{P}^3$, and $\mathfrak{M}^{3\times 1}$ is apparent from Examples 7 and 8; The subspace $\mathfrak{W}$ of Example 10, however, is equivalent to $\mathfrak{M}^{2\times 1}$ rather than $\mathfrak{M}^{3\times 1}$, even though the elements of $\mathfrak{W}$ are polynomials in $\mathfrak{P}^3$. The key to the equivalence lies not in the nature of the elements, but rather in the number of "degrees of freedom" in each space (the number of scalars which must be specified in order to specify a vector); more to the point, the key lies in the number of vectors in a basis for each space.

Definition. A vector space is **finite dimensional** if it is spanned by a finite number of vectors. It is intuitively clear that all bases for a finite-dimensional space contain the same number of vectors. The number of vectors in a basis for a finite-dimensional space $\mathfrak{V}$ is called the **dimension** of $\mathfrak{V}$ and is denoted by $\dim(\mathfrak{V})$.

Thus $\mathfrak{R}^3$ and $\mathfrak{P}^3$ are both three-dimensional spaces. The subspace $\mathfrak{W}$ of Example 10 has dimension 2. Knowledge of the dimension of a space (or a subspace) is obtained in the course of determining a basis for the space (subspace). Since the space $\mathfrak{O} \overset{\Delta}{=} \{\, \boldsymbol{\theta} \,\}$ has no basis, we assign it dimension zero.

Example 11. A Basis for a Space of Random Variables. A vector space $\mathfrak{V}$ of random variables, defined on the possible outcomes of a single die-throwing experiment, is described in Example 11 of Section 2.1. A natural basis for $\mathfrak{V}$ is the set of random variables $\mathfrak{X} \overset{\Delta}{=} \{\mathbf{x}_i, i = 1,\dots,6\}$, where

$$\mathbf{x}_i(\sigma) \overset{\Delta}{=} 1 \text{ for } \sigma = i \text{ (the die equals } i)$$

$$\overset{\Delta}{=} 0 \text{ for } \sigma \neq i \text{ (the die does not equal } i)$$

That $\mathfrak{X}$ is a basis for $\mathfrak{V}$ can be seen from an attempt to determine the coordinates with respect to $\mathfrak{X}$ of an arbitrary random variable $\mathbf{z}$ defined on the experiment. If

$$\mathbf{z}(\sigma) \overset{\Delta}{=} \quad c_1 \quad \text{for} \quad \sigma = 1$$
$$\vdots$$
$$\overset{\Delta}{=} \quad c_6 \quad \text{for} \quad \sigma = 6$$

then $[\mathbf{z}]_{\mathfrak{X}} = (c_1 \cdots c_6)^{\mathrm{T}}$; a unique representation exists.

The random variables $\{x_1, \ldots, x_6\}$ are linearly independent. However, they are not *statistically* independent. **Statistical independence** of two random variables x and y means that knowledge of the *value* of one variable, say, x, does not tell us anything about the outcome of the experiment which determines the value of the other variable y, and therefore it tells us nothing about the value of y. The random variables $\{x_i\}$ are related by the underlying die-throwing experiment. If we know $x_1 = 0$, for instance, then we know $\sigma \neq 1$ (the die is not equal to 1); the probability mass functions for $x_2, \ldots, x_6$ and for all other vectors in $\mathcal{V}$ are modified by the information concerning the value of x_1. The new probability mass functions for x and y of Example 11, Section 2.1, given that $x_1 = 0$, are

$$\omega_x(x; x_1 = 0) = \tfrac{3}{5} \quad \text{for } x = 0 \qquad \omega_y(y; x_1 = 0) = 1 \quad \text{for } y = 0$$
$$= \tfrac{2}{5} \quad \text{for } x = 1 \qquad\qquad\qquad = 0 \quad \text{for } y = 1$$

The space l_2 of square-summable sequences described in Example 4 of Section 2.1 is obviously *infinite dimensional.* A direct extension of the standard basis for $\mathcal{R}^n$ seems likely to be a basis for l_2. It is common knowledge that functions f in $\mathcal{C}(0, 2\pi)$, the space of functions continuous on $[0, 2\pi]$, can be expanded uniquely in a Fourier series of the form $f(t) = b_0 + \sum_{k=1}^{\infty} (a_k \sin kt + b_k \cos kt)$. This fact leads us to suspect that the set of functions

$$\mathcal{F} \overset{\Delta}{=} \{1, \sin t, \cos t, \sin 2t, \cos 2t, \ldots\} \tag{2.19}$$

forms a basis for $\mathcal{C}(0, 2\pi)$, and that the coordinates of f relative to this basis are

$$(b_0, a_1, b_1, a_2, b_2, \ldots)$$

This suspicion is correct. The coordinates (or Fourier coefficients) actually constitute a vector in l_2. We show in Example 11 of Section 5.3 that l_2 serves as a convenient standard space of coordinate vectors for infinite-dimensional spaces; in that sense, it plays the same role that $\mathcal{M}^{n \times 1}$ does for n-dimensional spaces. Unfortunately, the concepts of independence, spanning sets, and bases do not extend easily to infinite-dimensional vector spaces. The concept of linear combination applies only to the combination of a finite number of vectors. We cannot add an infinite number of vectors without the concept of a limit; this concept is introduced in Chapter 5. Hence detailed examination of infinite-dimensional function spaces is left for that chapter.

Summary

There is no inherent basis in any space-one basis is as good as another. Yet a space may have one basis which appears more convenient than others. The standard basis for $\mathcal{R}^n$ is an example. By picking units of measurement in a physical system (e.g., volts, feet, degrees centigrade) we tie together the system and the model; our choice of units may automatically determine convenient or standard basis vectors for the vector space of the model (based on, say, 1 V, 1 ft, or 1 ° C).

By choosing a basis for a space, we remove the most distinguishing feature of that space, the nature of its elements, and thus tie each vector in the space to a unique coordinate matrix. Because of this unique connection which a basis establishes between the elements of a particular vector space and the elements of the corresponding space of coordinate matrices, we are able to carry out most vector manipulations in terms of coordinate matrices which represent the vectors. We have selected $\mathcal{M}^{n \times 1}$, rather than $\mathcal{R}^n$, as our standard n-dimensional space because matrix operations are closely tied to computer algorithms for solving linear algebraic equations (Section 1.5). Most vector space manipulations lead eventually to such equations.

Because coordinate matrices are themselves vectors in a vector space ($\mathcal{M}^{n \times 1}$), we must be careful to distinguish vectors from their coordinates. The confusion is typified by the problem of finding the coordinate matrix of a vector $\mathbf{x}$ from $\mathcal{M}^{n \times 1}$ relative to the standard basis for $\mathcal{M}^{n \times 1}$. In this instance $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$; the difference between the vector and its coordinate matrix is only conceptual. A vector is simply one of a set of elements, although we may use it to represent the physical condition of some system. The coordinate matrix of the vector, on the other hand, is the unique set of multipliers which specifies the vector as a linear combination of arbitrarily chosen basis vectors.

2.3 System Models

The concept of a vector as a model for the condition or change in condition of a system is explored in Sections 2.1 and 2.2. We usually separate the variables which pertain to the condition of the system into two broad sets: the independent (or input) variables, the values of which are determined outside of the system, and the dependent (or output) variables, whose values are determined by the system together with the independent variables. A model for the system itself consists in expressions of relations among the variables. In this section we identify properties of system models.

Example I. An Economic System   Let **x** represent a set of inputs to the U. S. national economy (tax rates, interest rates, reinvestment policies, etc.); let **y** represent a set of economic indicators (cost of living, unemployment rate, growth rate, etc.). The system model **T** must describe the economic laws which relate **y** to **x.**

Example 2. A Baking Process.   Suppose **x** is the weight of a sample of clay before a baking process and **y** is the weight after baking. Then the system model **T** must describe the chemical and thermodynamic laws insofar as they relate **x** and **y.**

Example 3. A Positioning System.   Suppose the system of interest is an armature-controlled motor which is used to position a piece of equipment. Let **x** represent the armature voltage, a function of time; let **y** be the shaft position, another function of time. The system model **T** should describe the manner in which the dynamic system relates the function **y** to the function **x.**

  The variables in the economic system of Example 1 clearly separate into input (or independent) variables and output (or system condition) variables. In Example 2, both the independent and dependent variables describe the condition of the system. Yet we can view the condition before baking as the input to the system and view the condition after baking as the output. The dynamic system of Example 3 is reciprocal; **x** and **y** are mutually related by **T.** Since the system is used as a motor, we view the armature voltage **x** as the input to the system and the shaft position **y** as the output. We could, as well, use the machine as a dc generator; then we would view the shaft position as the input and the armature voltage as the output.
  The notation **Tx = y** that we introduced in (1.1) implies that the model **T** does something to the vector **x** to yield the vector **y.** As a result, we may feel inclined to call **x** the input and **y** the output. Yet in Section 1.3 we note that equations are sometimes expressed in an inverse form. The positions of the variables in an equation do not determine whether they are independent or dependent variables. Furthermore, we can see from Example 3 that the input and output of a system in some instances may be determined arbitrarily. In general, we treat one of the vectors in the equation **Tx = y** as the input and the other as the output. However, unless we are exploring a problem for which the input is clearly defined, we use the terms input and output loosely in reference to the known and unknown variables, respectively.

Transformations on Vector Spaces

Our present purpose is to make more precise the vaguely defined model **T** introduced in (1.1) and illustrated above.

Definition. A **transformation** or **function** **T**: $\mathcal{S}_1 \to \mathcal{S}_2$ is a rule that

associates with each element of the set $\mathcal{S}_1$ a unique element from the set $\mathcal{S}_2^*$. The set $\mathcal{S}_1$ is called the **domain** of $\mathbf{T}$; $\mathcal{S}_2$ is the **range of definition** of $\mathbf{T}$.

   Our attention is directed primarily toward transformations where $\mathcal{S}_1$ and $\mathcal{S}_2$ are linear spaces. We speak of $\mathbf{T}\colon \mathcal{V} \to \mathcal{W}$ as a transformation from the vector space $\mathcal{V}$ into the vector space $\mathcal{W}$. An **operator** is another term for a transformation between vector spaces. We use this term primarily when the domain and range of definition are identical; we speak of $\mathbf{T}\colon \mathcal{V} \to \mathcal{V}$ as an operator **on** $\mathcal{V}$. If $\mathcal{S}_{\mathcal{V}}$ is a subset of $\mathcal{V}$, we denote by $\mathbf{T}(\mathcal{S}_{\mathcal{V}})$ the set of all vectors $\mathbf{Tx}$ in $\mathcal{W}$ for which $\mathbf{x}$ is in $\mathcal{S}_{\mathcal{V}}$; we refer to $\mathbf{T}(\mathcal{S}_{\mathcal{V}})$ as the **image of $\mathcal{S}_{\mathcal{V}}$ under T.** The **range of T** is $\mathbf{T}(\mathcal{V})$, the image of $\mathcal{V}$ under $\mathbf{T}$. The **nullspace of T** is the set of all vectors $\mathbf{x}$ in $\mathcal{V}$ such that $\mathbf{Tx} = \boldsymbol{\theta}_{\mathcal{W}}$ ($\boldsymbol{\theta}_{\mathcal{W}}$ is the zero vector in the space $\mathcal{W}$). If $\mathcal{S}_{\mathcal{W}}$ is a subset of $\mathcal{W}$, we call the set of vectors $\mathbf{x}$ in $\mathcal{V}$ for which $\mathbf{Tx}$ is in $\mathcal{S}_{\mathcal{W}}$ the **inverse image** of $\mathcal{S}_{\mathcal{W}}$. Thus the nullspace of $\mathbf{T}$ is the inverse image of the set $\{\boldsymbol{\theta}_{\mathcal{W}}\}$. See Figure 2.6.
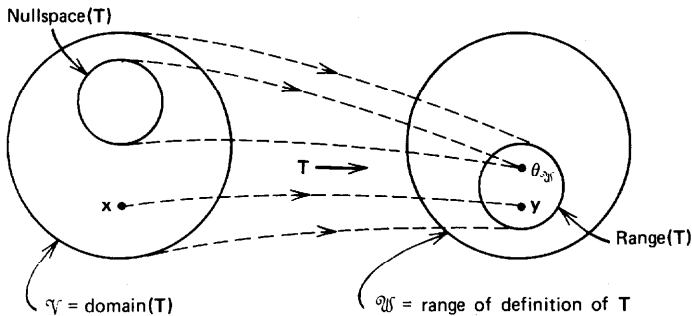

Figure 2.6.   Abstract illustration of a transformation **T**.

Example 4. A Transformation Define $\mathbf{T}\colon \mathcal{R}^2 \to \mathcal{R}^1$ by

$$\mathbf{T}(\xi_1, \xi_2) \overset{\Delta}{=} \sqrt{\xi_1^2 + \xi_2^2} - 1 \quad \text{for } \xi_1^2 + \xi_2^2 > 1$$
$$\overset{\Delta}{=} 0 \quad \text{for } \xi_1^2 + \xi_2^2 < 1 \tag{2.20}$$

Physically, the vector $\mathbf{Tx}$ can be interpreted as the distance between $\mathbf{x}$ and the unit circle in the two-dimensional arrow space. The variables $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are "dummy" variables; they merely assist us in cataloguing the "values" of $\mathbf{T}$ in the defining

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*In the modeling process we use the function concept twice: once as a vector—a model for the condition of a system—and once as a relation between input and output vectors—a model for the system itself. In order to avoid confusion, we use the term function in referring to vectors in a vector space, but the term transformation in referring to the relation between vectors.

equation; we can use any other symbols in their place without changing the definition of **T**. The range of **T** is the set of positive numbers in $\mathcal{R}^1$. The nullspace of **T** is the set consisting of all vectors in the domain $\mathcal{R}^2$ which satisfy $\xi_1^2 + \xi_2^2 \leqslant 1$.

Suppose we wish to solve the equation **Tx** = 1 for the transformation of Example 4. In effect, we ask which points in the arrow space are a unit distance from the unit circle—all points on the circle of radius 2. The solution is not unique because **T** assigns to the single number 1 in $\mathcal{R}^1$ more than one vector in $\mathcal{R}^2$. The equation **Tx** = -1, on the other hand, has no solution because **T** does not assign the number -1 in $\mathcal{R}^1$ to any vector in $\mathcal{R}^2$. We now proceed to specify the properties of a transformation which are necessary in order that the transformation be uniquely reversible.

Definition.   Let **T**: $\mathcal{V} \rightarrow \mathcal{W}$. Then **T** is **one-to-one** if

$$\mathbf{x}_1 \neq \mathbf{x}_2 \implies \mathbf{Tx}_1 \neq \mathbf{Tx}_2 \tag{2.21}$$

for all $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathcal{V}$; that is, if **T** does not assign more than one **x** in $\mathcal{V}$ to a single **y** in $\mathcal{W}$.

If **T** is one-to-one, any solution to **Tx** = **y** is unique. It might appear that the effect of **T** is reversible if **T** is one-to-one. The nonreversibility of **T** in Example 4, however, arises only in part because **T** is not one-to-one. In general, there may be vectors in the range of definition $\mathcal{W}$ which are not associated in any way with vectors in $\mathcal{V}$. In point of fact, **range(T)** consists precisely of those vectors **y** in $\mathcal{W}$ for which the equation **Tx** = **y** is solvable. Unless we know which vectors are in **range(T)**, we cannot reverse the transformation.

Definition.   Let **T**: $\mathcal{V} \rightarrow \mathcal{W}$. Then **T** is **onto** if

$$\mathbf{range(T)} = \mathcal{W} \tag{2.22}$$

That is, **T** is onto if every vector **y** in $\mathcal{W}$ is associated with at least one vector **x** in $\mathcal{V}$.

Definition.   If a transformation is one-to-one and onto, then it is **invertible** —it can be reversed uniquely. If T: $\mathcal{V} \rightarrow \mathcal{W}$ is invertible, we define the **inverse of T** to be the transformation $\mathbf{T}^{-1} : \mathcal{W} \rightarrow \mathcal{V}$ which associates with each **y** in $\mathcal{W}$ the unique vector **x** in $\mathcal{V}$ for which **Tx** = **y.** See (2.29) for another characterization of $\mathbf{T}^{-1}$.

Example 5. The Identity Operator, **I.**   Let $\mathcal{V}$ be a vector space. Define the operator **I** on $\mathcal{V}$ by

$$\mathbf{Ix} \overset{\Delta}{=} \mathbf{x} \tag{2.23}$$

for all **x** in $\mathcal{V}$. The nullspace of **I** is $\boldsymbol{\theta}_{\mathcal{V}}$. Range $(\mathbf{I}) = \mathcal{V}$; thus **I** is onto. Furthermore, **I** is one-to-one. Therefore, the identity operator is invertible.

Example 6. The Zero Transformation, $\boldsymbol{\Theta}$. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Define $\boldsymbol{\Theta} \colon \mathcal{V} \to \mathcal{W}$ by

$$\boldsymbol{\Theta}\mathbf{x} \stackrel{\Delta}{=} \boldsymbol{\theta}_{\mathcal{W}} \tag{2.24}$$

for all **x** in $\mathcal{V}$. The nullspace of $\boldsymbol{\Theta}$ is $\mathcal{V}$. The range of $\boldsymbol{\Theta}$ is $\boldsymbol{\theta}_{\mathcal{W}}$. The zero transformation is neither one-to-one nor onto. It is clearly not invertible.

Example 7. A Transformation on a Function Space. Define $\mathbf{T} \colon \mathcal{C}\,(a,b) \to \mathcal{R}^1$ by

$$\mathbf{Tf} \stackrel{\Delta}{=} \int_a^b \mathbf{f}^2(t)\, dt \tag{2.25}$$

for all **f** in $\mathcal{C}\,(a,\,b)$. This transformation specifies an integral-square measure of the size of the function **f**; this measure is used often in judging the performance of a control system. The function **f** is a dummy variable used to define **T**; the scalar t is a dummy variable used to define **f**. In order to avoid confusion, we must carefully distinguish between the concept of the function **f** in the vector space $\mathcal{C}(a,\,b)$ and the concept of the transformation **T** which relates each function **f** in $\mathcal{C}(a,\,b)$ to a vector in $\mathcal{R}^1$. The transformation acts on the whole function **f**—we must use all values of **f** to find **Tf**. The range of **T** is the set of positive numbers in $\mathcal{R}^1$; thus **T** is not onto the range of definition $\mathcal{R}^1$. The nullspace of **T** is the single vector $\boldsymbol{\theta}_{\mathcal{V}}$. If we define $\mathbf{f}_1$ and $\mathbf{f}_2$ by $\mathbf{f}_1(t) = 1$ and $\mathbf{f}_2(t) = -1$, then $\mathbf{Tf}_1 = \mathbf{Tf}_2$; therefore **T** is not one-to-one.

The transformations of Examples 4 and 7 are scalar valued; that is, the range of definition in each case is the space of scalars. We call a scalar-valued transformation a **functional.** Most functionals are not one-to-one.

Example 8. A Transformation for a Dynamic System. Let $\mathcal{C}^2(a,\,b)$ be the space of functions which have continuous second derivatives on $[a,\,b]$. Define $\mathbf{L} \colon \mathcal{C}^2(a,\,b) \to \mathcal{C}\,(a,b)$ by

$$(\mathbf{Lf})(t) \stackrel{\Delta}{=} \mathbf{f}''(t) + \alpha\big(\mathbf{f}(t) + 0.01\mathbf{f}^3(t)\big) \tag{2.26}$$

for all **f** in $\mathcal{C}^2(a,\,b)$ and all t in $[a,\,b]$. This transformation is a model for a particular mass-spring system in which the spring is nonlinear. The comments under Example 7 concerning the dummy variables **f** and t apply here as well. As usual, the definition is given in terms of scalars, functions evaluated at t. Again, **L** acts on the whole function **f**. Even in this example we cannot determine any value of the function **Lf** without using an "interval" of values of **f**, because the derivative

function $\mathbf{f}'$ is defined in terms of a limit of values of $\mathbf{f}$ in the neighborhood of t:

$$\mathbf{f}'(t) \stackrel{\Delta}{=} \lim_{\Delta t \to 0} \frac{\mathbf{f}(t + \Delta t) - \mathbf{f}(t)}{\Delta t}$$

The nullspace of $\mathbf{L}$ consists in all solutions of the nonlinear differential equation, $\mathbf{Lf} = \boldsymbol{\theta}_{\mathfrak{W}}$; restated in terms of the values of $\mathbf{Lf}$, this equation is

$$\mathbf{f}''(t) + \alpha\left(\mathbf{f}(t) + 0.01\mathbf{f}^3(t)\right) = 0 \qquad a \leqslant t \leqslant b$$

To determine these solutions is not a simple task. By selecting $\mathcal{C}(a, b)$ as the range of definition, we ask that the function $\mathbf{Lf}$ be continuous; since $\mathbf{Lf}$ represents a force in the mass-spring system described by (2.26) continuity seems a practical assumption. By choosing $\mathcal{C}^2(a, b)$ as the domain, we guarantee that $\mathbf{Lf}$ is continuous. Yet the range of $\mathbf{L}$ is not clear. It is in the range of definition, but is it equal to the range of definition? In other words, can we solve the nonlinear differential equation $\mathbf{Lf} = \mathbf{u}$ for *any* continuous $\mathbf{u}$? The function $\mathbf{f}$ represents the displacement versus time in the physical mass-spring system. The function $\mathbf{u}$ represents the force applied to the system as a function of time. Physical intuition leads us to believe that for given initial conditions there is a unique displacement pattern $\mathbf{f}$ associated with each continuous forcing pattern $\mathbf{u}$. Therefore, $\mathbf{L}$ should be onto. On the other hand, since no initial conditions are specified, we expect two degrees of freedom in the solution to $\mathbf{Lf} = \mathbf{u}$ for each continuous $\mathbf{u}$. Thus the dimension of nullspace $(\mathbf{L})$ is two, and $\mathbf{L}$ is not one-to-one.

Combining Transformations

The transformation introduced in Example 8 is actually a composite of several simpler transformations. In developing a model for a system, we usually start with simple models for portions of the system, and then combine the parts into the total system model. Suppose $\mathbf{T}$ and $\mathbf{U}$ are both transformations from $\mathcal{V}$ into $\mathcal{W}$. We define the transformation $a\mathbf{T} + b\mathbf{U}$: $\mathcal{V} \to \mathcal{W}$ by

$$(a\mathbf{T} + b\mathbf{U})\mathbf{x} \stackrel{\Delta}{=} a\mathbf{Tx} + b\mathbf{Ux} \qquad (2.27)$$

for all $\mathbf{x}$ in $\mathcal{V}$. If $\mathbf{G}: \mathcal{W} \to \mathcal{U}$, we define the transformation $\mathbf{GT}: \mathcal{V} \to \mathcal{U}$ by

$$(\mathbf{GT})\mathbf{x} \stackrel{\Delta}{=} \mathbf{G}(\mathbf{Tx}) \qquad (2.28)$$

for all $\mathbf{x}$ in $\mathcal{V}$. Equations (2.27) and (2.28) define **linear combination** and **composition** of transformations, respectively.

Example 9. ***Composition of Matrix Multiplications.***   Define $\mathbf{G}: \mathfrak{R}^3 \to \mathfrak{R}^2$ by

$$\mathbf{G}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \triangleq \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

and $\mathbf{T}: \mathfrak{R}^2 \to \mathfrak{R}^3$ by

$$\mathbf{T}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \triangleq \begin{pmatrix} 2 & 2 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

Then $\mathbf{GT}: \mathfrak{R}^2 \to \mathfrak{R}^2$ is described by

$$\mathbf{GT}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \mathbf{G}\begin{pmatrix} 2 & 2 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 \\ 14 & 9 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

Exercise 1. Let $\mathbf{T}: \mathcal{V} \to \mathcal{W}$. Show that $\mathbf{T}$ is invertible if and only if $\mathcal{V} = \mathcal{W}$ and there is a transformation $\mathbf{T}^{-1} : \mathcal{W} \to \mathcal{V}$ such that

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{T}\mathbf{T}^{-1} = \mathbf{I} \tag{2.29}$$

Exercise 2.   Suppose $\mathbf{G}$ and $\mathbf{T}$ of (2.26) are invertible. Show that

$$(\mathbf{GT})^{-1} = \mathbf{T}^{-1}\mathbf{G}^{-1} \tag{2.30}$$

The composition (or product) of two transformations has two nasty characteristics. First, unlike scalars, transformations usually **do not commute;** that is, $\mathbf{GT} \neq \mathbf{TG}$. As illustrated in Example 9, $\mathbf{G}$ and $\mathbf{T}$ generally do not even act on the same vector space, and $\mathbf{TG}$ has no meaning. Even if $\mathbf{G}$ and $\mathbf{T}$ both act on the same space, we must not expect commutability, as demonstrated by the following matrix multiplications:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Commutable operators do exist. In fact, since any operator commutes with itself, we can write $\mathbf{G}^2$, as we do in Example 10 below, without being ambiguous. Operators which commute act much like scalars in their behavior toward each other (see P&C 4.29).

If two scalars satisfy $ab = 0$, then either $a = 0$, $b = 0$, or both. The second matrix multiplication above demonstrates that this property does not extend even to simple transformations. This second difficulty with the composition of transformations is sometimes called the existence of **divisors of zero.** If $\mathbf{GT} = \mathbf{\Theta}$ and $\mathbf{G} \neq \mathbf{\Theta}$, we cannot conclude that $\mathbf{T} = \mathbf{\Theta}$; the cancellation laws of algebra do not apply to transformations. The difficulty lies in the fact that for transformations there is a "gray" region between being invertible and being zero. The range of $\mathbf{T}$ can lie in the nullspace of $\mathbf{G}$.

Example 10. Linear Combination and Composition of Transformations. The space $\mathcal{C}^n \ (a, b)$ consists in all functions with continuous nth derivatives on $[a, b]$. Define $\mathbf{G}: \mathcal{C}^n \ (a, b) \to \mathcal{C}^{n-1}(a, b)$ by $\mathbf{Gf} \overset{\Delta}{=} \mathbf{f}'$ for all $\mathbf{f}$ in $\mathcal{C}^n \ (a, b)$. Then $\mathbf{G}^2: \mathcal{C}^2(a, b) \to \mathcal{C} \ (a, b)$ is well defined. Let $\mathbf{U}: \mathcal{C}^2(a, b) \to \mathcal{C} \ (a, b)$ be defined by $(\mathbf{Uf})(t) \overset{\Delta}{=} \mathbf{f}(t) + 0.01\mathbf{f}^3(t)$ for all $\mathbf{f}$ in $\mathcal{C}^2(a, b)$ and all t in $[a, b]$. The transformation $\mathbf{L}$ of Example 8 can be described by $\mathbf{L} \overset{\Delta}{=} \mathbf{G}^2 + \alpha\mathbf{U}$.

As demonstrated by the above examples, the domain and range of definition are essential parts of the definition of a transformation. This importance is emphasized by the notation $\mathbf{T}: \mathcal{V} \to \mathcal{W}$. The spaces $\mathcal{V}$ and $\mathcal{W}$ are selected to fit the structure of the situation we wish to model. If we pick a domain that is too large, the operator will not be one-to-one. If we pick a range of definition that is too large, the operator will not be onto. Thus both $\mathcal{V}$ and $\mathcal{W}$ affect the invertibility of $\mathbf{T}$. We apply loosely the term *finite (infinite) dimensional transformation* to those transformations that act on a finite (infinite) dimensional domain.

2.4 Linear Transformations

One of the most common and useful transformations is the matrix multiplication introduced in Chapter 1. It is well suited for automatic computation using a digital computer. Let $\mathbf{A}$ be an $m \times n$ matrix. We define $\mathbf{T}: \mathcal{M}^{n \times 1} \to \mathcal{M}^{m \times 1}$ by

$$\mathbf{Tx} \overset{\Delta}{=} \mathbf{Ax} \tag{2.3 1}$$

for all $\mathbf{x}$ in $\mathcal{M}^{n \times 1}$. We distinguish carefully between $\mathbf{T}$ and $\mathbf{A}$. $\mathbf{T}$ is not $\mathbf{A}$, but rather *multiplication* by $\mathbf{A}$. The nullspace of $\mathbf{T}$ is the set of solutions to