the matrix equation $\mathbf{Ax}=\boldsymbol{\theta}$. Even though $\mathbf{T}$ and $\mathbf{A}$ are conceptually different, we sometimes refer to the nullspace of $\mathbf{T}$ as the nullspace of $\mathbf{A}$. Similarly, we define $\text{range}(\mathbf{A}) \triangleq \text{range}(\mathbf{T})$.

Suppose $\mathbf{A}$ is square $(m=n)$ and invertible; then the equation $\mathbf{Tx} = \mathbf{Ax} = \mathbf{y}$ has a unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$ for each $\mathbf{y}$ in $\mathfrak{M}^{n\times 1}$. But $\mathbf{T}^{-1}$ is defined as precisely that transformation which associates with each $\mathbf{y}$ in $\mathfrak{M}^{n\times 1}$ the unique solution to the equation $\mathbf{Tx}=\mathbf{y}$. Therefore, $\mathbf{T}$ is invertible, and $\mathbf{T}^{-1}$: $\mathfrak{M}^{m\times 1}\to\mathfrak{M}^{n\times 1}$ is given by $\mathbf{T}^{-1}\mathbf{y} \triangleq \mathbf{A}^{-1}\mathbf{y}$.

The properties of matrix multiplication (Appendix 1) are such that $\mathbf{A}(a\mathbf{x}_1 + b\mathbf{x}_2) = a\mathbf{Ax}_1 + b\mathbf{Ax}_2$. That is, matrix multiplication preserves linear combinations. This property of matrix multiplication allows **superposition** of solutions to a matrix equation: if $\mathbf{x}_1$ solves $\mathbf{Ax}=\mathbf{y}_1$ and $\mathbf{x}_2$ solves $\mathbf{Ax}=\mathbf{y}_2$, then the solution to $\mathbf{Ax}=\mathbf{y}_1+\mathbf{y}_2$ is $\mathbf{x}_1+\mathbf{x}_2$. From one or two input-output relationships we can infer others. Many other familiar transformations preserve linear combinations and allow superposition of solutions.

Definition. The transformation $\mathbf{T}$: $\mathcal{V}\to\mathcal{W}$ is **linear** if

$$\mathbf{T}(a\mathbf{x}_1 + b\mathbf{x}_2) = a\mathbf{Tx}_1 + b\mathbf{Tx}_2 \tag{2.32}$$

for all vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathcal{V}$ and all scalars a and b.

Example 1. Integration. Define $\mathbf{T}$: $\mathcal{C}(0, 1)\to\mathcal{C}(0, 1)$ by

$$(\mathbf{Tf})(t) \triangleq \int_0^t \mathbf{f}(s)\,ds \tag{2.33}$$

for all $\mathbf{f}$ in $\mathcal{C}(0, 1)$ and all t in $[0, 1]$. The linearity of this indefinite integration operation is a fundamental fact of integral calculus; that is,

$$\int_0^t [a\mathbf{f}_1(s)+b\mathbf{f}_2(s)]\,ds = a\int_0^t \mathbf{f}_1(s)\,ds + b\int_0^t \mathbf{f}_2(s)\,ds$$

The operator (2.33) is a special case of the linear integral operator $\mathbf{T}$: $\mathcal{C}(a, b)\to\mathcal{C}(c, d)$ defined by

$$(\mathbf{Tf})(t) \triangleq \int_a^b k(t,s)\mathbf{f}(s)\,ds \tag{2.34}$$

for all $\mathbf{f}$ in $\mathcal{C}(a,b)$ and all t in $[c,d]$. We can substitute for the domain $\mathcal{C}(a,b)$ any other space of functions for which the integral exists. We can use any range of definition which includes the integrals (2.34) of all functions in the domain. The function k is called the **kernel** of the integral transformation. Another special case of (2.34) is $\mathbf{T}$: $\mathcal{L}_2(-\infty,\infty)\to\mathcal{L}_2(-\infty,\infty)$ defined by

$$(\mathbf{Tf})(t) \triangleq \int_{-\infty}^{\infty} \mathbf{g}(t-s)\mathbf{f}(s)\,ds$$

for some $\mathbf{g}$ in $\mathcal{L}_2(-\infty, \infty)$, all $\mathbf{f}$ in $\mathcal{L}_2(-\infty, \infty)$, and all t in $(-\infty, \infty)$. This $\mathbf{T}$ is known as the convolution of $\mathbf{f}$ with the function $\mathbf{g}$. It arises in connection with the solution of linear constant-coefficient differential equations (Appendix 2).

The integral transformation (2.34) is the analogue for function spaces of the matrix multiplication (2.31). That matrix transformation can be expressed

$$(\mathbf{Tx})_i \triangleq \sum_{j=1}^{n} \mathbf{A}_{ij}\xi_j \qquad i=1,\ldots,m \tag{2.35}$$

for all vectors $\mathbf{x}$ in $\mathfrak{M}^{n\times 1}$. The symbol ξ_j represents the jth element of $\mathbf{x}$; the symbol $(\mathbf{Tx})_i$ means the ith element of $\mathbf{Tx}$. In (2.35) the matrix is treated as a function of two discrete variables, the row variable i and the column variable j. In analogy with the integral transformation, we call the matrix multiplication [as viewed in the form of (2.35)] a **summation transformation;** we refer to the function $\mathbf{A}$ (with values $\mathbf{A}_{ij}$) as the **kernel** of the summation transformation.

Example 2. Differentiation Define $\mathbf{D}$: $\mathcal{C}^1(a, b)\to\mathcal{C}(a, b)$ by

$$(\mathbf{Df})(t) \triangleq \mathbf{f}'(t) \triangleq \lim_{\Delta t\to 0} \frac{\mathbf{f}(t+\Delta t)-\mathbf{f}(t)}{\Delta t} \tag{2.36}$$

for all $\mathbf{f}$ in $\mathcal{C}^1(a,b)$ and all t in $[a,b]$; $\mathbf{f}'(t)$ is the slope of the graph of $\mathbf{f}$ at t; $\mathbf{f}'$ (or $\mathbf{Df}$) is the whole "slope" function. We also use the symbols $\dot{\mathbf{f}}$ and $\mathbf{f}^{(1)}$ in place of $\mathbf{Df}$. We can substitute for the above domain and range of definition any pair of function spaces for which the derivatives of all functions in the domain lie in the range of definition. Thus we could define $\mathbf{D}$ on $\mathcal{C}(a, b)$ if we picked a range of definition which contains the appropriate discontinuous functions. The nullspace of $\mathbf{D}$ is $\operatorname{span}\{\mathbf{1}\}$, where $\mathbf{1}$ is the function defined by $\mathbf{1}(t)=1$ for all t in $[a,b]$. It is well known that differentiation is linear; $\mathbf{D}(c_1\mathbf{f}_1 + c_2\mathbf{f}_2) = c_1\mathbf{Df}_1 + c_2\mathbf{Df}_2$.

We can define more general differential operators in terms of (2.36). The general linear constant-coefficient differential operator $\mathbf{L}$: $\mathcal{C}^n(a, b)\to\mathcal{C}(a, b)$ is defined, for real scalars $\{a_i\}$, by

$$\mathbf{L} \triangleq \mathbf{D}^n + a_1\mathbf{D}^{n-1} + \cdots + a_n\mathbf{I} \tag{2.37}$$

where we have used (2.27) and (2.28) to combine transformations. A **variable-coefficient** (or "time-varying") extension of (2.37) is the operator $\mathbf{L}$: $\mathcal{C}^n(a, b)\to\mathcal{C}(a,b)$ defined by*

$$(\mathbf{Lf})(t) \triangleq g_0(t)\mathbf{f}^{(n)}(t) + g_1(t)\mathbf{f}^{(n-1)}(t) + \cdots + g_n(t)\mathbf{f}(t) \tag{2.37}$$

*Note that we use boldface print for some of the functions in (2.38) but not for others. As indicated in the Preface, we use boldface print only to emphasize the vector or transformation interpretation of an object. We sometimes describe the same function both ways, $\mathbf{f}$ and f.

for all $\mathbf{f}$ in $\mathcal{C}^n(a,b)$ and all t in $[a,b]$. (We have denoted the kth derivative $\mathbf{D}^k\mathbf{f}$ by $\mathbf{f}^{(k)}$.) If the interval $[a,b]$ is finite, if the functions g_i are continuous, and if $g_0(t)\neq 0$ on $[a,b]$, we refer to (2.38) as a regular nth-order differential operator. [With $g_0(t)\neq 0$, we would lose no generality by letting $g_0(t)=1$ in (2.38).] We can apply the differential operators (2.37) and (2.38) to other function spaces than $\mathcal{C}^n(a,b)$.

Example 3. Evaluation of a Function. Define $\mathbf{T}\colon \mathcal{C}(a,b)\to \mathcal{R}^1$ by

$$\mathbf{T}\mathbf{f} \triangleq \mathbf{f}(t_1) \tag{2.39}$$

for all $\mathbf{f}$ in the function space $\mathcal{C}(a,b)$. In this example, $\mathbf{f}$ is a dummy variable, but t_1 is not. The transformation is a *linear functional* called "evaluation at t_1." The range of $\mathbf{T}$ is $\mathcal{R}^1$; $\mathbf{T}$ is onto. The nullspace of $\mathbf{T}$ is the set of continuous functions which pass through zero at t_1. Because many functions have the same value at t_1, $\mathbf{T}$ is not one-to-one. This functional can also be defined using some other function space for its domain.

Example 4. A One-Sided Laplace Transform, $\mathfrak{L}$. Suppose $\mathcal{W}$ is the space of complex-valued functions defined on the positive-real half of the complex plane. (See Example 10, Section 2.1.) Let $\mathcal{V}$ be the space of functions which are defined and continuous on $[0,\infty]$ and for which $e^{-ct}|f(t)|$ is bounded for some constant c and all values of t greater than some finite number. We define the one-sided Laplace transform $\mathfrak{L}\colon \mathcal{V}\to\mathcal{W}$ by

$$(\mathfrak{L}\mathbf{f})(s) \triangleq \int_0^\infty e^{-st}\,\mathbf{f}(t)\,dt \tag{2.40}$$

for all complex s with $\mathrm{real}(s)>0$. The functions in $\mathcal{V}$ are such that (2.40) converges for $\mathrm{real}(s)>0$. We sometimes denote the transformed function $\mathfrak{L}\mathbf{f}$ by $\mathbf{F}$. This integral transform, like that of (2.34), is linear. The Laplace transform is used to convert linear constant-coefficient differential equations into linear algebraic equations. •

Exercise 1. Suppose the transformations $\mathbf{T}$, $\mathbf{U}$, and G of (2.27) and (2.28) are linear and $\mathbf{T}$ is invertible. Show that the transformations $a\mathbf{T}+b\mathbf{U}$, $\mathbf{GT}$, and $\mathbf{T}^{-1}$ are also linear.

Exercise 2. Let $\mathcal{V}$ be an n-dimensional linear space with basis $\mathcal{X}$. Define $\mathbf{T}\colon \mathcal{V}\to \mathcal{M}^{n\times 1}$ by

$$\mathbf{T}\mathbf{x} \triangleq [\mathbf{x}]_{\mathcal{X}} \tag{2.41}$$

Show that $\mathbf{T}$, the process of taking coordinates, is a linear, invertible transformation.

*It can be shown that $[\mathfrak{L}(\mathbf{D}\mathbf{f})](s)=s(\mathfrak{L}\mathbf{f})(s)-\mathbf{f}(0^+)$, where $\mathbf{f}(0^+)$ is the limit of $\mathbf{f}(t)$ as $t\to 0$ from the positive side of 0.

The vector space $\mathcal{V}$ of Exercise 2 is equivalent to $\mathfrak{M}^{n\times 1}$ in every sense we might wish. The linear, invertible transformation is the key. We say two vector spaces $\mathcal{V}$ and $\mathcal{W}$ are **isomorphic** (or equivalent) if there exists an invertible linear transformation from $\mathcal{V}$ into $\mathcal{W}$. Each real n-dimensional vector space is isomorphic to each other real n-dimensional space and, in particular, to the real space $\mathfrak{M}^{n\times 1}$. A similar statement can be made using complex scalars for each space. Infinite-dimensional spaces also exhibit isomorphism. In Section 5.3 we show that all well behaved infinite-dimensional spaces are isomorphic to l_2.

Nullpace and Range—Keys to Invertibility

Even *linear* transformations may have troublesome properties. In point of fact, the example in which we demonstrate *noncommutability* and *noncancellation* of products of transformations uses linear transformations (matrix multiplications). Most difficulties with a linear transformation can be understood through investigation of the range and nullspace of the transformation.*

Let $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ be linear. Suppose $\mathbf{x}_h$ is a vector in the nullspace of $\mathbf{T}$ (any solution to $\mathbf{Tx} = \boldsymbol{\theta}$); we call $\mathbf{x}_h$ a homogeneous solution for the transformation $\mathbf{T}$. Denote by $\mathbf{x}_p$ a particular solution to the equation $\mathbf{Tx} = \mathbf{y}$. (An $\mathbf{x}_p$ exists if and only if $\mathbf{y}$ is in $\text{range}(\mathbf{T})$.) Then $\mathbf{x}_p + \alpha\mathbf{x}_h$ is also a solution to $\mathbf{Tx} = \mathbf{y}$ for any scalar α. One of the most familiar uses of the principle of superposition is in obtaining the general solution to a linear differential equation by combining particular and homogeneous solutions. The general solution to any linear operator equation can be obtained in this manner.

Example 5. The General Solution to a Matrix Equation. Define the linear operator $\mathbf{T}: \mathfrak{M}^{2\times 1} \to \mathfrak{M}^{2\times 1}$ by

$$\mathbf{T}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \triangleq \begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

Then the equation

$$\mathbf{Tx} = \begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}2\\ 2\end{pmatrix} \triangleq \mathbf{y} \tag{2.42}$$

has as its general solution $\mathbf{x} = (\xi_1 \quad 2-2\xi_1)$. A particular solution is $\mathbf{x}_p = (1 \quad 0)^{\mathrm{T}}$. The nullspace of $\mathbf{T}$ consists in the vector $\mathbf{x}_h = (-1 \quad 2)^{\mathrm{T}}$ and all its multiples. The general solution can be expressed as $\mathbf{x} = \mathbf{x}_p + \alpha\mathbf{x}_h$ where α is arbitrary. Figure 2.7 shows an

*See Sections 4.4 and 4.6 for further insight into noncancellation and noncommutability of linear operators.

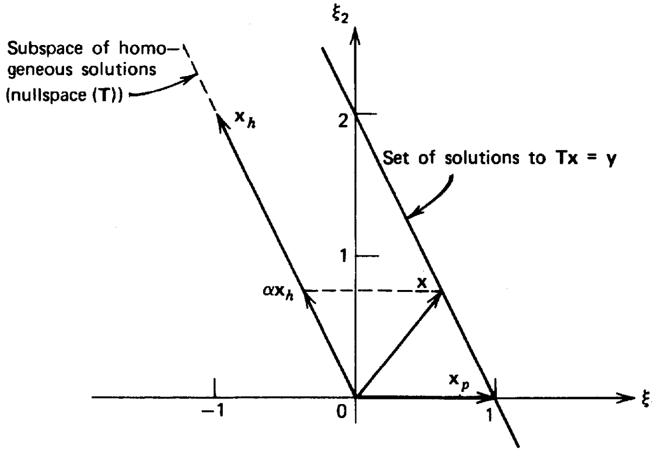

Figure 2.7. Solutions to the linear equation of Example 5.

arrow-space equivalent of these vectors. The nullspace of $\mathbf{T}$ is a subspace of $\mathfrak{M}^{2\times 1}$. The general solution (the set of all solutions to $\mathbf{Tx} = \mathbf{y}$) consists of a line in $\mathfrak{M}^{2\times 1}$; specifically, it is the nullspace of $\mathbf{T}$ shifted by the addition of any particular solution.

The nullspace of a linear transformation is always a subspace of the domain $\mathcal{V}$. The freedom in the general solution to $\mathbf{Tx} = \mathbf{y}$ lies only in **nullspace(T)**, the subspace of homogeneous solutions. For if $\hat{\mathbf{x}}_p$ is another particular solution to $\mathbf{Tx} = \mathbf{y}$, then

$$\mathbf{T}(\mathbf{x}_p - \hat{\mathbf{x}}_p) = \mathbf{Tx}_p - \mathbf{T}\hat{\mathbf{x}}_p = \mathbf{y} - \mathbf{y} = \boldsymbol{\theta}$$

The difference between $\mathbf{x}_p$ and $\hat{\mathbf{x}}_p$ is a vector in **nullspace(T)**. If **nullspace(T)** = $\boldsymbol{\theta}$, there is no freedom in the solution to $\mathbf{Tx}$ = y; it is unique.

Definition. A transformation $\mathbf{G}: \mathcal{V} \rightarrow \mathcal{W}$ is nonsingular if **nullspace(G)** = $\boldsymbol{\theta}$.

Exercise 3. Show that a *linear* transformation is one-to-one if and only if it is nonsingular.

Because a linear transformation $\mathbf{T}: \mathcal{V} \rightarrow \mathcal{W}$ preserves linear combinations, it necessarily transforms $\boldsymbol{\theta}_{\mathcal{V}}$ into $\boldsymbol{\theta}_{\mathcal{W}}$. Furthermore, $\mathbf{T}$ acts on the vectors in $\mathcal{V}$ by subspaces—whatever $\mathbf{T}$ does to $\mathbf{x}$ it does also to $c\mathbf{x}$, where c is any scalar. The set of vectors in $\mathcal{V}$ which are taken to zero, for example, is the subspace which we call **nullspace(T)**. Other subspaces of $\mathcal{V}$ are "rotated" or "stretched" by $\mathbf{T}$. This fact becomes more clear during our discussion of spectral decomposition in Chapter 4.

Example 6. The Action of a Linear Transformation on Subspaces. Define $\mathbf{T}$: $\mathcal{R}^3 \rightarrow \mathcal{R}^2$ by $\mathbf{T}(\xi_1,\xi_2,\xi_3) \triangleq (\xi_3,0)$. The set $\{\mathbf{x}_1 = (1,0,0), \mathbf{x}_2 = (0,1,0)\}$ forms a basis for **nullspace(T)**. By adding a third independent vector, say, $\mathbf{x}_3 = (1, 1, 1)$, we obtain a basis for the domain $\mathcal{R}^3$. The subspace spanned by $\{\mathbf{x}_1,\mathbf{x}_2\}$ is annihilated by $\mathbf{T}$. The subspace spanned by $\{\mathbf{x}_3\}$ is transformed by $\mathbf{T}$ into a subspace of $\mathcal{R}^2$—the range of $\mathbf{T}$. The vector $\mathbf{x}_3$ itself is transformed into a basis for **range(T)**. Because $\mathbf{T}$ acts on the vectors in $\mathcal{R}^3$ by subspaces, the dimension of **nullspace(T)** is a measure of the degree to which $\mathbf{T}$ acts like zero; the dimension of **range(T)** indicates the degree to which $\mathbf{T}$ acts invertible. Specifically, of the three dimensions in $\mathcal{R}^3$, $\mathbf{T}$ takes two to zero. The third dimension of $\mathcal{R}^3$ is taken into the one-dimensional **range(T)**.

The characteristics exhibited by Example 6 extend to any linear transformation on a finite-dimensional space, Let $\mathbf{T}: \mathcal{V} \rightarrow \mathcal{W}$ be linear with $\dim(\mathcal{V}) = n$. We call the dimension of **nullspace(T)** the **nullity of T**. The **rank of T** is the dimension of **range(T)**. Let $\{\mathbf{x}_1,\ldots,\mathbf{x}_k\}$ be a basis for **nullspace(T)**. Pick vectors $\{\mathbf{x}_{k+1},\ldots,\mathbf{x}_n\}$ which extend the basis for **nullspace(T)** to a basis for $\mathcal{V}$ (P&C 2.9). We show that $\mathbf{T}$ takes $\{\mathbf{x}_{k+1},\ldots,\mathbf{x}_n\}$ into a basis for **range(T)**. Suppose $\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$ is an arbitrary vector in $\mathcal{V}$. The linear transformation $\mathbf{T}$ annihilates the first k components of $\mathbf{x}$. Only the remaining $n-k$ components are taken into **range(T)**. Thus the vectors $\{\mathbf{T}\mathbf{x}_{k+1},\ldots,\mathbf{T}\mathbf{x}_n\}$ must span **range(T)**. To show that these vectors are independent, we use the test (2.11):

$$\xi_{k+1}(\mathbf{T}\mathbf{x}_{k+1}) + \cdots + \xi_n(\mathbf{T}\mathbf{x}_n) = \boldsymbol{\theta}_{\mathcal{W}}$$

Since $\mathbf{T}$ is linear,

$$\mathbf{T}(\xi_{k+1}\mathbf{x}_{k+1} + \cdots + \xi_n\mathbf{x}_n) = \boldsymbol{\theta}_{\mathcal{W}}$$

Then $\xi_{k+1}\mathbf{x}_{k+1} + \cdots + \xi_n\mathbf{x}_n$ is in **nullspace(T)**, and

$$\xi_{k+1}\mathbf{x}_{k+1} + \cdots + \xi_n\mathbf{x}_n = d_1\mathbf{x}_1 + \cdots + d_k\mathbf{x}_k$$

for some $\{d_i\}$. The independence of $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ implies $d_1 = \cdots = d_k = \xi_{k+1} = \ldots = \xi_n = 0$; thus $\{\mathbf{T}\mathbf{x}_{k+1},\ldots,\mathbf{T}\mathbf{x}_n\}$ is an independent set and is a basis for **range(T)**.

We have shown that a linear transformation T acting on a finite-dimensional space $\mathcal{V}$ obeys a "conservation of dimension" law:

$$\dim(\mathcal{V}) = \mathbf{rank}(\mathbf{T}) + \mathbf{nullity}(\mathbf{T}) \tag{2.43}$$

Nullity(T) is the "dimension" annihilated by $\mathbf{T}$. **Rank(T)** is the "dimension" $\mathbf{T}$ retains. If **nullspace(T)** $= \{\boldsymbol{\theta}\}$, then **nullity(T)** $= 0$ and **rank(T)** $= \dim(\mathcal{V})$. If, in addition, $\dim(\mathcal{W}) = \dim(\mathcal{V})$, then **rank(T)** $= \dim(\mathcal{W})$ (**T** is onto), and $\mathbf{T}$ is invertible. A linear $\mathbf{T}: \mathcal{V} \to \mathcal{W}$ cannot be invertible unless $\dim(\mathcal{W}) = \dim(\mathcal{V})$.

We sometimes refer to the vectors $\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n$ as **progenitors of the range of T**. Although the nullspace and range of $\mathbf{T}$ are unique, the space spanned by the progenitors is not; we can add any vector in nullspace to any progenitor without changing the basis for the range (see Example 6).

The Near Nullspace

In contrast to mathematical analysis, mathematical *computation* is not clear-cut. For example, a set of equations which is mathematically invertible can be so "nearly singular" that the inverse cannot be computed to an acceptable degree of precision. On the other hand, because of the finite number of significant digits used in the computer, a mathematically singular system will be indistinguishable from a "nearly singular" system. The phenomenon merits serious consideration.

The matrix operator of Example 5 is singular. Suppose we modify the matrix slightly to obtain the nonsingular, but "nearly singular" matrix equation

$$\begin{pmatrix} 2 & 1 \\ 2 & 1+\epsilon \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{2.44}$$

where ϵ is small. Then the arrow space diagram of Figure 2.7 must also be modified to show a pair of almost parallel lines. (Figure 1.7 of Section 1.5 is the arrow space diagram of essentially this pair of equations.) Although the solution (the intersection of the nearly parallel lines) is unique, it is difficult to compute accurately; the nearly singular equations are very ill conditioned. Slight errors in the data and roundoff during computing lead to significant uncertainty in the computed solution, even if the computation is handled carefully (Section 1.5). The uncertain component of the solution lies essentially in the nullspace of the operator; that is, it is almost parallel to the nearly parallel lines in the arrow-space diagram. The above pair of nearly singular algebraic equations might represent a nearly singular system. On the other hand, the underlying system might be precisely singular; the equations in the model of a singular system may be only *nearly singular* because of inaccuracies in the data. Regardless of which of these interpretations is correct, determining the "near nullspace" of the matrix is an important part of the analysis of the system. If the underlying system is singular, a description of the near nullspace is a description of the *freedom* in the solutions for the system. If the underlying system is just nearly singular, a description of the near nullspace is a description of the *uncertainty* in the solution.

Definition. Suppose $\mathbf{T}$ is a *nearly singular* linear operator on a vector space $\mathcal{V}$. We use the term **near nullspace of T** to mean those vectors that are taken *nearly* to zero by $\mathbf{T}$; that is, those vectors which $\mathbf{T}$ drastically reduces in "size."*

In the two-dimensional example described above, the near nullspace consists in vectors which are *nearly* parallel to the vector $\mathbf{x} = (-1\ \ 2)^{\mathrm{T}}$. The near nullspace of $\mathbf{T}$ is *not a subspace* of $\mathcal{V}$. Rather, it consists in a set of vectors which are *nearly* in a subspace of $\mathcal{V}$. We can think of the near nullspace as a "fuzzy" subspace of $\mathcal{V}$.

We now present a method, referred to as **inverse iteration,** for describing the near nullspace of a nearly singular operator $\mathbf{T}$ acting on a vector space $\mathcal{V}$. Let $\mathbf{x}_0$ be an arbitrary vector in $\mathcal{V}$. Assume $\mathbf{x}_0$ contains a component which is in the near nullspace of $\mathbf{T}$. (If it does not, such a component will be introduced by roundoff during the ensuing computation.) Since $\mathbf{T}$ reduces such components drastically, compared to its effect on the other components of $\mathbf{x}_0$, $\mathbf{T}^{-1}$ must drastically emphasize such components. Therefore, if we solve $\mathbf{T}\mathbf{x}_1 = \mathbf{x}_0$ (in effect determining $\mathbf{x}_1 = \mathbf{T}^{-1}\mathbf{x}_0$), the computed solution $\mathbf{x}_1$ contains a *significant* component in the near nullspace of $\mathbf{T}$. (This component is the error vector which appears during the solution of the nearly singular equation.) The inverse iteration method consists in iteratively solving $\mathbf{T}\mathbf{x}_{k+1} = \mathbf{x}_k$. After a few iterations, $\mathbf{x}_k$ is dominated by its near-nullspace component; we use $\mathbf{x}_k$ as a partial basis for the near nullspace of $\mathbf{T}$. (The number of iterations required is at the discretion of the analyst. We are not looking for a precisely defined subspace, but rather, a subspace that is fuzzy.) By repeating the above process for several different starting vectors $\mathbf{x}_0$, we usually obtain a set of vectors which spans the near nullspace of $\mathbf{T}$.

Example 7. Describing a Near Nullspace. Define a linear operator $\mathbf{T}$ on $\mathfrak{M}^{2\times 1}$ by means of the nearly singular matrix multiplication described above:

$$\mathbf{T}\mathbf{x} \triangleq \begin{pmatrix} 2 & 1 \\ 2 & 1+\epsilon \end{pmatrix}\mathbf{x}$$

For this simple example we can invert $\mathbf{T}$ explicitly

$$\mathbf{T}^{-1}\mathbf{x} = \frac{1}{2\epsilon}\begin{pmatrix} 1+\epsilon & -1 \\ -2 & 2 \end{pmatrix}\mathbf{x}$$

We apply the inverse iteration method to the vector $\mathbf{x}_0 = (1\ \ 1)^{\mathrm{T}}$; of course, we have no roundoff in our computations:

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \frac{1}{2\epsilon}\begin{pmatrix} (1+\epsilon)/2 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \frac{1}{(2\epsilon)^2}\begin{pmatrix} (\epsilon^2+2\epsilon+3)/2 \\ -(\epsilon+3) \end{pmatrix}, \ldots$$

*In Section 4.2 we describe the near nullspace more precisely as the eigenspace for the smallest eigenvalue of $\mathbf{T}$.