Strong Induction Principles in the Locally Nameless Representation of Binders (Work in Progress)

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Outline

"Once upon a time" the contenders in the POPLmark Challenge always made the claim that their approach to binding is the best and all the others are really, really horrible.

Jeremy Avigad asked me recently "...what was this POPLmark scandal about?"

Now, general techniques and tools (like Ott) seem to emerge that are independent of the representation of binders.

I will show that a nominal technique can be used in the locally nameless representation.

I did/do not know anything about locally nameless representation (Randy, Xavier, Arthur, James, Stephanie...).

Strong Induction Principles

Strong induction principles are designed to (only) deal with the variable convention in proofs.

Substitution Lemma: If $x \not\equiv y$ and $x \not\in FV(L)$, then $M[x := N][y := L] \equiv M[y := L][x := N[y := L]].$

Proving the case $\lambda z.M_1$: "...By the variable convention we may assume that $z \not\equiv x, y$ and z is not free in N, L."

(nominal_induct M avoiding: $x \ y \ N \ L$ rule: lam.induct)

Then in the lambda-case one can assume that $z \ \# \ (x,y,N,L)$ holds.

Strong induction principles are used all over the place in nominal verifications.

Strong Rule Inductions

Strong induction principles derived for structural and rule inductions.

Weakening Lemma: If $\Gamma \vdash t : T$, valid Γ' and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash t : T$.

(nominal_induct Γ t T avoiding: Γ' rule: typing.strong_induct)

Then in the typing rule for lambdas, one can assume that $x \ \# \ \Gamma'$ holds.

The main point of the strong induction principles: one does not prove the lemma for all binders, but only for some which satisfy additional freshnessconstraints (our take on the variable convention).

An Interesting Relation

- As far as I know, in the literature the variable convention concerns binders only.
- Crary, however, describes rules for equivalence-checking:

$$\begin{array}{c|c} \underbrace{s \Downarrow p \quad t \Downarrow q \quad \Gamma \vdash p \leftrightarrow q : tbase}{\Gamma \vdash s \Leftrightarrow t : tbase} \\ \hline \Gamma \vdash s \Leftrightarrow t : tbase \\ \hline x \# (\Gamma, s, t) \quad (x, T_1) :: \Gamma \vdash \mathsf{App} \, s \, (\mathsf{Var} \, x) \Leftrightarrow \mathsf{App} \, t \, (\mathsf{Var} \, x) : T_2 \\ \hline \Gamma \vdash s \Leftrightarrow t : T_1 \to T_2 \\ \hline \mathsf{valid} \, \Gamma \quad (x, T) \in \Gamma \\ \hline \Gamma \vdash \mathsf{Var} \, x \leftrightarrow \mathsf{Var} \, x : T \\ \hline \Gamma \vdash \mathsf{Var} \, x \leftrightarrow \mathsf{Var} \, x : T \\ \hline \Gamma \vdash \mathsf{p} \leftrightarrow q : T_1 \to T_2 \quad \Gamma \vdash s \Leftrightarrow t : T_1 \\ \hline \Gamma \vdash \mathsf{App} \, p \, s \leftrightarrow \mathsf{App} \, q \, t : T_2 \\ \hline \mathsf{valid} \, \Gamma \\ \hline \Gamma \vdash s \Leftrightarrow t : tunit \quad \hline \Gamma \vdash \mathsf{Const} \, n \leftrightarrow \mathsf{Const} \, n : tbase \\ \mathsf{Edinburgh, 29. May 2007 - p.5 (1/3)} \end{array}$$

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$$\frac{s \Downarrow p \quad t \Downarrow q \quad \Gamma \vdash p \leftrightarrow q : tbase}{\Gamma \vdash s \Leftrightarrow t : tbase}$$

$$x \# (\Gamma, s, t) \quad (x, T_1) :: \Gamma \vdash App \ s \ (Var \ x) \Leftrightarrow App \ t \ (Var \ x) : T_2$$

$$\Gamma \vdash s \Leftrightarrow t : T_1 \rightarrow T_2$$

$$\frac{valid \ \Gamma \quad (x, T) \in \Gamma}{\Gamma \vdash Var \ x \leftrightarrow Var \ x : T}$$

$$\frac{\Gamma \vdash p \leftrightarrow q : T_1 \rightarrow T_2 \quad \Gamma \vdash s \Leftrightarrow t : T_1}{\Gamma \vdash App \ p \ s \leftrightarrow App \ q \ t : T_2}$$

$$\frac{valid \ \Gamma}{\Gamma \vdash s \Leftrightarrow t : tunit} \qquad \frac{valid \ \Gamma}{\Gamma \vdash Const \ n \leftrightarrow Const \ n : tbase}$$

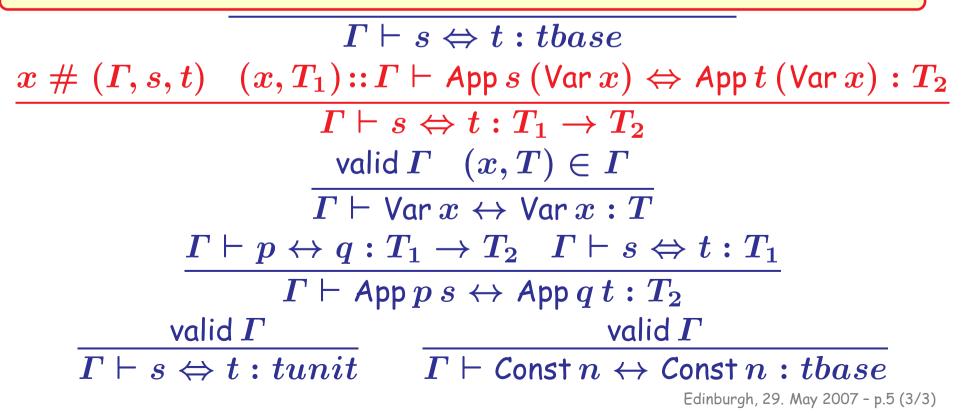
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... and proves monotonicity (kind of weakening) in a logical relation proof:

If $\Gamma \vdash s \Leftrightarrow t: T$, valid Γ' and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash s \Leftrightarrow t: T$ and

 $\text{ if } \Gamma \vdash s \leftrightarrow t:T \text{ , valid } \Gamma' \text{ and } \Gamma \subseteq \Gamma' \text{ , then } \Gamma' \vdash s \leftrightarrow t:T. \\$

In case of the extensionality rule, one needs the fact that x is fresh for Γ' (otherwise one has to rename).



VC-Compatibility

You can indeed use a variable convention for ${m x}$ in:

 $\frac{x \ \# \ (\Gamma, s, t) \quad (x, T_1) :: \Gamma \vdash \operatorname{App} s \ (\operatorname{Var} x) \Leftrightarrow \operatorname{App} t \ (\operatorname{Var} x) : T_2}{\Gamma \vdash s \Leftrightarrow t : T_1 \to T_2}$

- The reason is that x cannot appear freely in the conclusion of this rule.
- We identified conditions for when the variable convention is safe to use (described later on). These conditions also apply to non-binders.

Locally Nameless
 The lambda-calculus in the locally nameless approach:

Ilam = Var string
 Bnd nat
 App Ilam Ilam
 Lam Ilam

Has nice properties: e.g. represents alpha-equivalence classes in a canonical way; but needs a well-formed predicate

is a favourite with some people (not really with me, but this is not the point!!!)

Typing Relation in LN

Typing-rules in the locally nameless approach are specified as:

$$\frac{(x \colon T) \in \Gamma \ \text{ valid } \Gamma}{\Gamma \vdash \text{Var} \, x : T} \quad \frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash \text{App} \, t_1 \, t_2 : T_2}$$

$$x \# \Gamma$$
 valid Γ valid \varnothing valid $\{x:T\} \cup \Gamma$

 $t\{0 \leftarrow \operatorname{Var} x\}$ stands for "replacing" the 0-index with $\operatorname{Var} x$

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Proof of Weakening $x \# t \ \{x:T_1\} \cup \Gamma \vdash t\{0 \leftarrow Var x\}: T_2$ $\Gamma \vdash Lam t: T_1 \rightarrow T_2$

If $\Gamma_1 \vdash t : T$ then $\forall \Gamma_2$. valid $\Gamma_2 \land \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 \vdash t : T$

We know: $orall \Gamma_2$. valid $\Gamma_2 \wedge \{x:T'\} \cup \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 \vdash t\{0 \leftarrow \text{Var } x\}:T$ $x \ \# t$

We have to show: $orall \Gamma_2.$ valid $\Gamma_2 \wedge \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 dash Lam \, t : T' o T$

Proof of Weakeningx # t $\{x:T_1\} \cup \Gamma \vdash t\{0 \leftarrow Var x\} : T_2$ $\Gamma \vdash Lam t : T_1 \rightarrow T_2$

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Proof of Weakeningx # t $\{x:T_1\} \cup \Gamma \vdash t\{0 \leftarrow \forall ar x\} : T_2$ $\Gamma \vdash Lam t : T_1 \rightarrow T_2$

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We know:

$$\forall \Gamma_2 \cdot \text{valid } \Gamma_2 \wedge \{x:T'\} \cup \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 \vdash t\{0 \leftarrow \text{Var } x\}:T$$

 $x \ \# t$
 $\text{valid } \Gamma_2 \wedge \Gamma_1 \subseteq \Gamma_2 \Rightarrow \{x:T'\} \cup \Gamma_1 \subseteq \{x:T'\} \cup \Gamma_2$

Proof of Weakeningx # t $\{x:T_1\} \cup \Gamma \vdash t\{0 \leftarrow Var x\} : T_2$ $\Gamma \vdash Lam t : T_1 \rightarrow T_2$

If $\Gamma_1 \vdash t : T$ then $\forall \Gamma_2$. valid $\Gamma_2 \land \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 \vdash t : T$

We know: $\begin{array}{c} \Gamma_2 \mapsto \{x:T'\} \cup \Gamma_2 \\ \forall \Gamma_2. \text{ valid } \Gamma_2 \wedge \{x:T'\} \cup \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 \vdash t\{0 \leftarrow \text{Var } x\}:T \\ x \not = t \\ \text{valid } \Gamma_2 \wedge \Gamma_1 \subseteq \Gamma_2 \Rightarrow \{x:T'\} \cup \Gamma_1 \subseteq \{x:T'\} \cup \Gamma_2 \\ \text{valid } \{x:T'\} \cup \Gamma_2 ??? \end{array}$ We have to show:

 $\Gamma_2 \vdash \mathsf{Lam} \ t : T' \mathop{
ightarrow} T$

Existing Solutions

McKinna-Pollack introduce \vdash_s

 $rac{orall x.\,x \ \# \ \Gamma \Rightarrow \{x:T_1\} \cup \Gamma dash_s t\{0 \,{\leftarrow}\, ext{Var}\, x\}:T_2}{\Gamma dash_s ext{Lam}\, t:T_1 o T_2}$

They show $\vdash \Leftrightarrow \vdash_s$ and then prove: If $\Gamma_1 \vdash_s t: T$ then $\forall \Gamma_2$.valid $\Gamma_2 \land \Gamma_1 \subseteq \Gamma_2 \Rightarrow \Gamma_2 \vdash t: T$. Charguéraud et al introduce \vdash_c $\frac{\forall x \notin L. \{x: T_1\} \cup \Gamma \vdash_s t\{0 \leftarrow \text{Var } x\} : T_2}{\Gamma \vdash_s \text{Lam } t: T_1 \rightarrow T_2}$

where L is a (finite) list of names

Some Problems:

It is fair to say that it is still unclear to come up with ⊢_s and ⊢_c in the general case.

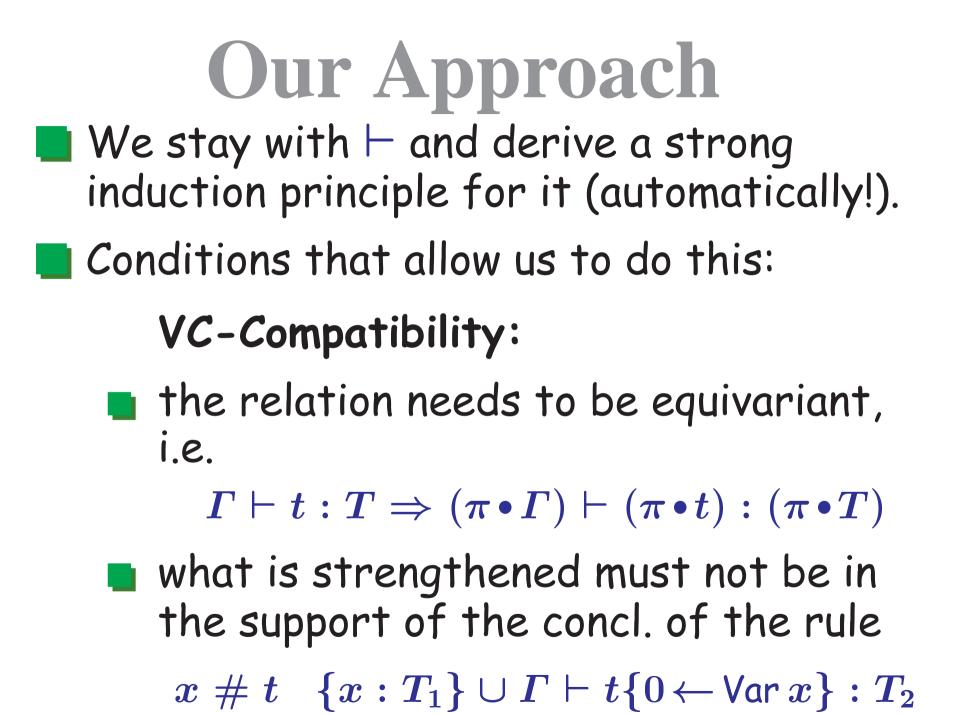
(Related) One likes to be sure to that \vdash_s and \vdash_c are equivalent to \vdash . It is annoying to prove this by hand.

If $I_1 \vdash_s t: T$ then $\forall I_2$, valid $I_2 \wedge I_1 \subseteq I_2 \Rightarrow I_2 \vdash t: T$.

Charguéraud et al introduce \vdash_c

 $rac{orall x \notin L. \{x: T_1\} \cup \Gamma dash_s t\{0 \leftarrow ext{Var} x\}: T_2}{\Gamma dash_s ext{Lam} t: T_1
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where L is a (finite) list of names



$$arGamma dash \mathsf{Lam} \ t: T_1 o T_2$$

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Our Conditions

- What happens when you violate the conditions?
 - or, in other words
 - Can the variable-convention lead you into trouble?

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 - or, in other words
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Yes!

 $egin{aligned} \overline{x\mapsto [],x} & ext{bind} [] \ t = t \ \overline{t_1 \ t_2 \mapsto [],t_1 \ t_2} & ext{bind} \ (x :: xs) \ t = \lambda x.(ext{bind} \ xs \ t) \ \overline{\lambda x.t \mapsto x :: xs,t'} \end{aligned}$

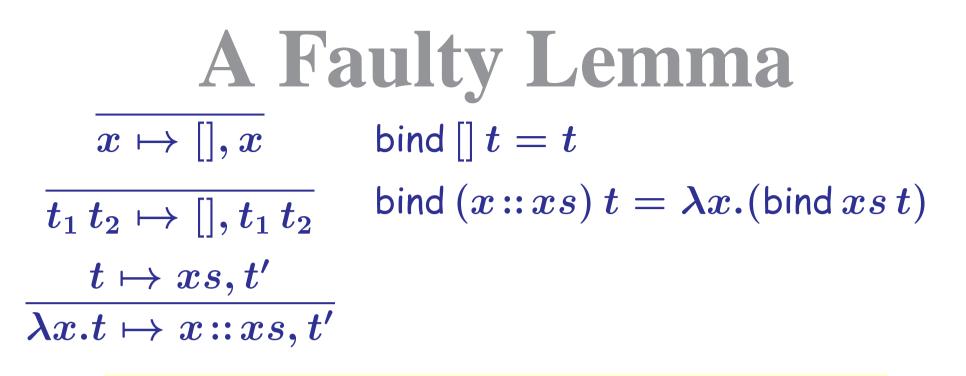
Our Conditions

Whyou can show: con If $t \mapsto xs, t'$ then $t =_{\alpha}$ bind xs t'. Can the variable-convention lead you into trouble?

Yes!

 $egin{aligned} \overline{x\mapsto [],x} & ext{bind} [] \ t = t \ \overline{t_1 \, t_2 \mapsto [], t_1 \, t_2} & ext{bind} \ (x :: xs) \ t = \lambda x. (ext{bind} \ xs \ t) \ \overline{\lambda x.t \mapsto x :: xs, t'} \end{aligned}$

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If $t \mapsto x :: xs, t'$ and $x \in FV(t')$ then also $x \in FV(bind xs t')$.

The faulty proof: using the variable convention you unbind a term to a list of distinct names

Two counter-examples

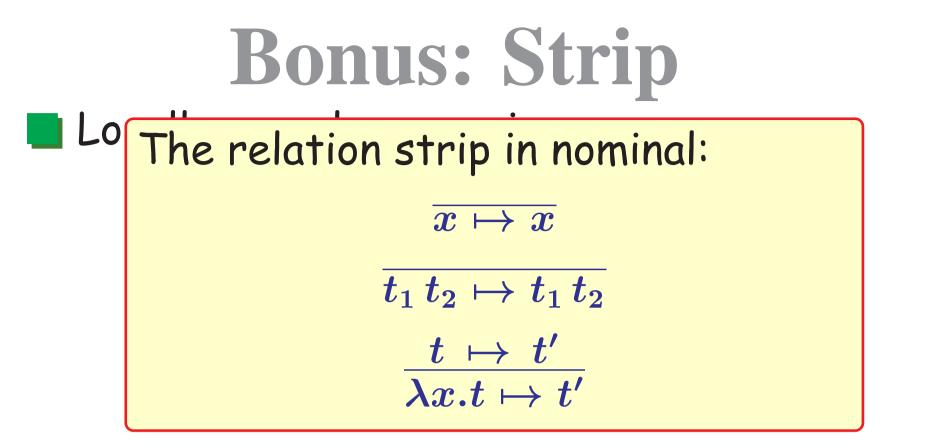
$$egin{aligned} \lambda x.\lambda x.x &\mapsto [x,x],x\ \lambda y.\lambda z.z &\mapsto [y,y],y \end{aligned}$$

Some Problems

- The proofs that use the strong induction principles in the nominal approach should also work in the locally nameless approach. For a number of proofs in the "locally nameless wild" the strong induction principles are of no help.
- Knowing that $\vdash \Leftrightarrow \vdash_s \Leftrightarrow \vdash_c$ is still needed in several instances. (There is no infrastructure available that could help you with such proofs.)

Conclusions

- Without modification a nominal technique applied to the locally nameless representation of binders.
- The strong induction principles are derived automatically in N and NL.
- We have conditions for when this possible (unbind is vc-incompatible).
- Bonus: A conjecture the cofinite rules of Charguéraud et al can be derived automatically provided the rules are variable-convention compatible.



The version according to Charguéraud et al $orall x \notin L. t\{0 \leftarrow \operatorname{Var} x\} \mapsto_c t'$ Lam $t \mapsto_c t'$



 $\operatorname{Var} x\mapsto \operatorname{Var} x$

$$egin{aligned} \overline{\operatorname{\mathsf{App}} t_1 \, t_2} &\mapsto \operatorname{\mathsf{App}} t_1 \, t_2 \ &x \ \# \, t \ t \{0 \, \leftarrow \, \operatorname{Var} x\} \ \mapsto \ t' \ & \operatorname{\mathsf{Lam}} t \ \mapsto t' \end{aligned}$$

The version according to Charguéraud et al $\frac{\forall x \notin L. t\{0 \leftarrow \operatorname{Var} x\} \mapsto_c t'}{\operatorname{Lam} t \mapsto_c t'}$

■ Locally-nameless version:

$$Var x \mapsto Var x$$

 $App t_1 t_2 \mapsto App t_1 t_2$

$$\frac{x \ \# \ t \ \ t\{0 \leftarrow \operatorname{Var} x\} \ \mapsto \ t'}{\operatorname{Lam} t \ \mapsto \ t'}$$

The version according to Charguéraud et al $\frac{\forall x \notin L. \ t\{0 \leftarrow \operatorname{Var} x\} \ \mapsto_c \ t'}{\operatorname{Lam} t \mapsto_c t'}$

 $\mathsf{Lam}\;(\mathsf{Bnd}\;0)\mapsto\mathsf{Var}\,x\quad\mathsf{but}\quad\mathsf{Lam}\;(\mathsf{Bnd}\;0)\not\mapsto_c\mathsf{Var}\,x$