GENERAL RELATIVITY – IRREDUCIBLE MINIMUM

ALAN L. MYERS

1. Vectors, contravariant and covariant

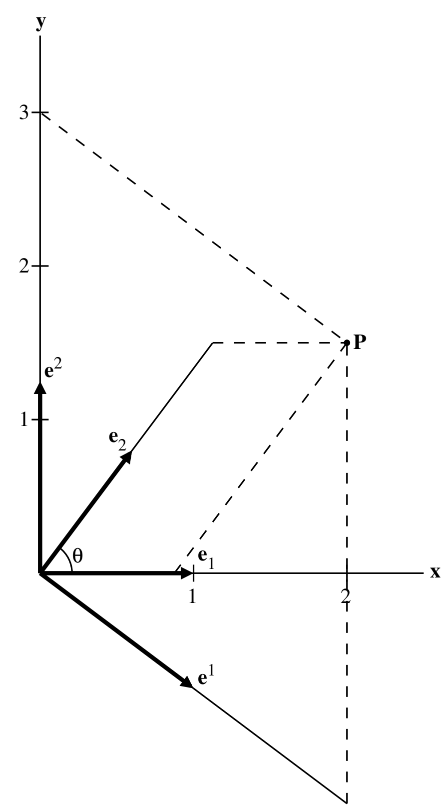

Figure 1. Non-orthogonal basis vectors in two dimensional flat space. Angle between basis vectors $\theta = 53.13°$. Basis vectors $\{e_1, e_2\}$ are set against a background of Cartesian coordinates $\{x, y\}$. The basis set for covectors is $\{e^1, e^2\}$.

Vectors are the simplest form of tensor. In 4-dimensional spacetime, tensors like the Riemann curvature tensor are of order 4 with $4^4 = 256$ components. It is helpful to begin the study of tensors with vectors, tensors of order 1 with only four components. Or simplify still further by working in 2-dimensional spacetime, with two components and two basis vectors. This simple two-dimensional case is adequate to illustrate the curvature of space (e.g., the surface of a sphere), the difference between contravariant and covariant vectors, and the metric tensor.

Unfortunately, terminology is confusing and inconsistent. The old-fashioned but still widely used names used to distinguish types of vectors are *contravariant* and *covariant*. The basis for these names will be explained in the next section, but at this stage it is just a name used to distinguish two types of vector. One is called the contravariant vector or just the vector, and the other one is called the covariant vector or dual vector or one-vector. A strict rule is that contravariant vector components are identified with superscripts like V^α, and covariant vector components are identified with subscripts like V_β. The mnemonic is: "Co- is low and that's all you need to know."

This discussion is focused on distinguishing contravariant and covariant vectors in the flat Cartesian space of Figure 1. A set of *basis vectors* $\{\vec{e}_1,\vec{e}_2\}$ is chosen so that any vector $\vec{V}$ can be expressed as:

$$\vec{V} = V^1\vec{e}_1 + V^2\vec{e}_2 = V^\alpha\vec{e}_\alpha$$

Notice that the path for locating a point traces a parallelogram (in two dimensions) or a parallelepiped (in three dimensions). Components are *not* determined by perpendicular projections onto the basis vector as for Cartesian components.

Already the usefulness of the Einstein summation rule is apparent: Any term containing a *dummy variable* forces a summation, in this case only over $i = 1, 2$ but in general over the four dimensions of spacetime. The dummy variable must be paired up and down, subscript and superscript, like α here. The basis vectors need be neither normalized nor orthogonal, it doesn't matter. In this case, the basis vectors $\{\vec{e}_1,\vec{e}_2\}$ are normalized for simplicity. Given the basis set $\{\vec{e}_1,\vec{e}_2\}$ for vectors, a basis set for dual vectors $\{\tilde{e}^1, \tilde{e}^2\}$ is defined by:

$$\tilde{e}^{\,\alpha}\,\vec{e}_\beta = \delta^\alpha_\beta \tag{1}$$

The $\vec{}$ symbol identifies vectors and their basis vectors, the $\tilde{}$ symbol identifies dual vectors and their basis vectors. As shown on Figure 1, the dual basis vectors are perpendicular to all basis vectors with a different index, and the scalar product of the dual basis vector with the basis vector of the same index is unity. The basis set for dual vectors enables any dual vector $\tilde{P}$ to be written:

$$\tilde{P} = P_1\tilde{e}^1 + P_2\tilde{e}^2 = P_\alpha\tilde{e}^\alpha$$

The set of basis vectors and their corresponding covectors for the example in Figure 1 are:

$$\begin{aligned} \vec{e}_1 &= (1, 0) \qquad & \tilde{e}^1 &= (1.0, -0.75) \\ \vec{e}_2 &= (0.6, 0.8) \qquad & \tilde{e}^2 &= (0, 1.25) \end{aligned}$$

which satisfy Eq. (1):

$$\vec{e}_1\tilde{e}^1 = 1; \qquad \vec{e}_2\,\tilde{e}^2 = 1; \qquad \vec{e}_1\tilde{e}^2 = \vec{e}_2\,\tilde{e}^1 = 0$$

Consider the vector OP from the origin to point P on Figure 1, which can be written in terms of its basis vectors:

$$OP = \vec{V} = V^1\vec{e}_1 + V^2\vec{e}_2 = 0.875\,\vec{e}_1 + 1.875\,\vec{e}_2$$

or in terms of its basis set for the covectors:

$$OP = \tilde{P} = P_1\tilde{e}^1 + P_2\tilde{e}^2 = 2\,\tilde{e}^1 + 2.4\,\tilde{e}^2$$

A vector may be thought of as an object that operates on a covector:

$$\vec{V}(\tilde{P}) = (0.875\,\vec{e}_1 + 1.875\,\vec{e}_2)(2\,\tilde{e}^1 + 2.4\,\tilde{e}^2) = 6.25$$

and yields the scalar product. Or vice versa for $\tilde{P}(\vec{V})$.

The values of these coordinates, $(V^1, V^2) = (0.875, 1.875)$ for the vector and $(P_1, P_2) = (2.0,\ 2.4)$ for the dual vector, can be verified by geometry. The metric tensor $g_{\alpha\beta}$ defined by its basis vectors:

$$g_{\alpha\beta} = \vec{e}_\alpha{\cdot}\vec{e}_\beta$$

The metric tensor provides the scalar product of a pair of vectors $\vec{A}$ and $\vec{B}$ by

$$\vec{A}\cdot\vec{B} = g_{\alpha\beta}V^\alpha V^\beta$$

The metric tensor for the basis vectors in Figure 1 is

$$g_{ij} = \begin{pmatrix} \vec{e}_1\cdot\vec{e}_1 & \vec{e}_1\cdot\vec{e}_2 \\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2\cdot\vec{e}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}$$

The inverse of g_{ij} is the raised-indices metric tensor for the covector space:

$$g^{ij} = \begin{pmatrix} \tilde{e}^1{\cdot}\tilde{e}^{\,1} & \tilde{e}^1{\cdot}\tilde{e}^{\,2} \\ \tilde{e}^2{\cdot}\tilde{e}^{\,1} & \tilde{e}^2{\cdot}\tilde{e}^{\,2} \end{pmatrix} = \begin{pmatrix} 1.5625 & -0.9375 \\ -0.9375 & 1.5625 \end{pmatrix}$$

The metric tensor for contravariant-covariant components is:

$$g^i_j = \begin{pmatrix} \tilde{e}^1 \cdot \vec{e}_1 & \tilde{e}^1 \cdot \vec{e}_2 \\ \tilde{e}^2 \cdot \vec{e}_1 & \tilde{e}^2 \cdot \vec{e}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The square of the vector $\vec{A}$ may be calculated from the metric in several ways:

$$\vec{V} \cdot \vec{V} = g_{ij} A^i A^j = g_{11} V^1 V^1 + g_{12} V^1 V^2 + g_{21} V^2 V^1 + g_{22} V^2 V^2 = 6.25$$
$$\tilde{P} \cdot \tilde{P} = g^{ij} P_i P_j = g^{11} P_1 P_1 + g^{12} P_1 P_2 + g^{21} P_2 P_1 + g^{22} P_2 P_2 = 6.25$$
$$\tilde{P} \cdot \vec{V} = g^i_j P_i V^j = \delta^i_j P_i V^j = P_i V^i = P_1 V^1 + P_2 V^2 = 6.25$$

Covariant components may be calculated from countervariant components using the metric

$$P_j = g_{ij} V^i$$

and countervariant components may be calculated from one-forms using the inverse metric

$$V^j = g^{ij} P_i$$

For example:

$$P_1 = g_{11} V^1 + g_{21} V^2 = (1)(0.875) + (0.6)(1.875) = 2.0$$
$$P_2 = g_{12} V^1 + g_{22} V^2 = (0.6)(0.875) + (1)(1.875) = 2.4$$

2. Change of Coordinates

2.1. **Contravariant vectors.**

$$\boxed{V'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} V^\beta} \qquad (2)$$

For spacetime, the derivative represents a four-by-four matrix of partial derivatives. A velocity V in one system of coordinates may be transformed into V' in a new system of coordinates. The upper index is the row and the lower index is the column, so for contravariant transformations, α is the row and β is the column of the matrix.

For example, for a 4-velocity vector in spacetime:

$$V'^\alpha = \frac{\partial x'^\alpha}{\partial \tau} = \frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial x_\beta}{\partial \tau} = \frac{\partial x'^\alpha}{\partial x^\beta} V^\beta$$

where τ is the proper time.

2.1.1. *Example:* $(r, \theta) \to (x, y)$.

$$x = r \cos\theta$$
$$y = r \sin\theta$$

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} \\ \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

2.1.2. *Example:* $(x, y) \to (r, \theta)$. The relations are

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} \\ \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

The transformation matrices are inverses.

2.2. **One-forms.**

$$\boxed{V'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} V_\beta} \tag{3}$$

The rule is that the upper index refers to the row and the lower index to the column, so for one-form transformations β is the row and α is the column. For example, a gradient V in one system of coordinates is transformed into a V' gradient in a new system of coordinates. The chain rule for a potential ϕ is:

$$V'_\alpha = \frac{\partial \phi}{\partial x'^\alpha} = \frac{\partial \phi}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^\alpha} = V_\beta \frac{\partial x^\beta}{\partial x'^\alpha}$$

Strictly V_β in this equation should be a row vector, but the order of matrices is generally ignored as in Eq. (3).

3. Tensors

3.1. **Tensor transformations.**

The rules for transformation of tensors of arbitrary rank are a generalization of the rules for vector transformation. For example, for a tensor of contravariant rank 2 and covariant rank 1:

$$T'^{\alpha\beta}_\gamma = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\gamma} T^{\mu\nu}_\rho$$

where the prime symbol identifies the new coordinates and the transformed tensor.

3.2. **Metric tensor.**

The metric tensor defined by:

$$g_{\mu\nu} = e_\mu \cdot e_\nu \tag{4}$$

Infinitesimal displacement vector:

$$d\vec{x} = dx^\mu e_\mu$$

$$dx^2 = (dx^\mu e_\mu)\cdot(dx^\nu e_\nu) = g_{\mu\nu} dx^\mu dx^\nu$$

More generally for vectors $\vec{V}$ and $\vec{W}$:

$$\vec{V}\cdot\vec{W} = g_{\mu\nu} V^\mu W^\nu$$

This is the "new" inner product, invariant under any linear transformation. It reproduces the "old" inner product in an orthonormal basis:

$$\mathbf{A}\cdot\mathbf{B} = (1\times A^1B^1) + (1\times A^2B^2) + (1\times A^3B^3)$$

3.3. **Contraction.**

Vector B is contracted to a scalar (S) by multiplication with a one-form A_α:

$$A_\alpha B^\alpha = S$$

Contraction of indices for a tensor works as follows:

$$T'^{\alpha\beta}_\alpha = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\alpha} T^{\mu\nu}_\rho = \frac{\partial x^\rho}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} T^{\mu\nu}_\rho = \delta^\rho_\mu \frac{\partial x'^\beta}{\partial x^\nu} T^{\mu\nu}_\rho = \frac{\partial x'^\beta}{\partial x^\nu} T^{\mu\nu}_\mu$$

Contracted tensors T' and T transform as contravariant vectors V' and V:

$$V'^\beta = \frac{\partial x'^\beta}{\partial x^\nu} V^\nu$$

Of fundamental importance is the use of the metric tensor $g_{\mu\nu}$ and its inverse (dual metric) $g^{\mu\nu}$ to raise and lower indices:

$$T_\nu = g_{\mu\nu} T^\mu$$

$$T^\nu = g^{\mu\nu} T_\mu$$

This process works for higher order tensors:

$$A^j_k = g^{ij} A_{ik}$$

$$C^i_{jk} = g_{jm} C^{im}_k$$

$$T^{ijk} = g^{im} T^{jk}_m$$

The Kronecker delta may be written as a tensor in terms of the metric and its inverse:

$$g^{\mu\nu}g_{\nu\lambda} = \delta^{\mu}_{\lambda}$$

The double contraction of a symmetric tensor $S^{\mu\nu}$ and an asymmetric tensor $A_{\mu\nu}$ is zero, that is $A_{\mu\nu}S^{\mu\nu} = 0$.

3.4. **Raising and Lowering Indices in E&M.** An electric field can refer to a gradient:

$$\mathbf{E} = -\boldsymbol{\nabla}\mathbf{V} \implies E_{\mu} = -\frac{\partial V}{\partial x^{\mu}}$$

or to a force:

$$\mathbf{E} = \frac{\mathbf{F}}{q} \implies E^{\mu} = \frac{F^{\mu}}{q} = \frac{ma^{\mu}}{q}$$

The index can be raised or lowered using the metric:

$$E^{\mu} = g^{\mu\nu}E_{\nu} \qquad E_{\mu} = g_{\mu\nu}E^{\nu}$$

In an orthonormal system, it makes no difference.

4. Christoffel Symbols from Metric Tensor

Definition of Christoffel symbol is

$$\Gamma^k_{ij}e_k = \frac{\partial e_i}{\partial x^j}$$

The Γ symbol by itself is not a tensor. Form dot product:

$$\Gamma^k_{ij}e_k \cdot e^m = e^m \cdot \frac{\partial e_i}{\partial x^j}$$

$$\Gamma^k_{ij}\delta^m_k = e^m \cdot \frac{\partial e_i}{\partial x^j}$$

$$\Gamma^m_{ij} = e^m \cdot \frac{\partial e_i}{\partial x^j}$$

Let $\frac{\partial e_i}{\partial x^j} = \frac{\partial e_j}{\partial x^i}$. This equality is for basis vectors and does not hold for unit vectors, for example, in spherical corrdinates $\frac{\partial \hat{r}}{\partial \theta} \neq \frac{\partial \hat{\theta}}{\partial r}$. The Christoffel symbol is symmetric in its two lower indices: $\Gamma^k_{ij} = \Gamma^k_{ji}$.

$$\Gamma^m_{ij} = \frac{1}{2}e^m \cdot \frac{\partial e_i}{\partial x^j} + \frac{1}{2}e^m \cdot \frac{\partial e_j}{\partial x^i}$$

$$\begin{aligned}\Gamma^m_{ij} = \frac{1}{2}e^m \cdot \frac{\partial e_i}{\partial x^j} &+ \left(\frac{1}{2}g^{km}\frac{\partial e_k}{\partial x^j} \cdot e_i - \frac{1}{2}g^{km}\frac{\partial e_j}{\partial x^k} \cdot e_i\right) \\ +\frac{1}{2}e^m \cdot \frac{\partial e_j}{\partial x^i} &+ \left(\frac{1}{2}g^{km}\frac{\partial e_k}{\partial x^i} \cdot e_j - \frac{1}{2}g^{km}\frac{\partial e_i}{\partial x^k} \cdot e_j\right)\end{aligned}$$

The terms on each line inside the parentheses sum to zero. Using $e^m = g^{km}e_k$:

$$\begin{aligned}\Gamma^m_{ij} = \frac{1}{2}g^{km}e_k \cdot \frac{\partial e_i}{\partial x^j} &+ \left(\frac{1}{2}g^{km}\frac{\partial e_k}{\partial x^j} \cdot e_i - \frac{1}{2}g^{km}\frac{\partial e_j}{\partial x^k} \cdot e_i\right) \\ +\frac{1}{2}g^{km}e_k \cdot \frac{\partial e_j}{\partial x^i} &+ \left(\frac{1}{2}g^{km}\frac{\partial e_k}{\partial x^i} \cdot e_j - \frac{1}{2}g^{km}\frac{\partial e_i}{\partial x^k} \cdot e_j\right)\end{aligned}$$

$$\Gamma^m_{ij} = \frac{1}{2}g^{km}\left[\left(e_k \cdot \frac{\partial e_i}{\partial x^j} + \frac{\partial e_k}{\partial x^j} \cdot e_i\right) + \left(e_k \cdot \frac{\partial e_j}{\partial x^i} + \frac{\partial e_k}{\partial x^i} \cdot e_j\right) - \left(e_i \cdot \frac{\partial e_j}{\partial x^k} + \frac{\partial e_i}{\partial x^k} \cdot e_j\right)\right]$$

$$\Gamma^m_{ij} = \frac{1}{2}g^{km}\left[\frac{\partial(e_k \cdot e_i)}{\partial x^j} + \frac{\partial(e_j \cdot e_k)}{\partial x^i} - \frac{\partial(e_i \cdot e_j)}{\partial x^k}\right]$$

$$\boxed{\Gamma^m_{ij} = \frac{1}{2}g^{km}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right]} \tag{5}$$

4.1. **Example for 2-Sphere.** The metric for a sphere is:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

For a 2-sphere of radius R:

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2$$

For a unit 2-sphere:

$$g_{ij} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$g^{ij} = \begin{pmatrix} g^{\theta\theta} & g^{\theta\phi} \\ g^{\phi\theta} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

There are 8 Christoffel symbols and only 3 of them are non-zero:

$$\begin{aligned} \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = &\quad \frac{1}{2} g^{\phi\phi} \frac{\partial g_{\phi\phi}}{\partial x^{\theta}} = \frac{1}{2} \left(\frac{1}{\sin^2\theta} \right) (2 \sin\theta \cos\theta) = \cot\theta \\ \Gamma^{\theta}_{\phi\phi} = &-\frac{1}{2} g^{\theta\theta} \frac{\partial g_{\phi\phi}}{\partial x^{\theta}} = -\frac{1}{2} (1)(2 \sin\theta \cos\theta) = -\sin\theta \cos\theta \end{aligned}$$

Eq. (5) contains 6 terms of type $g^{ij} \frac{\partial g_{ij}}{\partial x^k}$ for each of the 8 symbols, a total of 48 terms.

5. Non-Covariant Version

Consider in this section only non-curved coordinates on a flat manifold. The gradient of a scalar field is:

$$\boldsymbol{\nabla} f \implies \frac{\partial f}{\partial x^{\mu}} = \partial_{\mu} f$$

The gradient of a vector field is:

$$\boldsymbol{\nabla} \mathbf{V} \implies \frac{\partial V^{\mu}}{\partial x^{\nu}} = \partial_{\nu} V^{\mu}$$

and its divergence is:

$$\boldsymbol{\nabla} \cdot \mathbf{V} \implies \frac{\partial V^{\mu}}{\partial x^{\mu}} = \partial_{\mu} V^{\mu}$$

The vector product of $\mathbf{A}$ and $\mathbf{B}$ is:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \implies c_i = \epsilon_{ijk} a_j b_k$$

where ϵ is the permutation symbol (or alternating unit tensor) sometimes called the Levi-Civita pseudo-tensor, which is completely anti-symmetric. $\epsilon_{123} = 1$ and so does any even permutation of 123 like 231; $\epsilon_{132} = -1$ and so does any odd permutation of 123 like 132; and $\epsilon = 0$ if two of the three indices have the same value. Or imagine a clock with 1 at 12 o'clock, 2 at 4 o'clock and 3 at 8 o'clock. $\epsilon = +1$ for clockwise permutations and $\epsilon = -1$ for counter-clockwise permutations. For example, for the vector product:

$$c_1 = \epsilon_{1jk} a_j b_k = \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2$$

The following identify is useful:

$$e_{ijk} e_{i\ell m} = \delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{k\ell}$$

The scalar product is

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i$$

6. Covariant Derivative

We will need the connection coefficient:

$$\Gamma^k_{ij} e_k = \frac{\partial e_i}{\partial x^j}$$

For an abstract vector $\mathbf{V}$ expanded in countervariant basis vectors:

$$\mathbf{V} = V^i e_i$$

$$\frac{\partial \mathbf{V}}{\partial x^j} = \frac{\partial V^i}{\partial x^j} e_i + V^i \frac{\partial e_i}{\partial x^j} = \frac{\partial V^i}{\partial x^j} e_i + V^i \Gamma^k_{ij} e_k = \frac{\partial V^i}{\partial x^j} e_i + V^k \Gamma^i_{kj} e_i = \left(\frac{\partial V^i}{\partial x^j} + V^k \Gamma^i_{kj} \right) e_i$$

In terms of its coefficient tensor, the covariant derivative of a vector is

$$\boxed{\nabla_j V^i = \frac{\partial V^i}{\partial x^j} + V^k \Gamma^i_{kj}}$$

For an abstract vector $\mathbf{V}$ expanded in covariant basis vectors:

$$\mathbf{V} = V_i e^i$$

$$\frac{\partial \mathbf{V}}{\partial x^j} = \frac{\partial V_i}{\partial x^j} e^i + V_i \frac{\partial e^i}{\partial x^j} = \frac{\partial V_i}{\partial x^j} e^i + V_k \frac{\partial e^k}{\partial x^j}$$

A connection coefficient is needed for the derivative of a covariant base vector. Starting with:

$$e_i e^k = \delta^k_i$$

$$\frac{\partial e_i}{\partial x^j} e^k + \frac{\partial e^k}{\partial x^j} e_i = 0 \implies \Gamma^m_{ij} e_m e^k + \frac{\partial e^k}{\partial x^j} e_i = 0 \implies e_i \frac{\partial e^k}{\partial x^j} = -\Gamma^k_{ij}$$

Multiplying by e^i

$$e^i e_i \frac{\partial e^k}{\partial x^j} = \frac{\partial e^k}{\partial x^j} = -e^i \Gamma^k_{ij}$$

Substitution yields

$$\frac{\partial \mathbf{V}}{\partial x^j} = \frac{\partial V_i}{\partial x^j} e^i - V_k \Gamma^k_{ij} e^i = \left(\frac{\partial V_i}{\partial x^j} - V_k \Gamma^k_{ij} \right) e^i$$

The covariant derivative of a one-form is

$$\boxed{\nabla_j V_i = \frac{\partial V_i}{\partial x^j} - V_k \Gamma^k_{ij}}$$

Note the positive sign for vectors and the minus sign for one-forms.

6.1. **Extension to Higher-Order Tensors.**

$$\nabla_k T^{ij} = \frac{\partial T^{ij}}{\partial x^k} + T^{mj} \Gamma^i_{mk} + T^{im} \Gamma^j_{mk}$$
$$\nabla_k T_{ij} = \frac{\partial T_{ij}}{\partial x^k} - T_{mj} \Gamma^m_{ik} - T_{im} \Gamma^m_{jk}$$
$$\nabla_k T^i_j = \frac{\partial T^i_j}{\partial x^k} + T^m_j \Gamma^i_{mk} - T^i_m \Gamma^m_{jk}$$

7. Riemann Curvature Tensor

Parallel transport of a vector is defined as transport for which the covariant derivative is zero. The Riemann tensor is determined by parallel transport of a vector around a closed loop. Consider the commutator of covariant differentiation of a one-vector:

$$[\nabla_\gamma \nabla_\beta - \nabla_\beta \nabla_\gamma]\, V_\alpha$$

In a flat space, the order of differentiation makes no difference and the commutator is zero so that any non-zero result can be attributed to the curvature of the space.

$$\nabla_\beta V_\alpha = \frac{\partial V_\alpha}{\partial x^\beta} - \Gamma^\sigma_{\alpha\beta} V_\sigma \equiv V_{\alpha\beta}$$

$$\nabla_\gamma V_\alpha = \frac{\partial V_\alpha}{\partial x^\gamma} - \Gamma^\sigma_{\alpha\gamma} V_\sigma \equiv V_{\alpha\gamma}$$

$$\begin{aligned}\nabla_\beta \nabla_\gamma V_\alpha &= \frac{\partial V_{\alpha\beta}}{\partial x^\gamma} - \Gamma^\tau_{\alpha\gamma} V_{\tau\beta} - \Gamma^\eta_{\beta\gamma} V_{\alpha\eta} \\ &= \frac{\partial^2 V_\alpha}{\partial x^\gamma \partial x^\beta} - \frac{\partial \Gamma^\sigma_{\alpha\beta}}{\partial x^\gamma} V_\sigma - \Gamma^\sigma_{\alpha\beta} \frac{\partial V_\sigma}{\partial x^\gamma} - \Gamma^\tau_{\alpha\gamma}\left(\frac{\partial V_\tau}{\partial x^\beta} - \Gamma^\sigma_{\tau\beta} V_\sigma\right) - \Gamma^\eta_{\beta\gamma}\left(\frac{\partial V_\alpha}{\partial x^\eta} - \Gamma^\sigma_{\alpha\eta} V_\sigma\right)\end{aligned}$$

$$\begin{aligned}\nabla_\gamma \nabla_\beta V_\alpha &= \frac{\partial V_{\alpha\gamma}}{\partial x^\beta} - \Gamma^\tau_{\alpha\beta} V_{\tau\gamma} - \Gamma^\eta_{\gamma\beta} V_{\alpha\eta} \\ &= \frac{\partial^2 V_\alpha}{\partial x^\beta \partial x^\gamma} - \frac{\partial \Gamma^\sigma_{\alpha\gamma}}{\partial x^\beta} V_\sigma - \Gamma^\sigma_{\alpha\gamma} \frac{\partial V_\sigma}{\partial x^\beta} - \Gamma^\tau_{\alpha\beta}\left(\frac{\partial V_\tau}{\partial x^\gamma} - \Gamma^\sigma_{\tau\gamma} V_\sigma\right) - \Gamma^\eta_{\gamma\beta}\left(\frac{\partial V_\alpha}{\partial x^\eta} - \Gamma^\sigma_{\alpha\eta} V_\sigma\right)\end{aligned}$$

Each equation has 7 terms. In the commutator, the first terms cancel because the order of normal partial derivatives does not matter. The 3rd term of the first equation cancels with the 4th term of the second equation because the symbols used for dummy indices are irrelevant. The 4th term of the first equation cancels with the 3rd term of the second equation for the same reason. The 6th and 7th terms cancel because Christoffel symbols are symmetric in their lower indices. Only the 2nd and 5th terms survive:

$$[\nabla_\gamma \nabla_\beta - \nabla_\beta \nabla_\gamma]\, V_\alpha = \left(\frac{\partial \Gamma^\sigma_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial \Gamma^\sigma_{\alpha\beta}}{\partial x^\gamma} + \Gamma^\tau_{\alpha\gamma}\Gamma^\sigma_{\tau\beta} - \Gamma^\tau_{\alpha\beta}\Gamma^\sigma_{\tau\gamma}\right) V_\sigma$$

The terms within the parentheses define the Riemann curvature tensor:

$$R^\sigma_{\alpha\beta\gamma} \equiv \frac{\partial \Gamma^\sigma_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial \Gamma^\sigma_{\alpha\beta}}{\partial x^\gamma} + \Gamma^\tau_{\alpha\gamma}\Gamma^\sigma_{\tau\beta} - \Gamma^\tau_{\alpha\beta}\Gamma^\sigma_{\tau\gamma}$$

A necessary and sufficient condition for flat space is that the Riemann curvature is equal to zero.

8. Ricci Tensor

The first and fourth indices are contracted to give:

$$R_{\alpha\beta} = R^\gamma_{\alpha\beta\gamma} = \frac{\partial \Gamma^\gamma_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial \Gamma^\gamma_{\alpha\beta}}{\partial x^\gamma} + \Gamma^\tau_{\alpha\gamma}\Gamma^\gamma_{\tau\beta} - \Gamma^\tau_{\alpha\beta}\Gamma^\gamma_{\tau\gamma}$$

For spacetime coordinates:

$$R_{\mu\nu} = \frac{\partial \Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} - \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} + \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu} - \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha}$$

Some authors contract the first and third indices. Because of symmetries, other contractions give either zero or $\pm R_{\mu\nu}$.

The Ricci scalar is:

$$R = g^{ij} R_{ij}$$

9. Geodesics

In curved space the shortest distance between two points is called a geodesic. It can be defined geometrically as a path which "parallel transports" its own tangent vector. Think of an automobile tracing geodesics over sand dunes by locking the steering wheel into the straight-ahead direction. Let U be the tangent vector of a parameterized curve with components $dx^\beta/d\lambda$, where λ is the affine parameter. Let V be the parallel transport vector with the desired property that

$$\frac{dV}{d\lambda} = 0$$

Starting with

$$V = V^\alpha e_\alpha$$

take ordinary (not partial) derivatives:

$$\frac{dV}{d\lambda} = \frac{dV^\alpha}{d\lambda} e_\alpha + V^\alpha \frac{de_\alpha}{d\lambda}$$

Using the chain rule

$$\frac{de^\alpha}{d\lambda} = \frac{\partial e_\alpha}{\partial x^\beta} \frac{dx^\beta}{d\lambda}$$

and introducing the Christoffel symbol:

$$\frac{\partial e^\alpha}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta} e_\gamma$$

$$\begin{aligned}
\frac{dV}{d\lambda} &= \frac{dV^\alpha}{d\lambda} e_\alpha + V^\alpha \Gamma^\gamma_{\alpha\beta} e_\gamma \frac{dx^\beta}{d\lambda} \\
&= \frac{dV^\alpha}{d\lambda} e_\alpha + V^\gamma \Gamma^\alpha_{\gamma\beta} e_\alpha \frac{dx^\beta}{d\lambda} \\
&= \left(\frac{dV^\alpha}{d\lambda} + V^\gamma \Gamma^\alpha_{\gamma\beta} \frac{dx^\beta}{d\lambda} \right) e_\alpha
\end{aligned}$$

Using $dV/d\lambda = 0$ and $V = U = dx^\beta/d\lambda$

$$\boxed{\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0} \tag{6}$$

9.1. Example for Geodesic on 2-Sphere.

There are 3 non-zero Christoffel coefficients:

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta; \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

For $\alpha = \theta$:

$$\frac{d^2\theta}{d\lambda^2} + \Gamma^\theta_{\phi\phi} \frac{d\phi}{d\lambda} \frac{d\phi}{d\lambda} = \frac{d^2\theta}{d\lambda^2} - \sin\theta\cos\theta \left(\frac{d\phi}{d\lambda} \right)^2 = 0$$

For $\alpha = \phi$:

$$\frac{d^2\phi}{d\lambda^2} + \Gamma^\phi_{\theta\phi} \frac{d\theta}{d\lambda} \frac{d\phi}{d\lambda} + \Gamma^\phi_{\phi\theta} \frac{d\phi}{d\lambda} \frac{d\theta}{d\lambda} = \frac{d^2\phi}{d\lambda^2} + 2 \frac{\cos\theta}{\sin\theta} \frac{d\theta}{d\lambda} \frac{d\phi}{d\lambda} = 0$$

First consider a meridian from the north pole to the equator, parameterized as $\theta = \lambda$, $\phi = 0$, and $0 \le \lambda \le \frac{\pi}{2}$ for which

$$\frac{d\theta}{d\lambda} = 1; \qquad \frac{d^2\theta}{d\lambda^2} = \frac{d^2\phi}{d\lambda^2} = \frac{d\phi}{d\lambda} = 0$$

These values satisfy the two geodesic equations so the meridian is a geodesic.

Next consider a locus of constant latitude ($45°$), parameterized as $\theta = \frac{\pi}{4}$, $\phi = \lambda$, and $0 \le \lambda \le 2\pi$ for which

$$\frac{d\phi}{d\lambda} = 1; \qquad \frac{d^2\theta}{d\lambda^2} = \frac{d\theta}{d\lambda} = \frac{d^2\phi}{d\lambda^2} = 0$$

The second geodesic equation containing the term $d^2\phi/d\lambda^2$ is satisfied but the first geodesic equation containing the term $d^2\theta/d\lambda^2$ gives $0 - \sqrt{2}\sqrt{2}(1)^2 = -2 \neq 0$ is not satisfied, so the locus of constant $45°$ latitude is not a geodesic.

10. Euler-Lagrange Equation

Find the trajectory $q(t)$ for which the time integral of the Lagrangian is a minimum. Specifically

$$S = \int_{t_1}^{t_2} L\{q(t), \dot{q}(t), t\} dt$$

where S, which maps a function to a scalar, is called the action and $q(t)$ describes the trajectory. The Euler-Lagrange differential equation ensures the action is minimized with respect to all possible paths:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

For several coordinates:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

For conservative systems, $L = T - V$. The advantage is that we are dealing with scalars which have the same value in any coordinate system, which is the motivation for a special symbol q_i for general coordinates. Also, particles in GR move along geodesic paths of least distance in curved spacetime.

10.1. Example for Newtonian System in Cartesian Coordinates.

For the x-coordinate:

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$$

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}; \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\frac{\partial V}{\partial x} - \frac{d}{dt}m\dot{x} = -\frac{\partial V}{\partial x} - m\ddot{x} = 0$$

$$F_x = m\ddot{x}$$

10.2. Example for Newtonian System in 2-D Polar Coordinates.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$L = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 - V(r,\theta)$$

$$\begin{aligned}
&\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}; && \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\
&\frac{\partial L}{\partial \theta} = -\frac{\partial V}{\partial \theta}; && \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \\
&\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0; && \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \\
&mr\dot{\theta}^2 - \frac{\partial V}{\partial r} - m\ddot{r} = 0; && -\frac{\partial V}{\partial \theta} - \frac{d}{dt}(mr^2\dot{\theta}) = 0 \\
&F_r = m\ddot{r} - mr\dot{\theta}^2 && F_\theta = \frac{d}{dt}(mr^2\dot{\theta})
\end{aligned}$$

The term $mr\dot{\theta}^2$ is the centrifugal force and $mr^2\dot{\theta}$ is the angular momentum, which is constant in the absence of an applied torque F_θ.

10.3. **Generalization to Geodesic as Shortest Curve in Curved Spacetime.** In a curved space, the metric determines the infinitesimal length:

$$ds = \sqrt{g_{ab}dx^a dx^b}$$

A line integration gives finite length s:

$$s = \int ds = \int \frac{ds}{d\lambda} d\lambda = \int \sqrt{\left(\frac{ds}{d\lambda}\right)^2} d\lambda = \int L \, d\lambda$$

where L is the "Lagrangian" function of x and $\dot{x} \equiv dx/d\lambda$:

$$L = \sqrt{g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}} = L(x, \dot{x})$$

The Euler-Lagrange equation follows from the calculus of variations for the extremization of the path s with fixed end points:

$$\frac{\partial L}{\partial x^a} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^a} = 0$$

Substituting proper time for λ, It can be shown that the same Euler-Lagrange equation follows from a simpler Lagrangian of the form:

$$L(x, \dot{x}) = g_{ab} \dot{x}^a \dot{x}^b$$

Since the metric function g_{ab} depends on x but not on $\dot{x}$:

$$\frac{\partial L}{\partial \dot{x}^a} = 2 g_{ab} \dot{x}^b; \qquad \frac{\partial L}{\partial x^c} = \frac{\partial g_{ab}}{\partial x^c} \dot{x}^a \dot{x}^b$$

11. Newtonian Limit

Gauss' law for the Newtonian limit is:

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = -4\pi G \rho$$

or

$$\boldsymbol{\nabla}^2 \phi = 4\pi G \rho$$

which is equivalent to:

$$\boldsymbol{g} = -GM \frac{\boldsymbol{e}_r}{r^2}$$

Since

$$\boldsymbol{g} = \frac{\boldsymbol{F}}{m} = \frac{d^2 \boldsymbol{r}}{dt^2}$$

and because the gravitation force is conservative:

$$\boldsymbol{g} = -\boldsymbol{\nabla} \phi$$

we get:

$$\frac{d^2 \boldsymbol{r}}{dt^2} = -\boldsymbol{\nabla} \phi$$

How better to suggest that gravitation and acceleration are the same thing? Here it will be shown that the geodesic equation yields the above equation for a particle moving at nonrelativistic speed in a static, weak gravitational field ϕ. At the end, a "weak" gravitational field will be defined quantitatively.

For nonrelativistic speeds:

$$\frac{dx^i}{d\lambda} << c \frac{dt}{d\lambda}$$

$dt/d\lambda = dx^0/d\lambda$ is the dominant term of the geodesic Eq. (6):

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{00} \frac{dx^0}{d\lambda} \frac{dx^0}{d\lambda} = 0$$

The $\partial/\partial x^0$ terms in Eq. (5) vanish because the gravitational field is static so we have:

$$g_{\mu\nu} \Gamma^\nu_{00} = -\frac{1}{2} \frac{\partial g_{00}}{\partial x^\mu}$$

A small correction field $h_{\mu\nu}$ is added to the flat spacetime metric $\nu_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Since $\nu_{\mu\nu}$ is constant, $\partial g_{\mu\nu}/\partial x^\sigma = \partial h_{\mu\nu}/\partial x^\sigma$ and to leading order:

$$\nu_{\mu\nu}\Gamma^\nu_{00} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^\mu}$$

$$-\Gamma^0_{00} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^0} = 0; \qquad \Gamma^i_{00} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i}$$

Plugging these connections into the geodesic equation:

$$\frac{d^2x^0}{d\lambda^2} = 0 \quad \text{and} \quad \frac{dx^0}{d\lambda} = \text{const.}$$

$$\frac{d^2x^i}{d\lambda^2} - \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}\left(\frac{dx^0}{d\lambda}\right)^2 = 0$$

Using

$$\frac{d^2x_i}{d\lambda^2} = \left(\frac{dx^0}{d\lambda}\right)^2 \frac{d^2x^i}{d(x^\circ)^2}$$

the $(dx^0/d\lambda)^2$ factor cancels and

$$\frac{d^2x^i}{dt^2} = \frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i}$$

Comparing this with the Newtonian equation for acceleration:

$$h_{00} = -\frac{2\phi}{c^2}$$

so that

$$g_{00} = -\left(1 + \frac{2\phi}{c^2}\right)$$

For the gravitational field at the surface of the earth, $2\phi/c^2 = 1.4 \times 10^{-9}$. Therefore a *weak* gravitational field means much less than one billion g's.

12. Schwarzschild Metric

Birkhoff's theorem is that the Schwarzschild metric is the *only* spherically symmetric vacuum solution of Einstein's field equations:

$$\boxed{ds^2 = -\left(1 - \frac{r^*}{r}\right)c^2dt^2 + \left(1 - \frac{r^*}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}$$

$$r^* \equiv \frac{2GM}{c^2} = \text{Schwarzschild radius}$$

Consider the geometry of empty spacetime surrounding a spherically symmetric mass. The conditions are: (1) gravitational field is spherically symmetric; (2) field is static, meaning metric components are time-independent; (3) field is asymptotically flat, approaching flat spacetime at sufficiently large r.

The most general form which satisfies these requirements is:

$$ds^2 = -Uc^2dt^2 + V dr^2 + Wr^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

where U, V, W are unknown functions of r. Let $W = 1$, which alters the meaning of r but facilitates the introduction of spherical symmetry:

$$ds^2 = -Ac^2dt^2 + Bdr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

where A and B are new functions of r. Condition (2) is met. For fixed r and t, the line element for the surface of a sphere is recovered so condition (1) is satisfied. The functions A and B must approach unity as $r \to \infty$ to recover the Minkowski metric as demanded by condition (3).

The Einstein equation is:

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

In empty space $T_{\mu\nu} = 0$ and:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0$$

Multiplying both sides by $g^{\mu\nu}$:

$$g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}) = 0$$

$$R - \frac{1}{2} R\, \delta^{\nu}_{\nu} = R - \frac{1}{2} 4R = 0$$

Since the Ricci scalar R vanishes, $R_{\mu\nu} = 0$. That does not necessarily imply that the underlying Riemann tensor is zero, so the field equations can have non-trivial solutions even when the energy-momentum tensor is zero.

Substitute for the functions A and B:

$$ds^2 = -e^{2\nu} c^2 dt^2 + e^{2\lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

where μ and λ are unknown functions of r. The exponentials preserve the signature of the metric. Diagonal values for the metric $g_{\mu\nu}$ are:

$$g_{00} = -e^{2\nu}; \quad g_{11} = e^{2\lambda}; \quad g_{22} = r^2; \quad g_{33} = r^2 \sin^2\theta$$

Since $g_{\mu\nu}$ is a diagonal matrix, its inverse $g^{\mu\nu}$ is:

$$g^{00} = -e^{-2\nu}; \quad g^{11} = e^{-2\lambda}; \quad g^{22} = \frac{1}{r^2}; \quad g^{33} = \frac{1}{r^2 \sin^2\theta}$$

Connection coefficients are calculated from:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\frac{\partial g_{\rho\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right)$$

For $\sigma = \mu = 0$ and $\nu = 1$:

$$\Gamma^{0}_{01} = \frac{1}{2} g^{0\rho} \left(\frac{\partial g_{\rho 1}}{\partial x^{0}} + \frac{\partial g_{0\rho}}{\partial x^{1}} - \frac{\partial g_{01}}{\partial x^{\rho}} \right)$$

$g^{0\rho} = 0$ unless $\rho = 0$ so:

$$\Gamma^{0}_{01} = \frac{1}{2} g^{00} \left(\frac{\partial g_{01}}{\partial x^{0}} + \frac{\partial g_{00}}{\partial x^{1}} - \frac{\partial g_{01}}{\partial x^{0}} \right) = \frac{1}{2} \left(-e^{-2\nu}\right) \frac{\partial}{\partial r} \left(-e^{2\nu}\right) = \frac{1}{2} e^{-2\nu} e^{2\nu}\, 2 \frac{d\nu}{dr} = \frac{d\nu}{dr} \equiv \nu'$$

For $\mu = \nu = 0$ and $\sigma = 1$:

$$\Gamma^{1}_{00} = \frac{1}{2} g^{1\rho} \left(\frac{\partial g_{\rho 0}}{\partial x^{0}} + \frac{\partial g_{0\rho}}{\partial x^{0}} - \frac{\partial g_{00}}{\partial x^{\rho}} \right)$$

$g^{1\rho} = 0$ unless $\rho = 1$ so:

$$\Gamma^{1}_{00} = \frac{1}{2} g^{11} \left(\frac{\partial g_{10}}{\partial x^{0}} + \frac{\partial g_{01}}{\partial x^{0}} - \frac{\partial g_{00}}{\partial x^{1}} \right) = \frac{1}{2} e^{-2\lambda} (-1) \frac{\partial}{\partial r} \left(-e^{2\nu}\right) = \frac{1}{2} e^{-2\lambda} e^{2\nu} (2) \frac{d\nu}{dr} = \nu' e^{2(\nu - \lambda)}$$

For $\mu = \nu = \sigma = 1$:

$$\Gamma^{1}_{11} = \frac{1}{2} g^{1\rho} \left(\frac{\partial g_{\rho 1}}{\partial x^{1}} + \frac{\partial g_{1\rho}}{\partial x^{1}} - \frac{\partial g_{11}}{\partial x^{\rho}} \right)$$

$g^{1\rho} = 0$ unless $\rho = 1$ so:

$$\Gamma^{1}_{11} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^{1}} + \frac{\partial g_{11}}{\partial x^{1}} - \frac{\partial g_{11}}{\partial x^{1}} \right) = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^{1}} = \frac{1}{2} e^{-2\lambda} \frac{\partial}{\partial r} \left(e^{2\lambda}\right) = \frac{1}{2} e^{-2\lambda} e^{2\lambda} (2) \frac{d\lambda}{dr} = \lambda'$$

For $\mu = \nu = 2$ and $\sigma = 1$:

$$\Gamma^{1}_{22} = \frac{1}{2} g^{1\rho} \left(\frac{\partial g_{\rho 2}}{\partial x^{2}} + \frac{\partial g_{2\rho}}{\partial x^{2}} - \frac{\partial g_{22}}{\partial x^{\rho}} \right)$$

$g^{1\rho} = 0$ unless $\rho = 1$ so:

$$\Gamma^{1}_{22} = \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial x^{2}} + \frac{\partial g_{21}}{\partial x^{2}} - \frac{\partial g_{22}}{\partial x^{1}} \right) = \frac{1}{2} e^{-2\lambda} (-1) \frac{\partial}{\partial r} (r^2) = -r e^{-2\lambda}$$

For $\mu = \nu = 3$ and $\sigma = 1$:

$$\Gamma^{1}_{33} = \frac{1}{2} g^{1\rho} \left(\frac{\partial g_{\rho 3}}{\partial x^{3}} + \frac{\partial g_{3\rho}}{\partial x^{3}} - \frac{\partial g_{33}}{\partial x^{\rho}} \right)$$

$g^{1\rho} = 0$ unless $\rho = 1$ so:

$$\Gamma^1_{33} = \frac{1}{2} g^{11} \left(\frac{\partial g_{13}}{\partial x^3} + \frac{\partial g_{31}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^1} \right) = \frac{1}{2} e^{-2\lambda} (-1) \frac{\partial}{\partial r} (r^2 \sin^2 \theta) = -e^{-2\lambda} r \sin^2 \theta$$

[home page, Alan L. Myers]