NATURAL SYSTEM OF UNITS IN GENERAL RELATIVITY

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Abstract. The international system (SI) of units based on the kilogram, meter, and second is inconvenient for computations involving the masses of stars or the size of a galaxy. Papers about quantum gravity express mass, length, and time in terms of energy, usually powers of GeV. Geometrized units, in which all units are expressed in terms of powers of length are also prevalent in the literature of general relativity. Here equations are provided for conversions from geometrized or natural units to SI units in order to make numerical calculations.

1. International System of Units

The International System of Units (SI) is based on the meter-kilogram-second (MKS) system of units. A first course in any field of science or engineering usually begins with a discussion of SI units. Given an equation, students learn to check instinctively the units of the various terms for consistency. For example, in SI units for Newton's law of universal gravitation

$$F = \frac{GmM}{r^2}$$

m and M are the masses of the two bodies in kilograms, r is the distance between their centers in meters, F is the force of attraction in newtons (1 N = 1 kg m s^{-2}), and G = Newton's constant = 6.6743×10^{-11} m^3 kg^{-1} s^{-2}. This equation satisfies the consistency test because the terms on both sides of the equation have the same units (kg m s^{-2}).

Cosmologists toss the SI system of units out the window so that every variable is expressed in powers of energy. The equations are simplified by the absence of constants including Newton's constant (G) and the speed of light (c). However, for a person educated in the SI system of units, the lingo of "natural units" used by cosmologists is confusing and seems (incorrectly) to display a casual disregard of the importance of units in calculations.

First let us examine units term-by-term in the Einstein equation of general relativity

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}} \tag{1}$$

written in SI units. Popular articles and most textbooks on general relativity introduce this equation without discussing its units. The energy-momentum tensor $T_{\mu\nu}$ has units of energy density (J m^{-3}) or, equivalently, momentum flux density (kg m^{-1} s^{-2}). Multiplication of the units of $T_{\mu\nu}$ (kg m^{-1} s^{-2}) by the units of G/c^4 (s^2 kg^{-1} m^{-1}) yields units of m^{-2} for the RHS of the equation. The terms on the LHS must have the same units (m^{-2}). The metric tensor ($g_{\mu\nu}$) is dimensionless. The Riemann tensor is the second derivative with respect to distance of the metric tensor and therefore has units of m^{-2}. The index-lowered forms of the Riemann tensor, $R_{\mu\nu}$ and R, have units of m^{-2} so the units are the same on both sides of the equation, as required.

2. Natural units.

So-called natural units are used almost exclusively in cosmology and general relativity. In order to read the literature, it is necessary to learn how to write equations and perform calculations in natural units. In the version of natural units used in cosmology, four fundamental constants are set to unity:

$$c = \hbar = \epsilon_\circ = k_B = 1$$

where

$$\begin{aligned} c &= \text{speed of light} = 2.9979 \times 10^8 \text{ m/s} \\ \hbar &= \text{reduced Planck constant} = 1.0546 \times 10^{-34} \text{ J s} \\ \epsilon_\circ &= \text{electric constant} = 8.8542 \times 10^{-12} \text{ A}^2 \text{ s}^4 \text{ kg}^{-1} \text{ m}^{-3} \\ k_B &= \text{Boltzmann constant} = 1.3806 \times 10^{-23} \text{ J K}^{-1} \end{aligned}$$

As a consequence of these definitions, 1 s $= 2.9979 \times 10^8$ m and 1 s$^{-1} = 1.0546 \times 10^{-34}$ J. Length and time acquire the units of reciprocal energy; energy and mass have the same units. Any kinematical variable with SI units of (kg$^\alpha$ m$^\beta$ s$^\gamma$) may be expressed in SI units:

$$\boxed{(E)^{\alpha-\beta-\gamma} \; \hbar^{\beta+\gamma} \, c^{\beta-2\alpha}} \tag{2}$$

where E in an arbitrarily chosen energy unit. A popular choice is GeV (1 GeV $= 1.6022 \times 10^{-10}$ J). Setting the constants $\hbar$ and c equal to unity gives natural units of GeV$^{\alpha-\beta-\gamma}$. Given natural units of GeV$^{\alpha-\beta-\gamma}$, the desired SI unit can always be recovered by multiplying by the conversion factor $(\hbar^{\beta+\gamma} \, c^{\beta-2\alpha})$.

Consider, for example, momentum with SI units of (kg m s^{-1}). For $\alpha = 1$, $\beta = 1$, and $\gamma = -1$, the natural unit is GeV$^{\alpha-\beta-\gamma}$ = Gev^1 and the conversion factor is c^{-1}. Conversion from natural to SI units gives:

$$1\,\text{GeV} = \frac{E}{c} = \frac{1.6022 \times 10^{-10}\,\text{J}}{2.9979 \times 10^8\,\text{m}\,\text{s}^{-1}} = 5.3444 \times 10^{-19}\,\text{kg m s}^{-1}$$

Table 1 provides conversion factors for some of the variables encountered in cosmology. Energy, mass, and momentum have the same natural units. Velocity, angular momentum, and charge are dimensionless. Note that pressure (force per unit area) has the same units as energy density in both systems of units, as it must.

In the system of natural units, the factors in Table 1 are unity by definition. To make conversions, cosmologists multiply or divide by the factors $\hbar$ and c with appropriate exponents to obtain the units desired. This seems like a trial-and-error procedure! Rules for converting units for SI to natural units, and for the reverse conversion, are given in Table 1.

Table 1. Natural Units. Factor is $(\hbar^{\beta+\gamma} \, c^{\beta-2\alpha})$. To convert natural unit $\rightarrow$ SI, multiply by factor. To convert SI $\rightarrow$ natural unit, divide by factor. For SI units of (kg$^\alpha$ m$^\beta$ s$^\gamma$), natural units are E$^{\alpha-\beta-\gamma}$.

Variable	SI Unit	Natural Unit	Factor	Natural unit $\rightarrow$ SI unit
mass	kg	E	c^{-2}	1 GeV $\rightarrow 1.7827 \times 10^{-27}$ kg
length	m	E^{-1}	$\hbar c$	1 GeV^{-1} $\rightarrow 1.9733 \times 10^{-16}$ m
time	s	E^{-1}	$\hbar$	1 GeV^{-1} $\rightarrow 6.5823 \times 10^{-25}$ s
energy	kg m^2 s^{-2}	E	1	1 GeV $\rightarrow 1.6022 \times 10^{-10}$ J
momentum	kg m s^{-1}	E	c^{-1}	1 GeV $\rightarrow 5.3444 \times 10^{-19}$ kg m s^{-1}
velocity	m s^{-1}	dimensionless	c	1 $\rightarrow 2.9979 \times 10^8$ m s^{-1}
angular momentum	kg m^2 s^{-1}	dimensionless	$\hbar$	1 $\rightarrow 1.0546 \times 10^{-34}$ J s
area	m^2	E^{-2}	$(\hbar c)^2$	1 GeV^{-2} $\rightarrow 3.8938 \times 10^{-32}$ m^2
force	kg m s^{-2}	E^2	$(\hbar c)^{-1}$	1 GeV^2 $\rightarrow 8.1194 \times 10^5$ N
energy density	kg m^{-1} s^{-2}	E^4	$(\hbar c)^{-3}$	1 GeV^4 $\rightarrow 2.0852 \times 10^{37}$ J m^{-3}
charge	C = A$\cdot$s	dimensionless	1	1 $\rightarrow 5.2909 \times 10^{-19}$ C

The entry in Table 1 for charge requires an explanation. The dimensionless fine-structure constant (α) is:

$$\alpha = \frac{e^2}{4\pi\epsilon_\circ\hbar c} = 0.0072974$$

The elementary charge $e = 1.6022\times10^{-19}$ C and the value of α is calculated from the set of constants $\{e,\epsilon_\circ,\hbar,c\}$. In natural units, $\epsilon_\circ = \hbar = c = 1$ and the dimensionless elementary charge e is:

$$e = \sqrt{4\pi\alpha} = 0.30282$$

so that

$$0.30282 = 1.6022\times10^{-19}\,\mathrm{C} \qquad \Longrightarrow \qquad 1 = 5.2909\times10^{-19}\,\mathrm{C}$$

In natural units, Eq. (1) becomes:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\,T_{\mu\nu}$$

The Planck mass is

$$m_p = \sqrt{\frac{\hbar c}{G}} = 2.1764\times10^{-8}\,\mathrm{kg}$$

In natural units $m_p = 1.2209\times10^{19}$ GeV. Replacement of Newton's gravitational constant G in Einstein's equation with the Planck mass gives

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi}{m_p^2}\,T_{\mu\nu}} \tag{3}$$

Continuing with natural units, the energy-momentum tensor has units of energy density or GeV^4 and the Planck mass has units of GeV. The RHS of the equation therefore has units of GeV^2. On the LHS of the equation, the metric tensor $g_{\mu\nu}$ is dimensionless so the Ricci tensor $R_{\mu\nu}$, Ricci scalar R, and the cosmological constant Λ all have natural units of GeV^2, or mass squared since energy and mass are equivalent.

As an exercise in the manipulation of natural units, consider the cosmological constant (Λ) which is frequently characterized with the same units as the energy-momentum tensor $G_{\mu\nu}$ and called the energy density of a vacuum (ρ_{vac}):

$$\rho_{vac} \approx 3\times10^{-47}\,\mathrm{GeV}^4$$

Transfer of the quantity from the RHS to the LHS of Eq. (3) requires multiplication by the factor of $8\pi/m_p^2$:

$$\Lambda = \rho_{vac}\left(\frac{8\pi}{m_p^2}\right) = \frac{(8\pi)(3\times10^{-47}\mathrm{GeV}^4)}{(1.2209\times10^{19}\,\mathrm{GeV})^2} = 5.06\times10^{-84}\,\mathrm{GeV}^2$$

These natural units for Λ may be converted to the SI unit of m^{-2} using the conversion factor in Table 1:

$$\Lambda = 5.06\times10^{-84}\,\mathrm{GeV}^2\left(\frac{1\,\mathrm{GeV}^{-2}}{3.8938\times10^{-32}\,\mathrm{m}^2}\right) = 1.3\times10^{-52}\,\mathrm{m}^{-2}$$

Conversion of the vacuum energy density from natural to SI units gives:

$$\rho_{vac} = 3\times10^{-47}\,\mathrm{GeV}^4\left(\frac{2.0852\times10^{37}\,\mathrm{J\ m}^{-3}}{1\,\mathrm{GeV}^4}\right) = 6.3\times10^{-10}\,\mathrm{J\ m}^{-3}$$

In terms of the the mc^2 energy of mass, $\rho_{vac} \approx 4$ hydrogen atoms per cubic meter.

3. Sign conventions

The signs in Eq. (3) depend on certain conventions. The sign convention for the flat Minkowski metric $\eta_{\mu\nu}$ and general metric $g_{\mu\nu}$ is $(-,+,+,+)$. The other sign conventions are:

$$\begin{aligned} R^{\rho}{}_{\sigma\mu\nu} &= +\left[\partial_\mu\,\Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu\,\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\right] \\ R^{\alpha}{}_{\mu\alpha\nu} &= +R_{\mu\nu} \\ G_{\mu\nu} &= +8\pi G T_{\mu\nu} \end{aligned}$$

The above sign convention for the metric, the Riemann tensor, and the Einstein equation conforms with MTW (Misner, Thorne and Wheeler) and is widely used but other conventions such as $(+,-,-,-)$ for the metric can be found in textbooks.

Cosmologists still argue about signs. The "mostly minus" metric signature is called the "West Coast" metric and the "mostly plus" metric signature is called the "East Coast" metric, possibly because Feynman used the West Coast metric. Recent general relativity textbooks (Carroll, D'Inverno, Hartle, Schutz) adopted the mostly plus metric but Cheng used the mostly minus metric, as did Einstein.

4. Geometrized Units

The natural system of units prevails in 2016 because quantum gravity is at the forefront of research in cosmology, and quantum physics is based on natural units. In the late 20th century geometrized units were used extensively in cosmology. The tome "Gravitation" (1279 pages) published in 1973 by Misner, Thorne, and Wheeler, sometimes referred to as the "bible" of general relativity or simply MTW, used geometrized units for which

$$c = G = 1$$

where

$$c = \text{speed of light} = 2.9979 \times 10^{8} \text{ m s}^{-1}$$
$$G = \text{Newton's constant} = 6.6743 \times 10^{-11} \text{ m}^{3} \text{ kg}^{-1} \text{ s}^{-2}$$

Geometrized units are powers of length (m). For SI units of [kg$^{\alpha}$ m$^{\beta}$ s$^{\gamma}$], the conversion factor from geometrized units of m$^{\alpha+\beta+\gamma}$ to SI units is

$$(G^{-\alpha} c^{2\alpha-\gamma})$$

Consider, for example, energy density with SI units of (kg m^{-1} s^{-2}), for which $\alpha = 1$, $\beta = -1$ and $\gamma = -2$. The conversion to SI units from geometrized units of m$^{\alpha+\beta+\gamma}$ = m^{-2} is

$$(1\,\text{m}^{-2})(G^{-1}c^{4}) = (1\text{ m}^{-2})\frac{(2.9979 \times 10^{8}\text{ m s}^{-1})^{4}}{(6.6743 \times 10^{-11}\text{ m}^{3}\text{ kg}^{-1}\text{ s}^{-2})} = 1.2102 \times 10^{44}\,[\text{kg m}^{-1}\text{s}^{-2}]$$

Other conversion factors are listed in Table 2. In the geometrized system, mass, length, time, energy, and momentum are expressed in meters.

Table 2. Geometrized Units. Factor is $(G^{-\alpha}c^{2\alpha-\gamma})$. To convert geometrized unit → SI, multiply by factor. To convert SI unit → geometrized unit, divide by factor. For SI units of (kg$^{\alpha}$ m$^{\beta}$ s$^{\gamma}$), geometrized units are m$^{\alpha+\beta+\gamma}$.

Variable	SI Unit	Geom. Unit	Factor	Geometrized unit → SI unit
mass	kg	m	c^2G^{-1}	1 m → 1.3466×10^{27} kg
length	m	m	1	1 m → 1 m
time	s	m	c^{-1}	1 m → 3.3356×10^{-9} s
energy	kg m^2 s^{-2}	m	c^4G^{-1}	1 m → 1.2102×10^{44} kg m^2 s^{-2}
momentum	kg m s^{-1}	m	c^3G^{-1}	1 m → 4.0370×10^{35} kg m s^{-1}
velocity	m s^{-1}	dimensionless	c	1 → 2.9979×10^{8} m s^{-1}
angular momentum	kg m^2 s^{-1}	m^2	c^3G^{-1}	1 m^2 → 4.037×10^{35} kg m^2 s^{-1}
force	kg m s^{-2}	dimensionless	c^4G^{-1}	1 → 1.2102×10^{44} kg m s^{-2}
acceleration	m s^{-2}	m^{-1}	c^2	1 m^{-1} → 8.9875×10^{16} m s^{-2}
energy density	kg m^{-1} s^{-2}	m^{-2}	c^4G^{-1}	1 m^{-2} → 1.2102×10^{44} kg m^{-1} s^{-2}

For geometrized units the Einstein equation takes the form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

so that the energy-momentum tensor $T_{\mu\nu}$, Ricci tensor $R_{\mu\nu}$, Ricci scalar R, and cosmological constant Λ all have the same units (m^{-2}). The metric tensor $g_{\mu\nu}$ is dimensionless. As mentioned above, the cosmological constant $\Lambda = 1.3 \times 10^{-52}$ m^{-2}.

The mass of the sun in geometrized units is

$$\text{mass of sun} = (1.989 \times 10^{30}\ \text{kg})\frac{(1\ \text{m})}{(1.3466 \times 10^{27}\text{kg})} = 1480\ \text{m} = 1.48\ \text{km}$$

which is one-half of its Schwarzschild radius.

5. Units for special relativity.

Books and papers about special relativity commonly use a special system of units in which the speed of light is set equal to unity ($c = 1$). Units of mass and length retain their SI definitions in terms of kilograms and meters, but time is transformed into a length and velocity is dimensionless. Let us call the system of units in which $c = 1$ the special relativity (SR) system. Table 3 shows the rules for conversions between SR and SI units.

Table 3. SR Units for $c = 1$. Factor is $c^{-\gamma}$. To convert SR unit → SI, multiply by factor. To convert SI unit → SR unit, divide by factor. For SI units of ($\text{kg}^{\alpha}\ \text{m}^{\beta}\ \text{s}^{\gamma}$), SR units are ($\text{kg}^{\alpha}\ \text{m}^{\beta+\gamma}$).

Variable	SI Unit	SR Unit	Factor	SR unit	→ SI unit
mass	kg	kg	1	1 kg	→ 1 kg
length	m	m	1	1 m	→ 1 m
time	s	m	c^{-1}	1 m	→ 3.3356×10^{-9} s
energy	kg m^2 s^{-2}	kg	c^2	1 kg	→ 8.9875×10^{16} kg m^2 s^{-2}
momentum	kg m s^{-1}	kg	c	1 kg	→ 2.9979×10^{8} kg m s^{-1}
velocity	m s^{-1}	dimensionless	c	1	→ 2.9979×10^{8} m s^{-1}
angular momentum	kg m^2 s^{-1}	kg m	c	1 kg m	→ 2.9979×10^{8} kg m^2 s^{-1}
force	kg m s^{-2}	kg m^{-1}	c^2	1 kg m^{-1}	→ 8.9875×10^{16} kg m s^{-2}
acceleration	m s^{-2}	m^{-1}	c^2	1 m^{-1}	→ 8.9875×10^{16} m s^{-2}
energy density	kg m^{-1} s^{-2}	kg m^{-3}	c^2	1 kg m^{-3}	→ 8.9875×10^{16} kg m^{-1} s^{-2}

[home page, Alan L. Myers]