Stochastic systems analysis and simulation

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Martingale bet

- There is a certain game in a certain casino in which your chances of winning are $p > 1/2$
- You place $\$b$ bets,
  (a) With probability p you gain $\$b$ and
  (b) With probability $(1 - p)$ you loose your $\$b$ bet
- The catch is that you either
  (a) Play until you go broke (loose all your money)
  (b) Keep playing forever
- You start with an initial wealth of $\$w_0$
- Shall you play this game?

Modeling

- Let t be a time index (number of bets placed)
- Denote as $x(t)$ the outcome of the bet at time t
  - $x(t) = 1$ if bet is won (with probability p)
  - $x(t) = 0$ if bet is lost (probability $(1-p)$)
- $x(t)$ is called a bernoulli random varible with parameter p
- Denote as $w(t)$ the player's wealth at time t
- At time $t = 0$, $w(0) = w_0$
- At times $t > 0$ wealth $w(t)$ depends on past wins and losses
- More specifically we have
  - When bet is won $w(t) = w(t-1) + b$
  - When bet is lost $w(t) = w(t-1) - b$

Coding

$t = 1$; $w(t) = w_0$; ; `// Initialize variables`

% repeat while not broke up to time max_t

while $(w(t) > 0)$ & $(t < max_t)$ **do**

    $x(t) =$ random('bino',1,p); % Draw Bernoulli random variable

    if $x(t) == 1$ **then**

        $w(t+1) = w(t) + b$; % If $x = 1$ wealth increases by b

    else

        $x(t+1) = w(t) - b$; % If $x = 0$ wealth decreases by b

    end

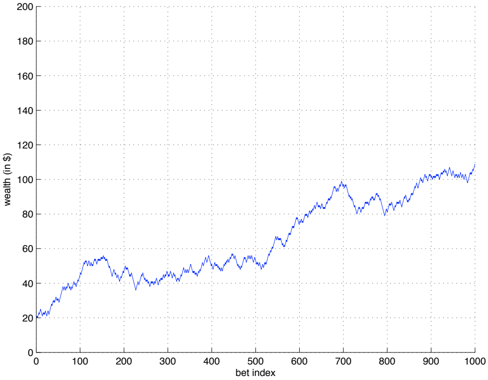
    $t = t + 1$;

end

- Initial wealth $w_0 = 20$, bet $b = 1$, win probability $p = 0.55$
- Shall we play ?

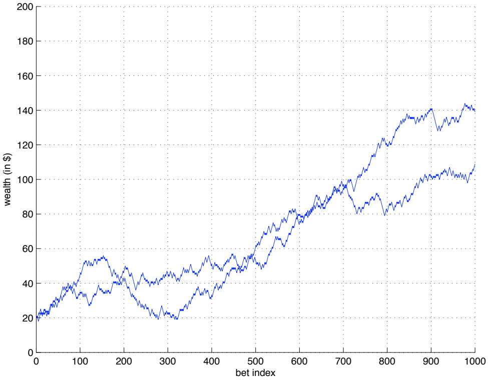
One lucky player

- She didn't go broke. After $t = 1000$ bets, her wealth is $w(t) = 109$
- Less likely to go broke now because wealth increased



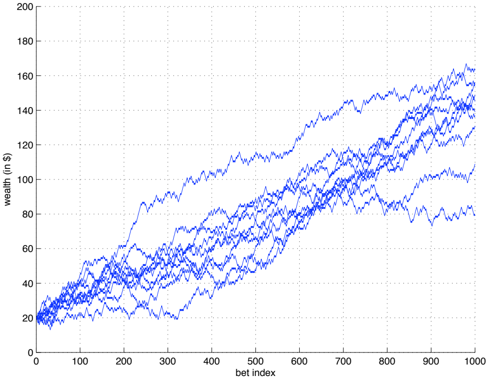
Two lucky players

- Wealths are $w_1(t) = 109$ and $w_2(t) = 139$
- Increasing wealth seems to be a pattern



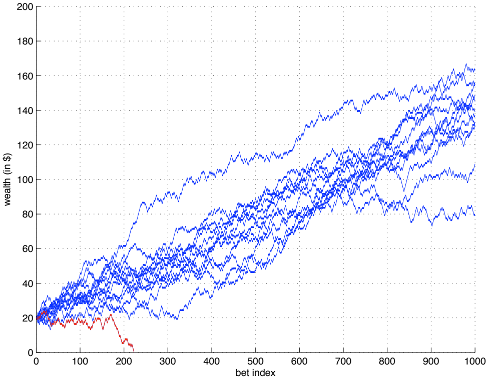
Ten lucky players

- Wealths $w_j(t)$ between 78 and 139
- Increasing wealth is definitely a pattern



One unlucky player

- But this does not mean that all players will turn out as winners
- The twelfth player $j = 12$ goes broke



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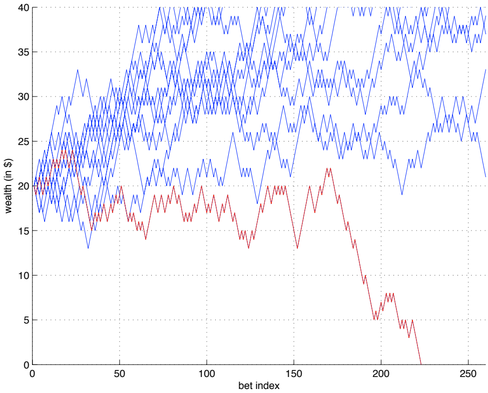
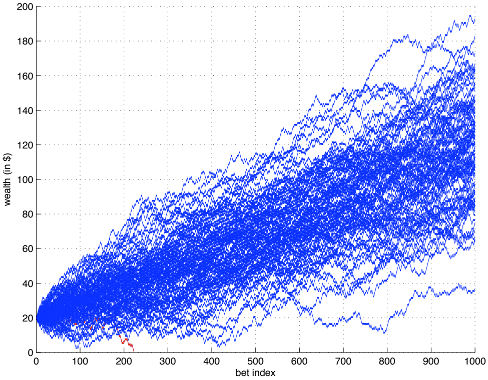


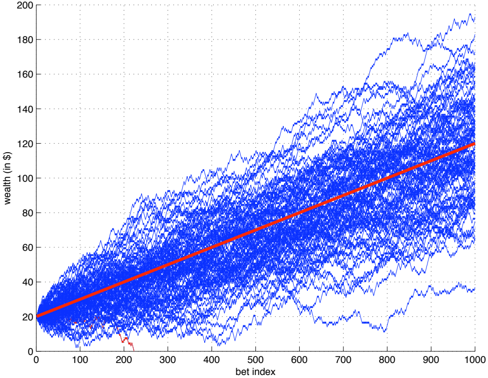
One hundred players

- Only one player ($j = 12$) goes broke
- All other players end up with substantially more money



Average tendency

- It is not difficult to find a line estimating the average of $w(t)$
- $\bar{w}(t) \approx w_0 + (2p-1)t \approx w_0 + 0.1t$



Where does the average tendency comes from?

- To discover average tendency $\bar{w}(t)$ assume $w(t-1) > 0$ and note

$$\begin{aligned}\mathbb{E}\left[w(t) \,\middle|\, w(t-1)\right] &= w(t-1) + b\Pr[x(t) = 1] - b\Pr[x(t) = -1] \\ &= w(t-1) + bp \qquad\qquad\quad - b(1-p) \\ &= w(t-1) + (2p-1)b\end{aligned}$$

- Now, condition on $w(t-2)$ and use the above expression once more

$$\begin{aligned}\mathbb{E}\left[w(t) \,\middle|\, w(t-2)\right] &= \mathbb{E}\left[w(t-1) \,\middle|\, w(t-2)\right] + (2p-1)b \\ &= w(t-2) + (2p-1)b \quad + (2p-1)b\end{aligned}$$

- Proceeding recursively t times yields

$$\mathbb{E}\left[w(t) \,\middle|\, w(0)\right] = w_0 + t(2p-1)b$$

- This analysis is not entirely correct because $w(t)$ might be zero

Analysis of outcomes: mean

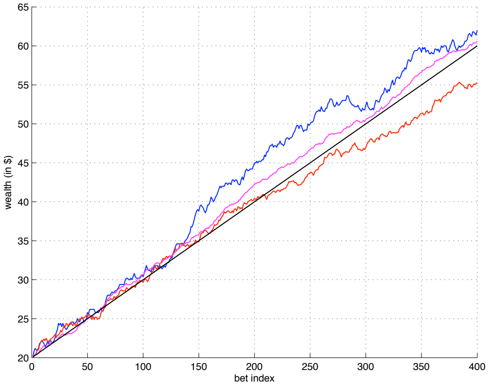
- For a more accurate analysis analyze simulation's outcome
- Consider J experiments
- For each experiment, there is a wealth history $w_j(t)$
- We can estimate the average outcome as

$$\bar{w}_J(t) = \sum_{j=1}^{J} w_j(t)$$

- $\bar{w}(t)$ is called the sample average
- Do not confuse $\bar{w}(t)$ with $\mathbb{E}[w(t)]$
  - $\bar{w}_J(t)$ is computed from experiments, it is a random quantity in itself
  - $\mathbb{E}[w(t)]$ is a property of the random variable $w(t)$
  - We will see later that for large J, $\bar{w}_J(t) \to \mathbb{E}[w(t)]$

Analysis of outcomes: mean

- Expected value $\mathbb{E}[w(t)]$ in black (approximation)
- Sample average for $J = 10$ (blue), $J = 20$ (red), and $J = 100$ (magenta)



Histogram

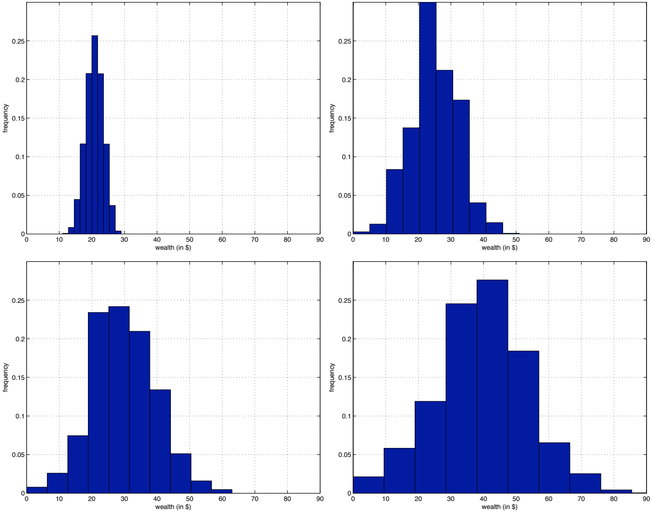
- There is more information in the simulation's output
- Estimate the probability distribution function (pdf) $\Rightarrow$ Histogram
- Consider a set of points $w^{(1)}, \ldots, w^{(N)}$
- Indicator function of the event $w^{(n)} \leq w_j < w^{(n+1)}$
  - $\mathbb{I}\left[w^{(n)} \leq w_j < w^{(n+1)}\right] = 1$ when $w^{(n)} \leq w_j < w^{(n+1)}$
  - $\mathbb{I}\left[w^{(n)} \leq w_j < w^{(n+1)}\right] = 0$ else
- Histogram is then defined as

$$H\left[t; w^{(n)}, w^{(n+1)}\right] = \frac{1}{J}\sum_{j=1}^{J} \mathbb{I}\left[w^{(n)} \leq w_j(t) < w^{(n+1)}\right]$$

- Fraction of experiments with wealth $w_j(t)$ between $w^{(n)}$ and $w^{(n+1)}$

Histogram

- The pdf broadens and shifts to the right ($t = 10, 50, 100, 200$)



What is this class about

- Analysis and simulation of stochastic system

  $\Rightarrow$ A system that evolves in time with some randomness

- They are usually quite complex $\Rightarrow$ Simulations

- We will learn how to model stochastic systems, e.g.,
  - $x(t)$ Bernoulli with parameter p
  - $w(t) = w(t-1) + b$ when $x(t) = 1$
  - $w(t) = w(t-1) - b$ when $x(t) = 0$

- ... how to analyze, e.g., $\mathbb{E}\left[w(t) \,\middle|\, w(0)\right] = w_0 + t(2p-1)b$

- ... and how to interpret simulations and experiments, e.g,
  - Average tendency through sample average