

Opportunistic control over shared wireless channels

Konstantinos Gatsis, Miroslav Pajic, Alejandro Ribeiro, and George J. Pappas

Abstract—We consider a wireless control architecture with multiple control loops and a shared wireless medium. A scheduler observes the random channel conditions that each control system experiences over the shared medium and opportunistically selects systems to transmit at a set of non-overlapping frequencies. The transmit power of each system also adapts to channel conditions and determines the probability of successfully transmitting and closing the loop. We formulate the problem of designing optimal channel-aware scheduling and power allocation mechanisms that minimize the total power consumption while meeting control performance requirements for all systems. In particular it is required that for each control system a given Lyapunov function decreases at a specified rate in expectation over the random channel conditions. We develop an offline algorithm to find the optimal communication design, as well as an online protocol which selects scheduling and power variables based on a random observed channel sequence and converges almost surely to the optimal operating point. We illustrate in simulations the power savings of our approach compared to other non-channel-aware schemes.

I. INTRODUCTION

Wireless control systems in e.g., smart buildings or industrial automation applications, are characterized by sensors, actuators, and controllers communicating information between different physical locations using wireless transceivers. As the number of independent wireless control systems that coexist in such environments increases, the need for efficiently sharing the available wireless medium becomes apparent. Scheduling access to the shared medium helps eliminate interferences between transmissions of different systems, but the rate at which a control system accesses the medium affects closed loop performance. This necessitates the development of wireless communication and resource management mechanisms that are control-aware. Moreover, since wireless devices in such applications are often battery-operated, these mechanisms are desired to be energy-efficient.

Scheduling in wired or wireless networked control systems has received a lot of attention in the past. Scheduling mechanisms usually examined are either static or dynamic. Typical examples of the first type are periodically protocols where the wireless devices transmit in a predefined repeating order, e.g., round-robin. Stability conditions under such scheduling protocols can be examined by converting the system in some form of a switching system, usually including other network phenomena such as delays, uncertain communication times, or packet drops – see, e.g., [2]–[4]. The problem of designing

static schedules suitable for control applications has also been addressed. Periodic sequences leading to stability [5], controllability and observability [6], or minimizing linear quadratic objectives [7] have been proposed. Deriving otherwise optimal scheduling sequences is recognized as a hard combinatorial problem [8], [9].

Dynamic schedulers on the other hand do not rely on a predefined sequence but decide access to the communication medium at each step, for example by dynamically assigning priorities to the competing tasks. Priorities commonly depend on the current plant/control system states, i.e., informally speaking, the subsystem with the largest state discrepancy is scheduled to communicate. Examples of such dynamic schedulers can be found in [3], [10]–[12]. Another approach, motivated by the problem of scheduling control tasks sharing a computation (CPU) rather than a communication resource, is to abstract control performance requirements in the time/frequency domain. Knowing for example how often a task needs access to the resource, to communicate and close the loop in our case, static/offline and dynamic/online schedules meeting the desired requirements can be obtained using algorithms from real-time scheduling theory – see, e.g., [13], [14] for more details on this approach.

However, in the case of multiple control tasks sharing a wireless communication medium the existing scheduling mechanisms in the control literature do not explicitly model or account for the wireless physical layer aspects of the problem. In particular, time-varying channel conditions cause large unpredictable variations in wireless channel interferences, referred to as fading [15, Ch. 3,4]. The problem of designing wireless communication networks to counteract such channel variability and maximize the utility to the users has received considerable attention [16]–[18]. The general approach is to allocate the available communication resources, e.g., medium access, power resources, channel capacity, by opportunistically adapting to the randomly varying channel conditions.

In this paper we propose a channel-aware approach for scheduling independent control tasks sharing a wireless communication medium. The channel conditions on the medium not only change randomly over time, but also differ among the control systems at a given time step. A channel-aware scheduling mechanism can opportunistically exploit channel information to, e.g., grant channel access to control loops experiencing favorable channel conditions, or equivalently avoid control loops transmitting under adverse conditions. The particular wireless control architecture we consider is shown in Fig. 1, where a scheduler selects at most one control system to transmit over each of a number of available frequencies at each time step. Moreover, following our previous work in [19], [20], we allow for the allocation of transmit power when a system is scheduled, which together with the channel fading

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determines the probability of successful message delivery at the receiver.

Furthermore, the wireless communication design of our setup, i.e., the opportunistic scheduling and power allocation, needs to serve a set of predesigned control tasks. To facilitate composition of the control tasks over the shared wireless medium, the control system dynamics and performance requirements need to be abstracted with an interface that is suitable for the wireless communication design problem. In this paper each control system is abstracted by some given Lyapunov function and control performance is specified as a desired decrease rate for this Lyapunov function (Section II). Our goal is to design how scheduling and power allocation should adapt to the random channel conditions so that *all* Lyapunov functions decrease at the specified rates at *every* time step. These constraints are expressed in a stochastic sense, *in expectation* over the channel conditions, since deterministically at most one loop can close at a time step. Also, this way the control performance requirements are expressed in a static single-time-step framework unlike, e.g., the timing/frequency abstractions in [13] or the periodic sequences in [6] which would be hard, if not intractable, to employ under random wireless channel conditions.

We formulate the problem of optimal channel-aware scheduling and power allocation that minimize the expected total power consumption subject to the expected Lyapunov decrease rate constraints (Section II-A). In Section III we develop an offline algorithm to solve the problem in the dual domain and obtain a characterization of the optimal solution. The optimal power allocation is decentralized among users and frequencies, while the optimal scheduling follows a channel-dependent assignment problem where control systems are opportunistically assigned to frequencies. The offline algorithm requires knowledge of the channel probability distribution which in most practical cases is not available, hence in Section IV we develop an online algorithm that utilizes only a channel sequence observed during execution. The online algorithm bears an intuitive pricing interpretation (Section IV-A), and we establish that if scheduling and power allocation are selected this way, the desired Lyapunov performance constraints are met in the limit in a strong sense (almost surely). Finally, simulations in Section V illustrate the opportunistic nature of the proposed channel-aware approach as well as the reduction in power consumption (at a magnitude of 80% in examples) compared to non-channel-aware mechanisms. We conclude with a discussion and future research directions in Section VI.

Notation: We denote the real m -dimensional non-negative orthant with \mathbb{R}_+^m , and the comparison with respect to the orthant (i.e., element-wise) with \geq , i.e., $x \geq y$ if and only if $x - y \in \mathbb{R}_+^m$. The cone of $n \times n$ real symmetric positive semi-definite matrices is denoted by S_+^n , and the comparison with respect to this cone with \succeq . The set of $n \times n$ real symmetric positive definite matrices is denoted by S_{++}^n .

II. PROBLEM DESCRIPTION

Consider the wireless control architecture of Fig. 1 consisting of m independent networked control systems. At each

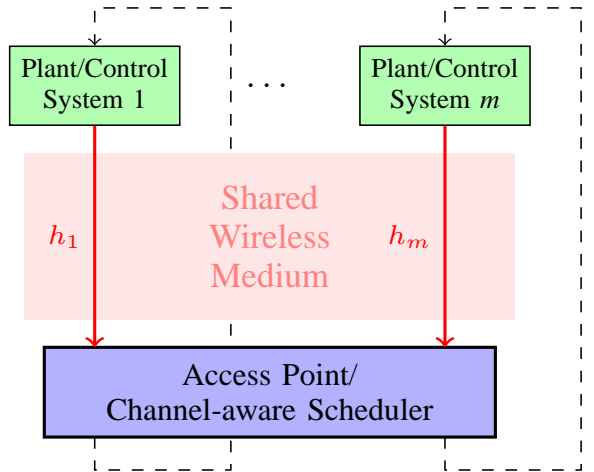


Fig. 1. Architecture for opportunistic scheduling of control tasks over a shared wireless medium. Independent control systems close the loop by transmitting over the shared wireless medium to a common receiver/access point. For simplicity the case of a single transmission frequency is shown, where each control system i experiences random channel conditions h_i . A centralized scheduler located at the access point observes all channel states and opportunistically decides which system is scheduled to transmit and close the loop.

time k , by $x_{i,k} \in \mathbb{R}^{n_i}$ we denote the state of the system i ($i = 1, 2, \dots, m$). To keep the framework general we assume that different descriptions for the evolution of each system i from $x_{i,k}$ to $x_{i,k+1}$ at time k are given depending on whether a transmission occurs at time k or not. Let us indicate with $\gamma_{i,k} \in \{0, 1\}$ the event that a successful transmission occurs at time k for the subsystem i . For simplicity then we describe the system evolution by a switched linear time invariant model,

$$x_{i,k+1} = \begin{cases} A_{c,i} x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 1 \\ A_{o,i} x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 0 \end{cases} \quad (1)$$

At a successful transmission the system dynamics are described by the matrix $A_{c,i} \in \mathbb{R}^{n_i \times n_i}$, where 'c' stands for closed-loop, and otherwise by $A_{o,i} \in \mathbb{R}^{n_i \times n_i}$, where 'o' stands for open-loop. We assume that $A_{c,i}$ is asymptotically stable, implying that if system i were to transmit at each slot its respective state evolution is stable. The open loop matrix $A_{o,i}$ could be unstable. The additive terms $w_{i,k}$ model an independent identically distributed (i.i.d.) noise process with mean zero and covariance $W_i \succeq 0$. Note that the noise terms are modeled as independent across time k for each i but also across plants i . Furthermore, it is worth noting that closed-loop dynamics for all of the m controllers are fixed (meaning that adequate controllers have been already designed). Thus in this work we focus on designing the wireless communication aspects of the control system. The above networked control system description (1) can model various control architectures, as shown in the following examples.

Example 1. Suppose each closed loop i consists of a linear plant of the form

$$x_{i,k+1} = A_i x_{i,k} + B_i u_{i,k} + w_{i,k}, \quad (2)$$

and a wireless sensor transmitting the plant state measurement $x_{i,k}$ to a controller/actuator which provides input $u_{i,k}$. Let then

the controller apply a linear feedback $u_{i,k} = K_i x_{i,k}$ when a measurement is received ($\gamma_{i,k} = 1$), otherwise apply for simplicity $u_{i,k} = 0$ when no measurement is received ($\gamma_{i,k} = 0$). The resulting closed loop system can be written as

$$x_{i,k+1} = \begin{cases} (A_i + B_i K_i) x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 1 \\ A_i x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 0 \end{cases} \quad (3)$$

which is of the form (1) with $A_{c,i} = A_i + B_i K_i$ and $A_{o,i} = A_i$. \square

Example 2. As a more general example consider again the plant in (2) and a wireless sensor measuring a system output of the form

$$y_{i,k} = C_i x_{i,k} + v_{i,k}, \quad (4)$$

where $v_{i,k}$ is some i.i.d. measurement noise with zero mean and finite covariance. A dynamic control law based on this plant output and adapted to the packet drops updates a local controller state according to

$$z_{i,k+1} = F_i z_{i,k} + \gamma_{i,k} (F_{c,i} z_{i,k} + G_i y_{i,k}), \quad (5)$$

i.e., corrects appropriately the local state whenever a measurement is received. For example $z_{i,k}$ may represent a local estimate of the plant state [2]. The control input applied by the controller can similarly be modeled as

$$u_{i,k} = K_i z_{i,k} + \gamma_{i,k} (K_{c,i} z_{i,k} + L_i y_{i,k}), \quad (6)$$

The overall closed loop system is obtained by joining plant and controller states into

$$\begin{bmatrix} x_{i,k+1} \\ z_{i,k+1} \end{bmatrix} = \begin{bmatrix} A_i & B_i K_i \\ 0 & F_i \end{bmatrix} \begin{bmatrix} x_{i,k} \\ z_{i,k} \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} w_{i,k} + \gamma_{i,k} \begin{bmatrix} B_i L_i C_i & B_i K_{c,i} \\ G_i C_i & F_{c,i} \end{bmatrix} \begin{bmatrix} x_{i,k} \\ z_{i,k} \end{bmatrix} + \gamma_{i,k} \begin{bmatrix} B_i L_i C_i \\ G_i \end{bmatrix} v_{i,k}, \quad (7)$$

which is again of the form (1). \square

Let us now describe the wireless communication system and model how it determines the packet transmission successes, i.e., the indicators $\gamma_{i,k}$. Suppose there are f different frequencies that each system may use to communicate and let the wireless channel conditions for a system i and frequency j at time k be denoted as $h_{ij,k}$. Channel conditions $h_{ij,k}$ can be described as the channel fading coefficient that system i experiences if it transmits at time k over frequency j . Due to propagation effects the channel gains $h_{ij,k}$ change unpredictably [15, Ch. 3] and take values in a subset $\mathcal{H} \subseteq \mathbb{R}_+$ of the positive reals. We adopt a block fading model whereby channel states $\{h_{ij,k}, 1 \leq i \leq m, 1 \leq j \leq f\}$ are modeled as random variables independent across different time slots k and identically distributed according to some joint distribution ϕ on $\mathcal{H}^{m \times f}$. They are also independent of the plant process noise $w_{i,k}$. We assume the channel states are available before transmission – see Remark 1 for a discussion about practical implementation. We also make the following technical assumption on their joint distribution to exclude the possibility of channel states becoming degenerate random variables.

Assumption 1. The joint distribution ϕ of channel states $\{h_{ij,k}, 1 \leq i \leq m, 1 \leq j \leq f\}$ has a probability density function on $\mathcal{H}^{m \times f}$.

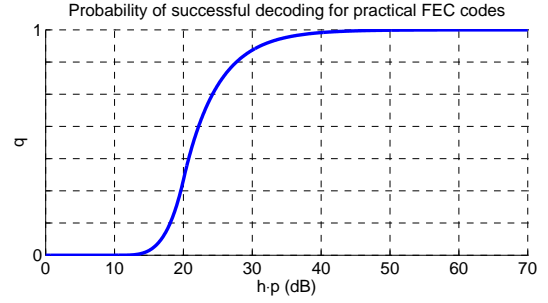


Fig. 2. Complementary error function for practical FEC codes. The probability of successful decoding q for a FEC code is a sigmoid function of the received SNR $\sim hp$.

If system i transmits at time k over frequency j it selects a transmit power level $p_{ij,k}$ taking values in $[0, p_{\max}]$. Then channel fading and transmit power affect the probability of successful decoding of the transmitted packet at the receiver. In particular given the forward error-correcting code (FEC) in use, the probability q that a packet is successfully decoded is a function of the received signal-to-noise ratio (SNR). The SNR is proportional to the received power level expressed by the product $h \cdot p$ of channel fading and the allocated transmit power. Overall we express the probability of success by a given relationship of the form $q(h_{ij,k} \cdot p_{ij,k})$ – for more details on this model, the reader is referred to [19]. An illustration of such a function is shown in Fig. 2. The following assumption on the form of the function $q(hp)$ will be helpful in the subsequent sections but as we will explicitly note it is not required for all of the results in this paper to hold.

Assumption 2. The function $q(\cdot)$ as a function of the product $r = hp$ for $r \geq 0$ satisfies:

- (a) $q(0) = 0$,
- (b) $q(r)$ is continuous, and strictly increasing when $q(r) > 0$, i.e., for $r' > r$ it holds that $q(r') > q(r) > 0$,
- (c) for any $\mu \geq 0$ and for almost all values $h \in \mathcal{H}$ the set $\operatorname{argmin}_{0 \leq p \leq p_{\max}} p - \mu q(hp)$ is a singleton.

Parts (a) and (b) of this assumption state that the probability of successful decoding $q(hp)$ is zero when the received power level hp is small, and it becomes positive $q(hp) > 0$ and strictly increasing for larger values of hp . These properties are verified for cases of practical interest as shown in Fig. 2. Part (c) is a more stringent requirement on the shape of function $q(hp)$ to facilitate the technical development of this paper, but is not restrictive in practice. The sigmoid shape in Fig. 2 guarantees the unique minimizer in (c). We also note that the minimizer set in (c) exists by the continuity assumption in (b).

Apart from packet drops due to low received SNR, packet collisions may occur if more than one of the control systems transmit at a given time slot on the same frequency over the shared wireless medium. For this reason we are interested in designing a mechanism to select which system accesses each of the available frequencies at the channel, i.e., which system is *scheduled* to transmit. We denote with $\alpha_{ij,k} = 1$ the decision to schedule system i on frequency j at time k , and $\alpha_{ij,k} = 0$ otherwise. To avoid packet collisions we let at most one system

transmit on each frequency j , that is $\sum_{i=1}^m \alpha_{ij,k} \leq 1$. We allow each system i to transmit on at most one frequency, that is $\sum_{j=1}^f \alpha_{ij,k} \leq 1$. Mathematically we may denote then the set $\Delta_{m,f}$ of all feasible scheduling decisions $\alpha_{ij,k}$ at each time k as

$$\Delta_{m,f} = \left\{ \alpha \in \{0, 1\}^{m \times f} : \begin{array}{l} \sum_{i=1}^m \alpha_{ij} \leq 1, 1 \leq j \leq f, \\ \sum_{j=1}^f \alpha_{ij} \leq 1, 1 \leq i \leq m \end{array} \right\}. \quad (8)$$

For compactness we group channel states, scheduling decisions, and power allocations of the overall communication system at time k into matrices $h_k \in \mathcal{H}^{m \times f}$, $\alpha_k \in \Delta_{m,f}$, and $p_k \in [0, p_{\max}]^{m \times f}$ respectively. We can then model the transmission event $\gamma_{i,k}$ of system i at time k given scheduling variables, power allocation, and channel state, as a Bernoulli random variable with success probability

$$\mathbb{P}[\gamma_{i,k} = 1 \mid h_k, \alpha_k, p_k] = \sum_{j=1}^f \alpha_{ij,k} q(h_{ij,k}, p_{ij,k}) \quad (9)$$

This expression states that the probability of a message for system i being successfully received equals the probability that the message is correctly decoded if system i is scheduled to transmit on any of the f available frequencies. Note that, by design of the scheduling variables, system i uses at most one frequency and we have made an implicit assumption that no interferences arise from other systems transmitting on different frequencies.

Our goal is to design the communication variables of the wireless control system, which are the scheduling and power allocation variables. Since the randomly varying channel affects the communication process, we are interested in selecting appropriate scheduling and power variables that adapt to channel states h_k in order to counteract these effects. Overall we express the scheduling and power decisions α_k, p_k respectively as mappings of the form

$$\begin{aligned} \mathcal{A} &= \{ \alpha : \mathcal{H}^{m \times f} \mapsto \Delta_{m,f} \}, \\ \mathcal{P} &= \{ p : \mathcal{H}^{m \times f} \mapsto [0, p_{\max}]^{m \times f} \}, \end{aligned} \quad (10)$$

so that $\alpha_k = \alpha(h_k)$, $p_k = p(h_k)$. Since channel states h_k are independent over time k these mappings do not need to change over time. Substituting the scheduling and power allocation mappings $\alpha(\cdot), p(\cdot)$ in our communication model described by (9) the probability of successful transmission for each system i at any given slot k becomes

$$\begin{aligned} \mathbb{P}(\gamma_{i,k} = 1) &= \mathbb{E}_{h_k} \left\{ \mathbb{P}[\gamma_{i,k} = 1 \mid h_k, \alpha(h_k), p(h_k)] \right\} \\ &= \mathbb{E}_h \sum_{j=1}^f \alpha_{ij}(h) q(h_{ij}, p_{ij}(h)). \end{aligned} \quad (11)$$

Here the expectation is with respect to the joint distribution ϕ of the channel realization h_k which we assumed to be identical for any time k , hence we drop the index k . Note also that the communication process modeled by the sequence $\{\gamma_{i,k}, 1 \leq i \leq m, k \geq 0\}$ depends only on variables related to the wireless communication counterpart of the overall system, and is in particular independent of the system evolutions $\{x_{i,k}, 1 \leq i \leq m, k \geq 0\}$.

Our primary goal in designing the communication variables of the system is to guarantee a level of closed loop performance for each subsystem. To formalize the problem description we consider Lyapunov-like performance requirements for the control systems. In particular suppose quadratic Lyapunov functions of the form

$$V_i(x) = x^T P_i x, x \in \mathbb{R}^{n_i}, \quad (12)$$

with $P_i \in S_{++}^{n_i}$ being positive definite matrices, are given for each system i . A Lyapunov-like requirement then states that these functions should decrease at given rates $\rho_i < 1$ during the execution of each subsystem i . This evolution however is random because of the stochastic nature of the overall wireless communication/control system, i.e., due to process noise, random channel states, randomized channel access, and packet drops. To take these effects into account we require that at any value $x_{i,k} \in \mathbb{R}^{n_i}$ of system state i at time k the Lyapunov functions at the next time step decrease at the desired rate $\rho_i < 1$ in expectation, that is

$$\mathbb{E} [V_i(x_{i,k+1}) \mid x_{i,k}] \leq \rho_i V_i(x_{i,k}) + \text{Tr}(P_i W_i). \quad (13)$$

The expectation over the next system state $x_{i,k+1}$ on the left hand side accounts via (1) for the randomness introduced by the process noise $w_{i,k}$ as well as the transmission success $\gamma_{i,k}$. The effect of process noise appears on the right hand side as the constant term $\text{Tr}(P_i W_i)$, while the transmission success is expressed in (11) and depends on the observed channel state h_k as well as the communication decisions α_k, p_k . The intuition behind requirement (13) is that if it holds on each time k it follows that

$$\mathbb{E} [V_i(x_{i,N}) \mid x_{i,0}] \leq \rho_i^N V_i(x_{i,0}) + \frac{1 - \rho_i^N}{1 - \rho_i} \text{Tr}(P_i W_i) \quad (14)$$

meaning that system states have second moments that decay exponentially and in the limit remain bounded by $\text{Tr}(P_i W_i)/(1 - \rho_i)$.

On the other hand, apart from control performance requirements an efficient communication design should make an efficient use of the available power resources at the devices. The induced overall expected power consumption on each slot k is given by

$$\mathbb{E}_{h_k} \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij,k}(h_k) p_{ij,k}(h_k), \quad (15)$$

summing up the transmit power of each system i and frequency j if the system is scheduled to transmit. The expectation here is with respect to the joint channel distribution $h_k \sim \phi$. The approach we take in designing scheduling and power allocation (cf. (10)) that are control-performance aware (cf. (13)) and also energy-efficient (cf. (15)) is through a stochastic optimization framework that we present next.

Remark 1. The centralized channel-aware scheduler can be implemented in a multiple access channel architecture as shown in Fig. 1, where each control system transmits to a common access point. For example, the access point can be collocated with a centralized controller which receives sensor measurements from the independent plants and is responsible

for providing inputs to each plant. The channel conditions for each system can be measured at the access point at the beginning of each time slot by pilot signals sent from the sensors to the access point. Depending on the measured channel states the access point decides which plant is scheduled to close the loop. \square

A. Scheduling and power allocation as stochastic optimization

We formulate the problem of designing scheduling and power allocation in an optimization framework as follows.

Problem 1 (Optimal Scheduling and Power Allocation Design). Consider a shared wireless control architecture with f frequencies and m systems of the form (1), quadratic Lyapunov performance requirements by (13), channel states $h_k \in \mathcal{H}^{m \times f}$ i.i.d. with distribution ϕ , and communication modeled by (9). The design of optimal scheduling and power allocation $\alpha_k = \alpha(h_k)$, $p_k = p(h_k)$ is posed as

$$\begin{aligned} & \underset{\alpha, p \in (\mathcal{A}, \mathcal{P})}{\text{minimize}} && \mathbb{E}_{h_k} \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij,k}(h_k) p_{ij,k}(h_k) && (16) \\ & \text{subject to} && \mathbb{E} [V_i(x_{i,k+1}) | x_{i,k}] \leq \rho_i V_i(x_{i,k}) + \text{Tr}(P_i W_i) \\ & && \text{for all } x_{i,k} \in \mathbb{R}^{n_i}, i = 1, \dots, m. \end{aligned}$$

\square

In other words, at each time step we seek to minimize the total expected power consumption (15) of the design while satisfying the Lyapunov requirements (13). To make explicit how the functions $\alpha(\cdot), p(\cdot)$ appear in the constraints of the problem, i.e., the Lyapunov requirements, observe that by (1) we have that

$$\begin{aligned} \mathbb{E} [V_i(x_{i,k+1}) | x_{i,k}] &= \mathbb{P}(\gamma_{i,k} = 1) x_{i,k}^T A_{c,i}^T P_i A_{c,i} x_{i,k} \\ &+ \mathbb{P}(\gamma_{i,k} = 0) x_{i,k}^T A_{o,i}^T P_i A_{o,i} x_{i,k} + \text{Tr}(P_i W_i), \end{aligned} \quad (17)$$

where we used the fact that the random variable $\gamma_{i,k}$ is independent of the system state $x_{i,k}$ as it depends only on the communication variables (cf. (9)-(11)). Plugging (17) at the left hand side of the constraints in (16) we get for $x_{i,k} \neq 0$

$$\mathbb{P}(\gamma_{i,k} = 1) \geq \frac{x_{i,k}^T (A_{o,i}^T P_i A_{o,i} - A_{c,i}^T P_i A_{c,i}) x_{i,k}}{x_{i,k}^T (A_{o,i}^T P_i A_{o,i} - \rho_i P_i) x_{i,k}}. \quad (18)$$

The decision variable in this constraint is $\mathbb{P}(\gamma_i = 1)$ at the left hand side which depends on $\alpha(\cdot), p(\cdot)$ by (11). Note then that according to problem 1 condition (18) needs to hold at any value of $x_{i,k} \in \mathbb{R}^{n_i}$. We can rewrite all these constraints by intersecting them to get $c_i \leq \mathbb{P}(\gamma_i = 1)$ where

$$c_i = \sup_{y \in \mathbb{R}^{n_i}, y \neq 0} \frac{y^T (A_{o,i}^T P_i A_{o,i} - A_{c,i}^T P_i A_{c,i}) y}{y^T (A_{o,i}^T P_i A_{o,i} - \rho_i P_i) y}. \quad (19)$$

Computing c_i is a simple semidefinite programming optimization problem which can be easily solved using available convex optimization software. The value c_i represents the minimum probability of transmission for each system i that guarantees the desired Lyapunov decay rate ρ_i – see also Remark 2. It can alternatively be thought of as a minimum required utilization factor of the shared wireless channel, analogously to a utilization of a shared CPU in, e.g., [14].

Intuitively, large value of c_i implies that system i requires more resources, i.e., more frequent channel access and possibly higher power expenditures.

Summarizing, the Lyapunov constraints in optimization (16) can be simplified by solving the auxiliary problems (19) for each control loop i , so that the optimization (16) can be equivalently written as

$$\begin{aligned} & \underset{\alpha, p \in (\mathcal{A}, \mathcal{P})}{\text{minimize}} && \mathbb{E}_h \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij}(h) p_{ij}(h) && (20) \\ & \text{subject to} && c_i \leq \mathbb{E}_h \sum_{j=1}^f \alpha_{ij}(h) q(h_{ij}, p_{ij}(h)), i = 1, \dots, m \end{aligned}$$

Here we have dropped the time indices k from the variables h_k since they are identically distributed over time. We also make a final constraint qualification assumption that is typical in optimization theory, i.e., that a strictly feasible point for this optimization problem exists.

Assumption 3. There exist variables $\alpha' \in \mathcal{A}$ and $p' \in \mathcal{P}$ that satisfy the constraints of the optimization problem (20) with strict inequality, i.e.,

$$c_i < \mathbb{E}_h \sum_{j=1}^f \alpha'_{ij}(h) q(h_{ij}, p'_{ij}(h)), \quad i = 1, \dots, m \quad (21)$$

By the equivalence between problems (16) and (20), condition (21) can be interpreted as a feasibility/schedulability assumption for the shared wireless control system. It requires that there exist some channel-aware scheduling and power allocation such that the control performance requirements (13) of *all* control systems are met. This assumption however does not provide any information on how to find such a solution.

In the rest of the paper we examine problem (20), which is equivalent to the optimal scheduling and power allocation design for the shared wireless control architecture in Problem 1. Since this problem is feasible by Assumption 3 let us denote the optimal value by P and an optimal solution pair by $\alpha^*(\cdot), p^*(\cdot)$. In the following section we characterize the form and properties of an optimal solution and describe a methodology to obtain it.

Remark 2. Since c_i is a required lower bound on the probability of successful transmission for system i , it must be that the value satisfies $c_i < 1$. Equivalently the right hand side of (19) needs to be less than one for all values of y , which in turn is equivalent to the condition $A_{c,i}^T P_i A_{c,i} \preceq \rho_i P_i$. This condition states that the closed-loop part of system (1) should satisfy the required decrease rate ρ_i for the given quadratic Lyapunov function V_i , or in other words that if system i transmit all the time the Lyapunov requirement is met. Since $A_{c,i}$ is stable by assumption, we may also assume that the given matrices P_i are selected appropriately for this to hold. \square

III. OPTIMAL SCHEDULING AND POWER ALLOCATION

In this section we examine how the optimal scheduling and power allocation for the wireless control system can be recovered by considering the optimization problem in the dual

domain. This allows us to develop an offline algorithm to solve the problem and provides an explicit characterization of the form of the optimal solution.

First let us derive the Lagrange dual problem of (20). Consider non-negative dual variables $\mu \in \mathbb{R}_+^m$ corresponding to each one of the m constraints of (20). The Lagrangian then is defined as

$$L(\alpha, p, \mu) = \mathbb{E}_h \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij}(h) p_{ij}(h) + \sum_{i=1}^m \mu_i \left[c_i - \mathbb{E}_h \sum_{j=1}^f \alpha_{ij}(h) q(h_{ij}, p_{ij}(h)) \right], \quad (22)$$

while the dual function is defined as

$$g(\mu) = \min_{\alpha, p \in (\mathcal{A}, \mathcal{P})} L(\alpha, p, \mu). \quad (23)$$

For future reference we also denote the set of functions $\alpha(\cdot), p(\cdot)$ that minimize the Lagrangian at μ by

$$(\mathcal{A}, \mathcal{P})(\mu) = \operatorname{argmin}_{\alpha, p \in (\mathcal{A}, \mathcal{P})} L(\alpha, p, \mu), \quad (24)$$

whenever the minimizers exist. This set might contain in general multiple solutions and we denote with $\alpha(\mu), p(\mu)$ an arbitrary element of the set.

The Lagrange dual problem is defined as follows.

$$D = \max_{\mu \in \mathbb{R}_+^m} g(\mu). \quad (25)$$

Lagrange duality theory informs us that the dual function $g(\mu)$ is a lower bound on the optimal cost P of problem (20) for any μ , so that the optimal dual value also satisfies $D \leq P$ (weak duality). The following proposition however establishes a strong duality result ($D = P$) for the problem under consideration and provides a relationship between the optimal primal and dual variables.

Proposition 1. *Let Assumptions 1 and 3 hold. Let P be the optimal value of the optimization problem (20) and (α^*, p^*) be an optimal solution, and let D be the optimal value of the dual problem (25) and μ^* be an optimal solution. Then*

- (a) $P = D$ (strong duality)
- (b) $\mu_i^* \left[c_i - \mathbb{E}_h \sum_{j=1}^f \alpha_{ij}^*(h) q(h_{ij}, p_{ij}^*(h)) \right] = 0$ for $i = 1, \dots, m$ (complementary slackness)
- (c) $(\alpha^*, p^*) \in (\mathcal{A}, \mathcal{P})(\mu^*)$

Proof: Statement (a) under assumptions 1 and 3 follows immediately from [18, Theorem 1] where a similar optimization setup is examined. The proof is omitted due to space limitations.

To show (b) observe that, by definition of the dual function in (23), at the point μ^* we have that

$$g(\mu^*) \leq L(\alpha^*, p^*, \mu^*) \quad (26)$$

Since μ^* is optimal for (25) and using part (a) we have for the left hand side of (26) that $g(\mu^*) = D = P$. On the other hand,

the right hand side of (26), by the definition of the Lagrangian at (22), equals

$$L(\alpha^*, p^*, \mu^*) = P + \sum_{i=1}^m \mu_i^* \left[c_i - \mathbb{E}_h \sum_{j=1}^f \alpha_{ij}^*(h) q(h_{ij}, p_{ij}^*(h)) \right], \quad (27)$$

because the objective of (20) at the optimal solution (α^*, p^*) equals the optimal value P . These expressions for the left and right hand sides of the inequality in (26) therefore give

$$P \leq P + \sum_{i=1}^m \mu_i^* \left[c_i - \mathbb{E}_h \sum_{j=1}^f \alpha_{ij}^*(h) q(h_{ij}, p_{ij}^*(h)) \right]. \quad (28)$$

This implies that the sum on the right hand side is non-negative. However all summands are non-positive, because $\mu^* \geq 0$ since it is feasible for the dual problem (25), and also the term in the brackets in (28) are non-positive because (α^*, p^*) are feasible for the primal problem (20). The only possibility then is that all summands in (28) are identically zero, which proves statement (b).

We have established that (28) holds with equality, so by tracing back our steps, we have that (26) holds with equality too, which, by the definition of the dual function on (23) translates to

$$\min_{\alpha, p \in (\mathcal{A}, \mathcal{P})} L(\alpha, p, \mu^*) = L(\alpha^*, p^*, \mu^*). \quad (29)$$

This verifies statement (c). ■

It is worth noting that this proposition states that strong duality holds even though the original problem is not convex, regardless also of the form of the function $q(h, p)$ (Assumption 2 was not imposed). More importantly, part (c) suggests the possibility of recovering the optimal primal variables α^*, p^* by solving first the dual problem for the optimal point μ^* . In other words, the design of scheduling and power allocation that meet the control performance specifications in Problem 1 is reduced to the problem of determining the optimal dual variables. A method to find the latter is presented next.

A. Dual subgradient method

To maximize the dual function $g(\mu)$ for the dual problem (25) we employ a dual projected subgradient algorithm [21, Ch. 8]. A subgradient direction for the (concave) function $g(\mu)$ with respect to $\mu \in \mathbb{R}_+^m$ is a vector, denoted here as $s(\mu) \in \mathbb{R}^m$, that satisfies

$$g(\mu') - g(\mu) \leq (\mu' - \mu)^T s(\mu) \quad \text{for all } \mu' \in \mathbb{R}_+^m. \quad (30)$$

If we pick $\alpha(\mu), p(\mu) \in (\mathcal{A}, \mathcal{P})(\mu)$ by (24) then a subgradient $s(\mu)$ can be found as the constraint slack of the primal problem (20) evaluated at these points, i.e.,

$$s_i(\mu) = c_i - \mathbb{E}_h \sum_{j=1}^f \alpha_{ij}(\mu; h) q(h_{ij}, p_{ij}(\mu; h)). \quad (31)$$

To show this observe that for any μ' in general we have $g(\mu') \leq L(\alpha(\mu), p(\mu), \mu')$ by the definition of the dual function in (23). Subtracting $g(\mu) = L(\alpha(\mu), p(\mu), \mu)$ from

both sides of this inequality and expanding the terms of the Lagrangian as in (22) we get

$$g(\mu') - g(\mu) \leq \sum_{i=1}^m (\mu'_i - \mu_i) \left[c_i - \mathbb{E}_h \sum_{j=1}^f \alpha_{ij}(\mu; h) q(h_{ij}, p_{ij}(\mu; h)) \right]. \quad (32)$$

Comparing this with the property of the subgradient in (30), we verify that (31) indeed gives a subgradient direction. We also note for future reference that for any μ the subgradients are bounded because at the right hand side of (31) the term c_i is bounded (cf.(19)) and the term in the expectation corresponds to a probability (cf.(11)).

A projected dual subgradient ascent method to maximize the (concave) dual function $g(\mu)$ then consists of the following steps:

- 1) At iteration t given $\mu(t)$ find primal optimizers of the Lagrangian at $\mu(t)$ according to (24),

$$p(\mu(t)), \alpha(\mu(t)) \in (\mathcal{A}, \mathcal{P})(\mu(t)) \quad (33)$$

- 2) Evaluate the subgradient vector $s(\mu(t))$ by (31) and update the dual variables by an ascent step

$$\mu(t+1) = [\mu(t) + \varepsilon(t)s(\mu(t))]_+ \quad (34)$$

where $[\]_+$ denotes the projection on the non-negative orthant and $\varepsilon(t) > 0$ is the stepsize.

The stepsizes are selected to be square summable but not summable, i.e.,

$$\sum_{t=1}^{\infty} \varepsilon(t)^2 < \infty, \quad \sum_{t=1}^{\infty} \varepsilon(t) = \infty. \quad (35)$$

Before stating the convergence properties of the algorithm, we note that in order to implement it we need an efficient way to compute primal Lagrange optimizers in (33) that solve (24). This problem also relates to our capability of finding the optimal primal variables of interest α^*, p^* as we have shown in Proposition 1(c). Hence we turn our focus to problem (24). A more convenient expression for the Lagrangian defined in (22) can be obtained by rearranging terms to get

$$L(\alpha, p, \mu) = \mu^T c + \mathbb{E}_h \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij}(h) [p_{ij}(h) - \mu_i q(h_{ij}, p_{ij}(h))]. \quad (36)$$

This form provides a useful separation of the primal variables across channel realizations h . We exploit this structure in the following proposition to obtain primal Lagrangian optimizers.

Proposition 2. For any $\mu \in \mathbb{R}_+^m$ the following hold true:

- (a) Solutions $\alpha(\mu), p(\mu) \in (\mathcal{A}, \mathcal{P})(\mu)$ of problem (24) can be obtained at each $h \in \mathcal{H}^m$ as

$$p_{ij}(\mu; h) = p_{ij}(\mu_i; h_{ij}) = \underset{0 \leq p \leq p_{\max}}{\operatorname{argmin}} p - \mu_i q(h_{ij}, p) \quad (37)$$

for any $i = 1, \dots, m$ and $j = 1, \dots, f$, and

$$\alpha(\mu; h) = \underset{\alpha \in \mathbb{R}_+^{m \times f}}{\operatorname{argmin}} \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij} \xi(h_{ij}, \mu_i) \quad (38)$$

subject to $\sum_{i=1}^m \alpha_{ij} \leq 1, \sum_{j=1}^f \alpha_{ij} \leq 1$

where

$$\xi(h_{ij}, \mu_i) = \min_{0 \leq p \leq p_{\max}} p - \mu_i q(h_{ij}, p). \quad (39)$$

- (b) If Assumptions 1 and 2 hold, then for any solution $\alpha(\mu), p(\mu) \in (\mathcal{A}, \mathcal{P})(\mu)$ the vector $s(\mu)$ defined in (31) has a unique value.

Proof: See Appendix A ■

The first part of the proposition provides through equations (37) and (38) a method to obtain primal Lagrange optimizers that can be used in step (31) of the subgradient algorithm. Interestingly a separability result for the optimal power allocation across systems i and frequencies j is revealed – see Remark 3 for more details. The second part of the proposition, which relies on Assumption 2, enables us to characterize the form of the optimal scheduling and power allocation variables in the following Theorem.

Theorem 1 (Optimal Scheduling and Power Allocation). Consider the design of channel-aware scheduling and power allocation variables in Problem 1 for the shared wireless control architecture of Fig. 1, and let Assumptions 1, 2, 3 hold. Then optimal scheduling α^* and power allocation p^* are obtained by (37)-(39) at a point $\mu^* \in \mathbb{R}_+^m$, which is an optimal solution of the dual problem (25). A point μ^* can be obtained by iterating (33)-(34), i.e., $\mu(t) \rightarrow \mu^*$, for stepsizes satisfying (35).

Proof: We first argue that all pairs $\alpha(\mu^*), p(\mu^*)$ that minimize the Lagrangian at the point μ^* are optimal solutions of the primal problem (20) (equivalently (16)). By Proposition 2(b), at the point μ^* we have that the vector $s(\mu^*)$ in (31), which is also the constraint slack in the primal problem (20) of any Lagrange optimizers $\alpha(\mu^*), p(\mu^*)$, is unique. But by Proposition 1(c) we have that the optimal primal variables α^*, p^* are also Lagrange optimizers at μ^* , and since they are primal feasible, then all other optimizers $\alpha(\mu^*), p(\mu^*)$ are primal feasible with the same constraint slack. Moreover all optimizers $\alpha(\mu^*), p(\mu^*)$ yield the same (minimum) Lagrangian value $L(\alpha(\mu^*), p(\mu^*), \mu^*)$. By the form of the Lagrangian in (22) it follows that all optimizers $\alpha(\mu^*), p(\mu^*)$ also give the same primal objective in (20) as the point α^*, p^* , i.e., the minimum P . Hence any optimizer pair $\alpha(\mu^*), p(\mu^*)$ is primal optimal. The first statement of the theorem follows because an optimal scheduling and power allocation pair can be obtained by Prop. 2(a) at μ^* .

The convergence of iterations (33)-(34) to the optimal dual variable μ^* for stepsizes (35) follows from a standard subgradient method argument – for a proof see, e.g., [21, Prop. 8.2.6]. ■

The theorem provides a characterization of the optimal scheduling and power allocation variables that meet the control

performance specifications in the shared wireless control architecture we examine. More details about the form of the optimal communication policy is given in the following remarks. It is worth noting that the optimal policy need not be unique. More precisely, there might be many optimal dual solutions μ^* , each one corresponding to a different scheduling and power allocation policy according to the theorem. However all such policies will have the same objective value in (16).

The theorem also establishes a methodology to find the optimal communication policy by iterating (33)-(34). This can be viewed as an offline algorithm which requires knowledge of the channel distribution. In the next section we develop an online algorithm that solves for the optimal communication policy based instead only on a random sequence of channel realizations observed during system execution.

Remark 3. According to Theorem 1, the optimal scheduling and power allocation variables can be obtained for each value of channel states h by solving (37)-(39) at the point μ^* . In particular, the optimal power allocation $p_{ij}^*(h)$ by (37) depends only on the variables μ_i^*, h_{ij} pertinent to system i and frequency j and not on the whole vectors μ^* or h . This implies a decentralized power allocation rule among systems i and frequencies j , which is made explicit in (37) by the notation $p_{ij}(\mu_i; h_{ij})$. Similar separability results are also known in the context of resource allocation for wireless communication networks [18]. Moreover, this optimal power allocation can be easily implemented in practice. Each control system i can be given the value μ_i^* and then adapt transmit power, whenever scheduled, based on the channel conditions it currently experiences. On the other hand, the optimal scheduling $\alpha^*(h)$ in (38) is centralized since it depends on the whole vector μ^* and all channel states h . \square

Remark 4. The problem of finding the optimal scheduling in (38) is posed as a linear program by relaxing the integer constraints of $\Delta_{m,f}$ in (8). As mentioned in the proof of the proposition there is no loss in doing so, as the optimal solution to the linear program is integer. It is worth noting that (38) solves a standard assignment problem¹. Besides the linear program method presented here, combinatorial algorithms with complexity polynomial in the number of systems m and frequencies f exist for such integer programming problems – see, e.g., [22, Ch. 7]. In the special case of a single frequency ($f = 1$) the complexity of the decision in (38) is linear in the number of systems ($O(m)$), since the scheduler needs to find and schedule the system i with the minimum value $\xi(h_i, \mu_i)$. \square

IV. ONLINE SCHEDULING AND POWER ALLOCATION

The algorithm presented in the previous section to obtain optimal scheduling and power allocation for the shared wireless control system is hard to implement in practice. In the primal step (33) one needs to obtain a solution pair $\alpha(h), p(h)$ for a continuum of channel variables $h \in \mathcal{H}^{m \times f}$, while for the

dual step in (34) one needs to compute the subgradient direction $s(\mu)$ in (31) by integrating over the channel distribution ϕ . A practical implementation would require drawing a large number of samples from ϕ and solving for primal variables at these samples to obtain an estimate of the actual subgradient direction. This is computationally intensive, does not scale for a large number of systems m and frequencies f , while also in most cases of practical interest the channel distribution is not available.

These drawbacks motivate us to develop an *online* algorithm to solve the optimal scheduling and power allocation problem. The algorithm is a stochastic version of the primal/dual steps (33), (34) of the offline subgradient method and does not rely on availability of the channel distribution. In particular, suppose that at time k a channel realization h_k is observed, and the current power and scheduling decision are selected as the ones solving (37)-(38) at the given h_k , i.e.,

$$\begin{aligned} p_{ij,k} &= p_{ij}(\mu_i; h_{ij,k}), \quad i = 1, \dots, m, j = 1, \dots, f, \\ \alpha_k &= \alpha(\mu_k; h_k). \end{aligned} \quad (40)$$

Then in contrast to updating the dual variables μ_k by (34) after computing the vector (31), suppose only the current channel measurement and power/scheduling choices are used. In particular, suppose we compute

$$s_{i,k} = c_i - \sum_{j=1}^f \alpha_{ij,k} q(h_{ij,k}, p_{ij,k}), \quad i = 1, \dots, m, \quad (41)$$

and update the variables μ_k by

$$\mu_{k+1} = [\mu_k + \varepsilon_k s_k]_+ \quad (42)$$

where $[\]_+$ is the projection on the non-negative orthant and $\varepsilon_k > 0$ is the stepsize.

To emphasize that this is an online algorithm we have explicitly indexed the variables with k corresponding to real time slots. This procedure, summarized in Algorithm 1, gives scheduling and power variables $\{\alpha_k, p_k, k \geq 0\}$ as well as (dual) variables $\{\mu_k, k \geq 0\}$ which are random because they depend on the random observed channel sequence $\{h_k, k \geq 0\}$. The main difference compared to the subgradient algorithm of the previous section is that it follows random directions s_k in (41) instead of the exact subgradient directions $s(\mu_k)$ by (31). Comparing these two expressions it is immediate that the expected value of s_k coincides with the subgradient $s(\mu_k)$, so it is reasonable to conjecture that the online algorithm is expected to move towards the maximum of the dual function, as the subgradient method does. The following proposition indeed establishes convergence in a strong sense.

Proposition 3. Consider the optimization problem (20) and its dual derived in (25) and let Assumption 3 hold. Let a sequence $\mu_k, k \geq 0$ be obtained by steps (40)-(42) based on a sequence $\{h_k, k \geq 0\}$ of i.i.d. random variables with distribution ϕ , and stepsizes ε_k satisfying (35). Then almost surely we have that

$$\lim_{k \rightarrow \infty} \mu_k = \mu^*, \quad \text{and} \quad \lim_{k \rightarrow \infty} g(\mu_k) = D \quad (47)$$

where μ^* is an optimal solution of the dual problem and D is the optimal value of the dual problem.

¹Technically the standard assignment problem requires equal number of systems and frequencies. This can be accomplished by introducing dummy systems or frequencies with zero values $\xi(h_{ij}, \mu_i)$.

Algorithm 1 Online Scheduling and Power Allocation

Input: $m, f, c \in [0, 1]^m$, $q : \mathcal{H} \times [0, p_{\max}] \mapsto [0, 1]$, $\varepsilon_k \in \mathbb{R}_+$, $k \geq 0$

1: Initialize $\mu_0 \in \mathbb{R}_+^m$, $k \leftarrow 0$

2: **loop**

3: At time k observe channel state h_k

4: Compute power allocation for all systems i and frequencies j by

$$p_{ij,k} \leftarrow \operatorname{argmin}_{0 \leq p \leq p_{\max}} p - \mu_{i,k} q(h_{ij}, p) \quad (43)$$

$$\xi_{ij,k} \leftarrow \min_{0 \leq p \leq p_{\max}} p - \mu_{i,k} q(h_{ij}, p) \quad (44)$$

5: Decide scheduling by solving

$$\alpha_k \leftarrow \operatorname{argmin}_{\alpha \in \Delta_{m,f}} \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij} \xi_{ij,k} \quad (45)$$

6: Compute for all $i = 1, \dots, m$

$$s_{i,k} \leftarrow c_i - \sum_{j=1}^f \alpha_{ij,k} q(h_{ij,k}, p_{ij,k}) \quad (46)$$

7: Update dual variables by $\mu_{k+1} \leftarrow [\mu_k + \varepsilon_k s_k]_+$

8: **end loop**

Proof: See Appendix B ■

The proposition states that the stochastic online algorithm yields a random sequence of dual variables μ_k that converge to the optimal dual variables μ^* in a strong sense, i.e., almost surely for any sequence of channel realizations that is observed. However the real problem of interest is the primal problem (20), or equivalently Problem 1. This is the problem of optimal design of scheduling and power allocation policies that satisfy the given Lyapunov performance requirements (13) for each control system i , while also minimizing the expected power expenditures of the communication process. Hence it is important to characterize how the control systems would actually perform if the communication variables are selected according to the proposed online algorithm. This characterization is provided in the following theorem.

Theorem 2 (Online Scheduling and Power Allocation). *Consider a shared wireless control architecture composed of m systems of the form (1), f frequencies, and communication modeled by (9) depending on channel states $h_k \in \mathcal{H}^{m \times f}$ which are i.i.d. with distribution ϕ , and scheduling and power allocation variables $\alpha_k \in \Delta_{m,f}$, $p_k \in [0, p_{\max}]^{m \times f}$. Also consider given quadratic Lyapunov performance requirements (13) for each system and let Assumptions 1, 2, 3 hold. If α_k, p_k are chosen according to (40)-(42), then almost surely with respect to the channel sequence $\{h_k, k \geq 0\}$ the control performances for all systems $i = 1, \dots, m$ satisfy*

$$\limsup_{k \rightarrow \infty} \mathbb{E}[V_i(x_{i,k+1}) \mid x_{i,k} = x_i, h_0, \dots, h_{k-1}] \leq \rho_i V_i(x_i) + \operatorname{Tr}(P_i W_i), \quad (48)$$

for any state values $x_i \in \mathbb{R}^{n_i}$. In addition, the power

consumption almost surely satisfies

$$\limsup_{k \rightarrow \infty} \mathbb{E} \left[\sum_{i=1}^m \sum_{j=1}^f \alpha_{ij,k} p_{ij,k} \mid h_0, \dots, h_{k-1} \right] \leq P \quad (49)$$

where P is the optimal value of the optimization problem (16).

Proof: See Appendix C. ■

According to the theorem after a sufficiently large time horizon the scheduling and power allocation variables obtained online by the proposed stochastic algorithm perform arbitrarily close to the optimal ones obtained by the optimization problem (20). In particular the Lyapunov requirements of all control systems are satisfied in the limit and the optimal power expenditure is reached. Before proceeding to simulations of the stochastic online algorithm, we present an intuitive interpretation of the algorithm from a resource allocation point of view.

A. Pricing interpretation of online scheduling and power allocation algorithm

In this section we provide an interpretation of the problem variables as well as the online Algorithm 1 in economic terms. In particular we may view each transmitter in the wireless control architecture as an agent that utilizes some scarce resource, namely transmit power, to produce some 'good', namely the probability of successfully transmitting and closing the corresponding control loop. Our development in Section II-A shows that each closed loop has a Lyapunov control performance requirement (cf. (13) that can be translated as requiring c_i units of good (cf. (19)). Under this view, the dual variables μ_i can be interpreted as the 'unit price' at which each agent can 'sell' the produced good. In this context the role of Algorithm 1 is to determine unit prices such that all demand levels c_i are met and in the most profitable manner from the agents' perspective.

More specifically, consider a time step k where prices are set to μ_k and the current channel conditions are described by h_k . If agent i gets access to the channel at frequency j , the agent can spend an amount $p_{ij,k}$ to produce $q(h_{ij,k}, p_{ij,k})$ units of good, which can be sold at a price of $\mu_{i,k}$ per unit. In this case the total profit for the agent can be expressed as

$$\mu_{i,k} q(h_{ij,k}, p_{ij,k}) - p_{ij,k}, \quad (50)$$

i.e., the difference between the total revenue $\mu_{i,k} q(h_{ij,k}, p_{ij,k})$ and the total cost $p_{ij,k}$. The optimal resource allocation $p_{ij,k}$ is the one maximizing the profit (50), matching exactly the optimization over power provided in (43). The optimal profit if agent i gets access to the channel at frequency j under conditions $h_{ij,k}$ equals $-\xi_{ij,k}$ given in (44).

The role of the scheduler is to opportunistically assign agents to the available frequencies in a way that maximizes the total aggregated profit. In particular the scheduler observes current conditions $h_{ij,k}$ for all agents i and frequencies j , computes the possible profit $-\xi_{ij,k}$ of all agent/frequency pairs, and searches for the scheduling $\alpha \in \Delta_{m,f}$ defined by (8) that maximizes the total profit

$$\sum_{i,j} \alpha_{ij} (-\xi_{ij,k}) \quad (51)$$

aggregated over all agents. This optimal scheduling matches the one implemented by Algorithm 1 – see line (45).

After the current scheduling α_k and power p_k decisions have been made, optimal in the sense of maximizing the overall system profit, the unit prices for the next step μ_{k+1} are adjusted depending on the current production levels. In particular, if the production for system i exceeds the required level c_i , meaning that $s_{i,k} < 0$ in (41), then the unit price for system i is reduced to $\mu_{i,k} + \varepsilon_k s_{i,k}$ – see line 7 in Algorithm 1. If on the other hand the production for system i does not meet c_i , i.e., $s_{i,k} > 0$, then the unit price i increases to $\mu_{i,k} + \varepsilon_k s_{i,k}$.

The goal of the online algorithm is to find the optimal prices μ^* , under which the expected production meets demand, where expectation is with respect to the channel conditions. Proposition 3 establishes that the online algorithm converges almost surely to the optimal prices in the limit. Moreover Theorem 2 establishes that in the limit the expected production at each time step meets the demand, while also the expected total production cost (cf. the objective of problem (20)) becomes optimal in the limit.

Note however that Theorem 2 does not provide theoretical guarantees on how fast the solution converges to the optimal one. We discuss this issue along with other limitations of the algorithm in Section VI. In the following section we present simulations verifying our theoretical results, and also indicating that the convergence of the algorithm is relatively fast so that online control performance is not severely affected.

V. SIMULATIONS

A. Advantages of opportunistic scheduling and power allocation

We first illustrate through simulations the opportunistic nature of the resource allocation mechanism for wireless control systems obtained in Section III, in particular how scheduling and power decisions adapt appropriately to channel conditions to meet the control performance goals. Moreover we compare the resulting performance with other simple non channel-adaptive allocation mechanisms. Recall that by solving the auxiliary problems (19), control systems with vector states are converted to scalar constraints in optimization problem (20). Hence without loss of generality we present an example with scalar control systems.

Consider a heating system application controlling the temperature in two independent rooms of a building. Assuming the wireless control architecture of Fig. 1 with $m = 2$, wireless sensors transmit the temperatures of each room to a central location (the access point in Fig. 1) responsible for adjusting the heating in the rooms. For simplicity suppose both systems have identical dynamics of the form (1) with state $x_{i,k}$ denoting the difference between current and some desired temperature for room i . In particular suppose that when system i transmits ($\gamma_{i,k} = 1$), heating is activated for system i and results in stable dynamics $A_{c,i} = 0.4$ in (1). Otherwise if $\gamma_{i,k} = 0$ the system is open loop unstable with $A_{o,i} = 1.1$ in (1), e.g., because heating is deactivated.

For simplicity we assume there is one ($f = 1$) available frequency and for symmetry let channel states $h_{1,k}$ and $h_{2,k}$

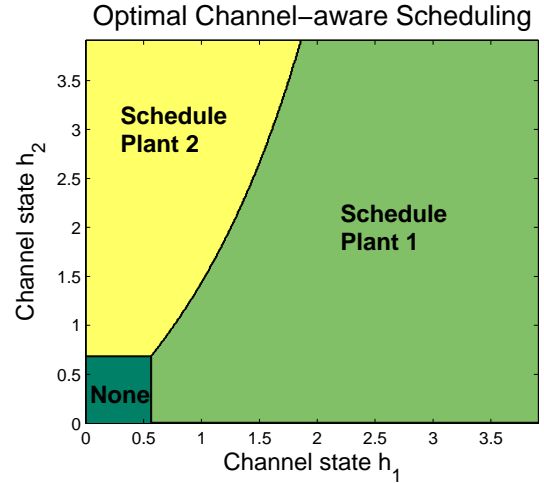


Fig. 3. Optimal channel-aware scheduling for the example presented in Section V. System 1 has a harder Lyapunov decrease rate requirement and is scheduled to transmit for most observed channel states h_1, h_2 . System 2 is scheduled only if its channel conditions h_2 are much more favorable than those of system 1. When both channels are very adverse systems select zero transmit powers so scheduling is irrelevant.

be independent for each system, both having an exponential distribution with mean 1. The function $q(h, p)$ is shown in Fig. 2. For these scalar systems it suffices to consider Lyapunov functions $V_i(x) = x^2$. We require then that system 1 guarantees a high Lyapunov decrease $\rho_1 = 0.75$ rate according to (13), while system 2 only requires $\rho_2 = 0.90$. For these choices we get a higher required success of transmission $c_1 \approx 0.44$ according to (19) for system 1, compared to a lower $c_2 \approx 0.30$ of system 2.

After solving problem (20) offline according to the sub-gradient method of Section III, the optimal channel-aware scheduling and power allocation variables are depicted in Fig. 3 and Fig. 4 respectively. We observe in Fig. 3 that System 1, which requires higher transmission success c_1 , is scheduled to transmit for most values of the channel states h_1, h_2 . System 2, which has a lower requirement, is scheduled only if its channel h_2 is sufficiently favorable *and* system 1 experiences an adverse channel h_1 . This illustrates how the scheduler exploits opportunistically the channel conditions to select which system will transmit to close the loop, in order to meet the Lyapunov constraints in a power efficient manner. We also note that when both systems experience very adverse channels the scheduling decision becomes irrelevant because, as we will see in Fig. 4, the optimal transmit powers then are zero (no transmission).

The optimal power allocation is decentralized as we noted in Remark 3, i.e., the transmit power p_i for system i depends only on the channel h_i that system i experiences, and thus we plot in Fig. 4 the power allocation for both systems on same axes. For both systems, when the channel conditions are adverse it is not worth to spend transmit power. System 1, which has a more demanding control constraint, requires in general higher transmit power since, as we saw in Fig. 3, it is scheduled to transmit even under adverse channel conditions. This is also captured in the expected power consumption of each system

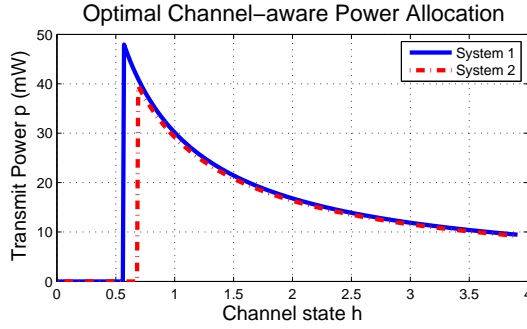


Fig. 4. Optimal channel-aware power allocation for the example presented in Section V. Under adverse channel conditions systems do not transmit. The channel threshold for transmission for system 1 is lower than that of system 2 because the former has a higher Lyapunov decrease rate requirement. System 1 also requires higher transmit power.

computed numerically as $\mathbb{E}_h \alpha_1^*(h) p_1^*(h_1) \approx 11mW$ and $\mathbb{E}_h \alpha_2^*(h) p_2^*(h_2) \approx 6.5mW$. The minimum total power budget required to meet the control objectives then is approximately $17.5mW$.

To demonstrate the power savings obtained by the opportunistic resource allocation we compare to a simple non-channel-aware communication mechanism. In particular suppose that at each step a system is chosen randomly to access the channel/frequency. With a slight abuse of notation suppose systems 1 and 2 are chosen with probabilities α_1 and $\alpha_2 = 1 - \alpha_1$ respectively. When a system is selected, we suppose it transmits with a constant power level p_c . The control performance requirements (cf. (20)) in this case become $\alpha_i \mathbb{E}_{h_i} q(h_i p_c) \geq c_i$ for $i = 1, 2$ and the total power cost is $(\alpha_1 + \alpha_2) p_c = p_c$. We briefly comment then on possible designs for α_1 and p_c .

First, observe from the channel-aware design in Fig. 4 that a system never transmits with power level larger than $50mW$. Suppose then we select the power budget $p_c = 50mW$. It turns out that the two control performance requirements cannot be achieved in this case, because we compute

$$\sum_{i=1,2} \alpha_i \mathbb{E}_{h_i} q(h_i p_c) = \mathbb{E}_{h_i} q(h_i p_c) \approx 0.65 < c_1 + c_2 \approx 0.74 \quad (52)$$

meaning that the constraints are infeasible. Searching numerically for a value p_c where the random access scheme meets the control objectives, we find $p_c \approx 73mW$. Contrasting this amount with the optimal power budget of the opportunistic case above, in this example the channel-aware resource allocation succeeded almost a 80% decrease in power budget compared to a not channel-aware random access scheme.

B. Stochastic online scheduling and power allocation

Next we implement the stochastic online algorithm of Section IV in a setup with three ($m = 3$) control loops sharing two ($f = 2$) frequencies. For example consider again the room heating system of the previous section including three rooms/systems with identical dynamics, $A_{o,i} = 1.1$ and $A_{c,i} = 0.4$ as before. We set the desired Lyapunov decrease rates as $\rho_1 = 0.75$, $\rho_2 = \rho_3 = 0.9$, implying that

	Control objective ρ	Mean Fading h_{i1}	Mean Fading h_{i2}	Transmit Rate at Freq. 1	Transmit Rate at Freq. 2
Plant 1	0.75	1	1	0.25	0.23
Plant 2	0.9	1	1	0.18	0.15
Plant 3	0.9	1	2	0.07	0.25

TABLE I
SYSTEM PARAMETERS & ONLINE TRANSMISSION RATES

system 1 is more demanding in communication resources. We assume channel states h_{ij} are independent across systems i and frequencies j , and have exponential distributions with means given in Table I. In particular we model that system 2 experiences better channel quality (higher channel fading gain) in frequency 2.

The evolution of the dual variables μ_k during Algorithm 1 is shown in Fig. 5. After a number of iterations (time k in this example corresponds to seconds) they remain in a small neighborhood around the optimal μ^* , as anticipated by the theoretical a.s. convergence in Prop. 3. Consequently, the scheduling and power allocation decisions taken online are almost feasible for the constraints of problem (20) after a number of iterations. We observe that the dual variable corresponding to system 1 is the largest, consistent with the fact that it has a harder control requirement to meet. Using the economic interpretation of Section IV-A about the dual variables, the price at which agent 1 can sell its produced good is higher, giving the incentive to schedule agent 1 to produce more often. On the other hand, systems 2 and 3 have the same control requirements but the dual variable for system 2 is larger. The reason is that system 2 experiences worse channel conditions than system 3 (cf. Table I), which imply higher required transmit power, or in economic terms a higher production cost in (50). By setting a higher selling price μ_2 , system 2 becomes profitable enough so that it is scheduled to produce at a sufficient rate to meet the requirement.

In Table I we show the average transmission rates that the online algorithm selected during system execution. In particular we evaluate the average number of time slots where each system i was selected to transmit (with a positive power level) at each frequency j as $1/N \sum_{k=1}^N \alpha_{ij,k} \mathbb{I}(p_{ij,k} > 0)$. System 3 was scheduled mainly at frequency 2, exploiting its better channel quality. This forced systems 1 and 2 to use frequency 1 more often. Also system 1, which has higher control requirement, transmitted more often than the other systems. We note that this behavior resulted from the online algorithm using only an observed channel sequence, not any prior knowledge on the channel quality distribution.

Finally, we examine the evolution of the three heating control systems when the online algorithm is employed for scheduling and power decisions. Suppose that for all systems i the states x_i , which measure deviations from reference room temperatures, are perturbed by disturbances $w_{i,k}$ as in (1), which we model as independent Gaussian with mean zero and variance $W_i = 1$ (at some normalized units of temperature). We plot in Fig. 6 the evolution of the empirical quadratic averages $1/N \sum_{k=1}^N x_{i,k}^2$. Recall that when the Lyapunov condition (13) is satisfied, we get from (14) that the

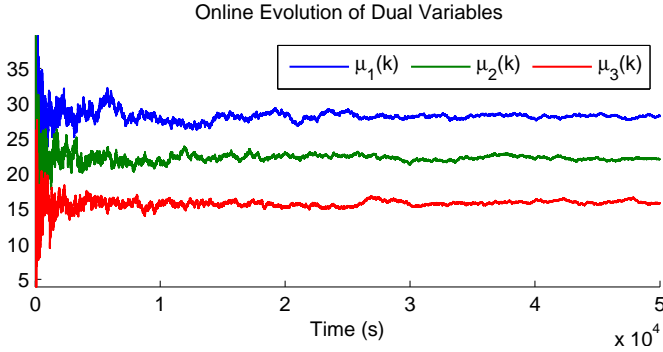


Fig. 5. After a number of steps during the online algorithm the dual variables μ_k remain in a neighborhood around the optimal μ^* .

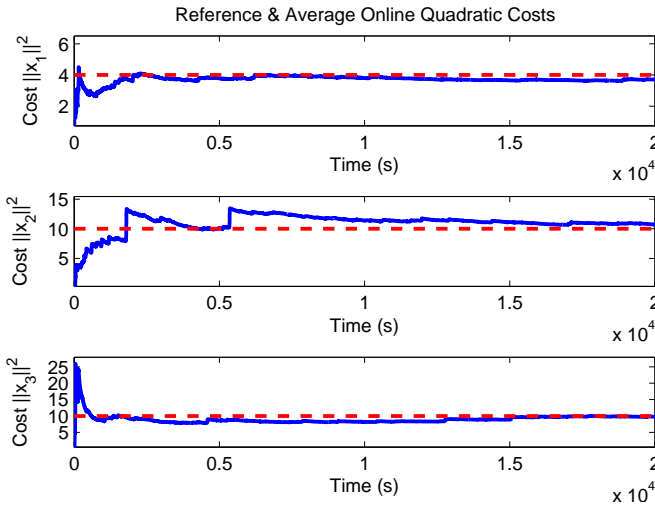


Fig. 6. Average quadratic costs during the online scheduling and power allocation algorithm. The stochastic algorithm keeps the average quadratic cost of each control system close to the upper bound of the limit expected cost, shown with dashed lines, induced theoretically by the required Lyapunov decrease rates.

expected limit quadratic costs are bounded by $W_i/(1 - \rho_i)$. We observe from Fig. 6 that after some initial transient the online communication algorithm keeps the empirical average quadratic costs close to the theoretical required upper bounds.

VI. DISCUSSION AND CONCLUSIONS

In this paper we presented a framework for designing opportunistic channel-aware schedulers for wireless control systems with multiple loops closing over a shared wireless medium. We showed that a stochastic optimization formulation is suitable for this setup. In particular, we considered scheduling and transmit power designs that minimize the total expected power expenditures while guaranteeing that given Lyapunov functions for each of the control systems exhibit a desired decrease rate for stability and performance. We developed an offline optimization algorithm, as well as an online stochastic algorithm utilizing a random observed channel sequence to solve the problem.

While the proposed online algorithm guarantees almost sure convergence to the optimal solution, it does not provide a

theoretical characterization of the convergence rate. This could potentially introduce a long transient control system behavior before the desired performance is reached. Another drawback is that the online algorithm uses decreasing step sizes, which limits the ability to adapt to an environment where the channel distributions are not stationary but vary with time. These issues will be the focus of future work. The case of scheduling interdependent control tasks is considered in [23]. A different research direction is to include, apart from channel states, the measured plant system states when making the scheduling decisions, as in, e.g., [10]–[12], or in the power management paradigm for a single closed loop in [19].

APPENDIX

A. Proof of Proposition 2

We first show part (a) of the proposition. Consider the problem of minimizing the Lagrangian as given at the form (36) over variables $\alpha(\cdot), p(\cdot)$ for some $\mu \in \mathbb{R}_+^m$. Since $\mu^T c$ is constant the problem is equivalent to

$$\min_{\alpha, p \in (\mathcal{A}, \mathcal{P})} \mathbb{E}_h \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij}(h) [p_{ij}(h) - \mu_i q(h_{ij}, p_{ij}(h))]. \quad (53)$$

Without loss of generality we can exchange the expectation over h and the minimization over functions $\alpha(\cdot), p(\cdot)$ in (53) to equivalently solve for each $h \in \mathcal{H}^{m \times f}$

$$\min_{\substack{\alpha(h) \in \Delta_{m,f} \\ p_{ij}(h) \in [0, p_{\max}]}} \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij}(h) [p_{ij}(h) - \mu_i q(h_{ij}, p_{ij}(h))] \quad (54)$$

This step is valid because any pair of functions α, p that does not minimize the objective in (54) on a set of values of variables h with ϕ -positive measure must yield a strictly larger expected value in the objective of (53). In other words, the minimizers of (53) can only differ from the minimizers of (54) at a set of values for h with measure zero.

Then note that at any $h \in \mathcal{H}^{m \times f}$ and any choice for the variable $\alpha(h)$ we have that $\alpha_{ij}(h) \geq 0$. Hence the optimization over $p(h)$ in (54) can be rearranged to

$$\min_{\alpha(h) \in \Delta_{m,f}} \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij}(h) \min_{p_{ij}(h) \in [0, p_{\max}]} p_{ij}(h) - \mu_i q(h_{ij}, p_{ij}(h)). \quad (55)$$

The optimization over power variables $p_{i,j}(h)$ in this expression corresponds exactly to (37). Using the notation introduced in (39), the minimization over scheduling variables $\alpha(h)$ in (55) becomes

$$\min_{\alpha(h) \in \Delta_{m,f}} \sum_{i=1}^m \sum_{j=1}^f \alpha_{ij}(h) \xi(h_{ij}, \mu_i), \quad (56)$$

The expression given in (38) is obtained by relaxing the integer constraint $\alpha_{ij} \in \{0, 1\}$ of the set $\Delta_{m,f}$ (cf.(8)) in problem (56) with $\alpha_{ij} \geq 0$. The resulting problem (38) is a linear program,

but the optimal solution will be integer (see, e.g., [22, Th. 7.5]) and feasible with respect to $\Delta_{m,f}$.

Now let us prove part (b) of the proposition. We need to show that any pair $\alpha(\mu), p(\mu)$, which are functions of h , that solves (53) gives a unique evaluation of $s(\mu)$ given in (31). Since $s_i(\mu)$ involves integrating the term

$$\sum_{j=1}^f \alpha_{ij}(\mu; h) q(h_{ij}, p_{ij}(\mu; h)) \quad (57)$$

with respect to the distribution ϕ of $h \in \mathcal{H}^{m \times f}$, it suffices to show that (57) is unique ϕ -a.s.

By the argument presented already, minimizing (53) is a.s. equivalent to minimizing (54). The latter is again equivalent to the problem (55) since all $\alpha_{ij}(h) \geq 0$. Note that the only case where the optimizers in (54) can differ from the ones obtained in (55) is if $\alpha_{ij}(\mu; h) = 0$ for some i, j is optimal at some values $h \in \mathcal{H}^{m \times f}$ and the power minimizer $p_{ij}(\mu; h)$ in (54) can be chosen arbitrarily. But this does not affect the computation of $s_i(\mu)$ since (57) will equal zero. Hence we only need to show that the minimizers $\alpha(\mu; h), p(\mu; h)$ in (55) imply a.s. uniqueness of (57).

For values of h where the minimizers $\alpha(\mu; h), p(\mu; h)$ of problem (55) are unique it is immediate that (57) has a unique value, hence we only need to consider h where the minimizers are not unique. By Assumption 2(c) the minimizer $p(\mu; h)$, which is given in (37), is unique for almost all h , therefore we only need to focus on the set of values for h where the minimizer $\alpha(\mu; h)$, described by (38), is not unique.

Let us denote by E the set of interest, i.e., the set of $h \in \mathcal{H}^{m \times f}$ where $\alpha(\mu; h)$ in (38) is not unique. By considering all possible pairs of multiple solutions $\alpha' \neq \alpha''$ in the finite set $\Delta_{m,f}$, we can rewrite E as a union

$$E = \bigcup_{\alpha' \neq \alpha'' \in \Delta_{m,f}} E_{\alpha', \alpha''} \quad (58)$$

where²

$$E_{\alpha', \alpha''} = \left\{ h \in \mathcal{H}^{m \times f} : \alpha', \alpha'' \in \operatorname{argmin}_{\alpha \in \Delta_{m,f}} \sum_{i,j} \alpha_{ij} \xi(h_{ij}, \mu_i) \right\}. \quad (59)$$

In other words, the set $E_{\alpha', \alpha''}$ is the set of values h where both α', α'' are optimal for (38). The rest of the proof shows that on any $E_{\alpha', \alpha''}$ the value of (57) is almost surely unique.

The set $E_{\alpha', \alpha''}$ depends on the shape of the function ξ defined in (39), so next we point out two properties of $\xi(h_{ij}, \mu_i)$.

Fact 1: For almost all h_{ij} where the optimal value of problem (39) is $\xi(h_{ij}, \mu_i) = 0$, the optimal solution is unique and equals $p_{ij}(\mu; h) = 0$.

Proof of Fact 1: First we note that for any h_{ij} , the choice $p = 0$ is feasible for problem (39) and by Assumption 2(a) it gives an objective $p - \mu_i q(h_{ij}, p) = 0$. So whenever the optimal value of problem (39) is 0, then $p = 0$ is an optimal solution. This optimal solution is unique for almost all h_{ij} because of Assumption 2(c). ■

Fact 2: If at some h_{ij} the optimal value of problem (39) is $\xi(h_{ij}, \mu_i) < 0$, then for $h'_{ij} > h_{ij}$ we have that $\xi(h'_{ij}, \mu_i) < \xi(h_{ij}, \mu_i)$.

Proof of Fact 2: First note that at the given h_{ij} it must be that the optimal solution $p_{ij}(\mu; h)$ of problem (39) satisfies $q(h_{ij}, p_{ij}(\mu; h)) > 0$. This is true because otherwise $q(h_{ij}, p_{ij}(\mu; h)) = 0$ implies $\xi(h_{ij}, \mu_i) = p_{ij}(\mu; h) \geq 0$. Second by Assumption 2(b) when $q(\cdot) > 0$, it is strictly increasing in its argument. Thus we have for $h'_{ij} > h_{ij}$ that

$$\begin{aligned} \xi(h_{ij}, \mu_i) &= p_{ij}(\mu; h) - \mu_i q(h_{ij}, p_{ij}(\mu; h)) \\ &> p_{ij}(\mu; h) - \mu_i q(h'_{ij}, p_{ij}(\mu; h)) \geq \xi(h'_{ij}, \mu_i). \end{aligned} \quad (60)$$

Let us now fix some $\alpha' \neq \alpha'' \in \Delta_{m,f}$ and consider the set $E_{\alpha', \alpha''}$. Pick indices i, j where α', α'' differ, i.e., without loss of generality, $\alpha'_{i,j} = 1, \alpha''_{i,j} = 0$. Consider first the case of $h \in E_{\alpha', \alpha''}$ where $\xi(h_{i,j}, \mu_i) = 0$. By Fact 1 above we know that this implies $p_{i,j}(\mu; h) = 0$ is almost surely the unique optimizer of (37). But in that case $q(h_{i,j}, p_{i,j}(\mu; h)) = 0$, and the choice of $\alpha_{i,j}(h)$ does not affect the value of (57), which is zero.

Second, we examine the set $h \in E_{\alpha', \alpha''}$ where $\xi(h_{i,j}, \mu_i) < 0$. We will show that this event happens with ϕ -probability zero. In particular by Assumption 1 ϕ has a probability density function on $\mathcal{H}^{m \times f}$, or more formally ϕ is absolutely continuous with respect to the Lebesgue measure on $\mathcal{H}^{m \times f}$. Hence to show that the discussed event has ϕ -measure zero, it suffices to show that it has Lebesgue measure zero. Note that we can upper bound the set as follows

$$\begin{aligned} E_{\alpha', \alpha''} &\cap \{h : \xi(h_{i,j}, \mu_i) < 0\} \\ &\subseteq \{h : \sum_{i,j} (\alpha''_{ij} - \alpha'_{ij}) \xi(h_{ij}, \mu_i) = 0, \quad \xi(h_{i,j}, \mu_i) < 0\} \\ &= \{h : \sum_{i \neq i, j \neq j} (\alpha''_{ij} - \alpha'_{ij}) \xi(h_{ij}, \mu_i) = \xi(h_{i,j}, \mu_i) < 0\} \end{aligned} \quad (61)$$

The subset in the first step is justified from the fact that, in contrary to the definition of $E_{\alpha', \alpha''}$ in (59), we do not take α', α'' to be optimal for problem (38). We only require that they yield the same objective in the problem. The second step follows by the appropriately selected indices i, j .

We will now argue that the last set in (61) has Lebesgue measure zero. If we fix the values of all the variables/coordinates $h_{ij}, i \neq i, j \neq j$, there is at most one value for the variable/coordinate $h_{i,j}$ that belongs in the set. The reason is that for values of the $h_{i,j}$ coordinate where $\xi(h_{i,j}, \mu_i) < 0$, Fact 2 above states that $\xi(h_{i,j}, \mu_i)$ is strictly monotonic in $h_{i,j}$. Hence there can be at most one value $h_{i,j}$ that equals the sum within the last set of (61). This means that the last set in (61) can be equivalently described by a mapping from an $m \cdot f - 1$ dimensional space to the space $\mathcal{H}^{m \times f}$, or in other words it is a lower-dimensional subset of $\mathcal{H}^{m \times f}$. Hence it has Lebesgue measure zero. This implies that the first set in (61) has Lebesgue (and ϕ) measure zero as well.

The above procedure can be iterated for any pair α', α'' in (58) to conclude that on the set E the value of the subgradient vector is almost surely unique.

²Within this proof we denote $\sum_{i=1}^m \sum_{j=1}^f$ as $\sum_{i,j}$ for compactness.

B. Proof of Proposition 3

We begin by noting that at every time k the vector s_k computed by (41) is a stochastic subgradient for the dual function $g(\mu)$ at the point μ_k , i.e.,

$$g(\mu') - g(\mu_k) \leq (\mu' - \mu_k)^T \mathbb{E}[s_k | \mu_k] \quad \text{for all } \mu' \in \mathbb{R}_+^m. \quad (62)$$

To show this fact compare equations (40)-(41) of the online algorithm with (31) to conclude that $\mathbb{E}[s_k | \mu_k] = s(\mu_k)$ because h_k is i.i.d for every k . Inequality (62) then follows directly from (30).

Then note that by Assumption 3 there exists a strictly feasible primal solution α', p' . Call P' the resulting objective value (20) at this point, and let a positive constant $\varepsilon' > 0$ denote the constraint slack of (21) at this point, i.e., $c_i + \varepsilon' \leq \mathbb{E}_h \sum_{j=1}^f \alpha'_{ij}(h) q(h_{ij}, p'_{ij}(h))$. Then we may bound the dual function (23) at any point μ by

$$\begin{aligned} g(\mu) &\leq L(\alpha', p', \mu) \\ &= P' + \sum_{i=1}^m \mu_i \left[c_i - \mathbb{E}_h \sum_{j=1}^f \alpha'_{ij}(h) q(h_{ij}, p'_{ij}(h)) \right] \\ &\leq P' - \sum_{i=1}^m \mu_i \varepsilon' \end{aligned} \quad (63)$$

Rearranging the terms in the above inequality, and since $\mu \geq 0$ it follows that for every i , $\mu_i \leq \sum_{\ell=1}^m \mu_\ell \leq (P' - g(\mu))/\varepsilon'$. In particular we find that the optimal dual variables are finite, $\mu_i^* \leq (P' - D)/\varepsilon'$.

Since the optimal dual variables are finite, the distance $\|\mu_k - \mu^*\|$ between any random μ_k obtained by Algorithm 1 and the set of optimal dual variables μ^* is a well-defined and bounded random variable. The following lemma gives an upper bound on this distance. Here recall that as we commented after (31) the subgradients $s(\mu)$ are always bounded in our problem.

Lemma 1. *Let D be the optimal value of the dual problem (25), μ^* be an optimal solution, and S be the bound on the subgradient $\|s(\mu)\| \leq S$ for any $\mu \in \mathbb{R}_+^m$. Then at each step k of Algorithm 1 the update of μ_{k+1} satisfies*

$$\mathbb{E}[\|\mu_{k+1} - \mu^*\|^2 | \mu_k] \leq \|\mu_k - \mu^*\|^2 + \varepsilon_k^2 S^2 - 2\varepsilon_k(D - g(\mu_k)) \quad (64)$$

Proof: First use the expression $\mu_{k+1} = [\mu_k + \varepsilon_k s_k]_+$ in Algorithm 1 to write

$$\|\mu_{k+1} - \mu^*\| = \|[\mu_k + \varepsilon_k s_k]_+ - \mu^*\| \leq \|\mu_k + \varepsilon_k s_k - \mu^*\|, \quad (65)$$

where the last inequality holds because when projecting on the positive orthant the distance from a point μ^* in the orthant can only decrease. Taking expectation on both sides given μ_k and expanding the square norm of the right hand side, we get

$$\begin{aligned} \mathbb{E}[\|\mu_{k+1} - \mu^*\|^2 | \mu_k] &\leq \|\mu_k - \mu^*\|^2 + \varepsilon_k^2 S^2 \\ &\quad + 2\varepsilon_k(\mu_k - \mu^*)^T \mathbb{E}[s_k | \mu_k] \end{aligned} \quad (66)$$

where we bounded $\|\mathbb{E}[s_k | \mu_k]\|^2 < S^2$. The statement (64) follows from (66) by applying inequality (62) with the substitution $\mu' = \mu^*$. ■

Our goal is to use (64) to show that $\|\mu_{k+1} - \mu^*\|^2 \rightarrow 0$ almost surely. To pursue this we will define a sequence that behaves as a supermartingale stochastic process and use the a.s. convergence results for such processes. In particular we will make use of the following result [24, Th. E7.4].

Theorem 3. *Suppose $\{a_k, k \geq 0\}$ and $\{b_k, k \geq 0\}$ are integrable non-negative stochastic processes adapted to a filtration \mathcal{F}_k , i.e., a_k, b_k measurable with respect to \mathcal{F}_k , and they also satisfy*

$$\mathbb{E}[a_{k+1} | \mathcal{F}_k] \leq a_k - b_k \quad (67)$$

Then a_k converges almost surely and b_k is almost surely summable, i.e., $\sum_{k=0}^{\infty} b_k < \infty$ a.s.

To make the connection between the above theorem and (64) define

$$a_k = \|\mu_k - \mu^*\|^2 + \sum_{\ell=k}^{\infty} \varepsilon_\ell^2 S^2, \quad (68)$$

$$b_k = 2\varepsilon_k(D - g(\mu_k)), \quad (69)$$

and let $\mathcal{F}_k = \{\mu_0, \dots, \mu_k\}$. Note that the process a_k is well defined because by assumption the stepsizes are square summable. Moreover $a_k \geq 0$ and also $b_k \geq 0$ because by definition D is the maximum value $g(\mu_k)$ can take (cf. (25)). Also a_k and b_k are bounded variables for every k because μ_k generated by Algorithm 1 is bounded at every k . Thus a_k and b_k are integrable, and trivially measurable with respect to \mathcal{F}_k . To check that condition (67) holds use the definition of a_k to write

$$\begin{aligned} \mathbb{E}[a_{k+1} | \mathcal{F}_k] &= \mathbb{E}[\|\mu_{k+1} - \mu^*\|^2 | \mu_k] + \sum_{\ell=k+1}^{\infty} \varepsilon_\ell^2 S^2 \\ &\leq \|\mu_k - \mu^*\|^2 + \varepsilon_k^2 S^2 - 2\varepsilon_k(D - g(\mu_k)) + \sum_{\ell=k+1}^{\infty} \varepsilon_\ell^2 S^2 \end{aligned} \quad (70)$$

where for the last inequality we used (64). It is immediate that the right hand side of (70) equals $a_k - b_k$ by our appropriately constructed processes. Hence all conditions of Theorem 3 hold true.

The theorem states that a_k converges almost surely to some random variable. Observe that the second summand $\sum_{\ell=k}^{\infty} \varepsilon_\ell^2 S^2$ of a_k in (68) is deterministic and converges to 0 because of square summability of the stepsizes. Thus we conclude that the random variable $\|\mu_k - \mu^*\|^2$ converges almost surely to some random variable.

To arrive at a contradiction suppose the limit random variable is not identically zero, i.e., it takes positive values with nonzero probability. Equivalently there exist $\delta > 0$ and $\varepsilon > 0$ such that with probability δ we have $\|\mu_k - \mu^*\| \geq \varepsilon$ for all sufficiently large k . This implies that μ_k are bounded away from the optimal, i.e., that for sufficiently large k we have $D - g(\mu_k) \geq \varepsilon'$ for some $\varepsilon' > 0$. Hence with probability δ we have

$$\sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} 2\varepsilon_k(D - g(\mu_k)) = +\infty \quad (71)$$

But this contradicts with Theorem 3 which states that $\sum_{k=0}^{\infty} b_k = \infty$ can only happen at a set of probability measure zero. Therefore $\|\mu_k - \mu^*\|$ must converge to zero with probability 1.

By continuity of the (concave) dual function $g(\mu)$ we also have that $g(\mu_k)$ converge to the optimal value $g(\mu^*) = D$ a.s.

C. Proof of Theorem 2

To show that (48) holds we first convert it into an equivalent one involving variables relating to the dual problem (25). Imitating the steps leading from problem (16) to problem (20), the statement of (48) becomes equivalent to

$$\limsup_{k \rightarrow \infty} c_i - \mathbb{E}_{h_k} \left[\sum_{j=1}^f \alpha_{ij,k} q(h_{ij,k}, p_{ij,k}) \mid \mu_k \right] \leq 0. \quad (72)$$

Here to suppress notation we have exploited the fact that according to the online algorithm the variables α_k, p_k depend just on the value of the variable μ_k and not on the whole observed channel history (but μ_k does depend on the whole history).

Then by the expression of s_k given in (41) condition (72) is equivalent to $\limsup_{k \rightarrow \infty} \mathbb{E}_{h_k} [s_k \mid \mu_k] \leq 0$. Also we already argued in the proof of Prop. 3 that $\mathbb{E}_{h_k} [s_k \mid \mu_k] = s(\mu_k)$ where $s(\mu_k)$ is given by (31) and expresses a subgradient of the dual function g at μ_k . To sum up, we have shown so far that (48) is equivalent to $\limsup_{k \rightarrow \infty} s(\mu_k) \leq 0$.

Under Assumption 3 we have established in Proposition 3 that for the online algorithm $\mu_k \rightarrow \mu^*$ almost surely with respect to the channel sequence $\{h_k, k \geq 0\}$. Then we note a convex analysis fact by [21, Prop. 4.2.3]. If g is concave, and $\mu_k \rightarrow \mu^*$, and $s(\mu_k)$ is selected as a subgradient of g at μ_k , then every limit point of $s(\mu_k)$ is a subgradient of g at μ^* . Hence for the sequence of μ_k obtained by the online algorithm we have that almost surely the sequence $s(\mu_k)$ converges to a subgradient of g at μ^* .

Also, as follows from Danskin's theorem [21, Prop. 4.5.1], the subgradients of the dual function g at any point μ belong in the convex hull of the vectors $s(\mu)$ obtained in (31). But under Assumptions 1, 2, and 3, Proposition 1(c) combined with Proposition 2(b) shows that all subgradient vectors $s(\mu^*)$ at μ^* satisfy $s(\mu^*) \leq 0$, as we argued in the proof of Theorem 1. Hence for the sequence of μ_k obtained by the online algorithm we have that almost surely $\limsup_{k \rightarrow \infty} s(\mu_k) \leq 0$, which verifies statement (48).

Finally let us prove (49). Recall that the dual function equals $g(\mu) = L(\alpha(\mu), p(\mu), \mu)$ where $\alpha(\mu), p(\mu)$ are chosen as Lagrange optimizers at μ according to (24). Using the definition of the Lagrangian at (22) and the interpretation of the subgradient $s(\mu)$ at (31) as the constraint slack, we have that for any μ_k

$$\begin{aligned} g(\mu_k) &= L(\alpha(\mu_k), p(\mu_k), \mu_k) \\ &= \mathbb{E}_h \sum_{i=1}^m \alpha_i(\mu_k; h) p_i(\mu_k; h) + \mu_k^T s(\mu_k) \end{aligned} \quad (73)$$

Now observe that the expectation in (49) equals the expectation given in (73) because by design of Algorithm 1 the

primal variables α_k, p_k are selected as Lagrange optimizers at μ_k . Therefore to show that (49) holds a.s. it suffices to show that the expectation in (73) converges a.s. to P which equals D by strong duality.

Proposition 3 establishes that the left hand side of (73) converges to $g(\mu_k) \rightarrow D$, and also that $\mu_k \rightarrow \mu^*$ a.s. We have also already argued that $s(\mu_k) \rightarrow s(\mu^*)$ a.s. Therefore also $\mu_k^T s(\mu_k) \rightarrow \mu^{*T} s(\mu^*)$ a.s. But by Prop. 1(b) $\mu^{*T} s(\mu^*) = 0$. This shows that the expectation at the right hand side of (73) converges to D , which completes the proof.

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