Navigation Functions for Convex Potentials in a Space with Convex Obstacles

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Abstract—**Consider a convex set of which we remove an arbitrarily number of disjoints convex sets – the obstacles. In this work we design a controller that ensures the goal of navigating in such a set from almost every initial position to the minimum of an unknown convex function. An artificial potential is proposed and we show that – under some restrictions over the size of the obstacles and its distance to the minimum of the objective function – is a navigation function. A gradient descent flow of this function ensures convergence to the minimum of the objective function for a set of initial position whose measure is one. The gradient controller uses only local information of the objective function – its value at a given point and its gradient. This controller pushes the agent along a linear combination of the gradient of the objective function and a direction pointing away from the obstacles. The weights of this linear combination are such that when we are away of the obstacles the dominant direction is the one of the gradient of the objective function. On the contrary, when we are close to an obstacle the repellent direction dominates, hence ensuring the avoidance of the obstacles. Numerical examples show that this controller achieves the minimum of the objective function without hitting any obstacle.**

I. Introduction

Consider a robot that must climb a hill while avoiding the trees on his way. The agent does not know where the location of the top of the hill is, however if he keeps moving in an ascending direction it will achieve the maximum. This observation suggest the use of a gradient controller to solve the problem. Yet, the ascent direction could be directed precisely to one of the trees causing that the robot crashes. Therefore, a modification to the classical gradient controller must be included to attain the top of the hill and avoid the obstacles. This paper shows that under some restrictions on the environment – the distribution of the trees on the hill – a gradient controller based on a navigation function, constructed similarly as in [1] achieves the goal of climbing the hill without encountering any obstacle.

More generically, we consider an agent that its trying to attain the minimum (maximum) of a convex (concave) function – the height profile of the hill– over a set that contains forbidden convex regions – the trees. We aim to construct an artificial potential based on the position of the obstacles and the objective function of interest such that for almost every initial condition we converge to the desired minimum (maximum). In order to succeed we need that the minimum of the artificial potential coincides with the minimum of the function that we are minimizing and its maximum must be in the boundary of the forbidden regions, so that a gradient descent controller never reaches these points. In addition it is necessary that the artificial potential does not have any other local minima than the minimum of the objective function. Otherwise for some initial positions the trajectory would be attracted to those other critical points.

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Navigation functions and artificial potentials have been widely used to address the path planning problem to achieve a desired configuration, see e.g. [1]–[7]. One of the advantages of this approach is that it provides a minimal energy trajectory as discussed in [8]. In particular in [1] the problem was solved when obstacles are spheres and an extension to star shaped obstacles has been made in [2]. In this constructions, the complete knowledge of the obstacles is assumed, further works relaxed this hypothesis [9]–[12]. The navigation function framework provide as well a manner to handle multi agent path planning problems [13]–[15]. The interest of using navigation functions is that the path planning problem is jointly solved with the problem of following the planned trajectory, hence the problem can be solved by a low level controller as opposed to other motion planning techniques as cell decomposition [16]–[21]. On the other hand the latter algorithms provide a collision free curve on the free space. Once such a trajectory is computed, the torques needed to follow the curve are computed by the inverse dynamics technique [22], [23]. However, while designing the curve, the robot's dynamics constraints are not taken into account, hence it is not guaranteed that the trajectory builded using this approach is feasible. Furthermore, inverse techniques rely on the cancelation of some of the terms present in the dynamic of the robot. However, this is not possible to achieve exactly.

In the navigation function approach a goal position –or configuration– must be specified. However if the desired goal is unknown – as in the problem of climbing a hill– this approach is not suitable. Convergence to the minimum of a function can be ensured using a gradient descent controller of the objective function see e.g. [24]. This approach and variations has been used in extremum seeking controllers in [25]–[27]. However in this cases the problem of avoiding obstacles has not been considered or it has been solved using an addition controller specifically to that task, see e.g. [28].

The novelty of this work is to design a controller that allows the robot to achieve the minimum of an unknown convex function using only its local information while avoiding obstacles in the configuration space. We build an artificial potential following the ideas in [1] and we show that under some restrictions regarding the eigenvalues of the Hessian of the objective function and the geometry of the free space this potential is a navigation function (Theorems 2 and 3 in Section III). The particular form of the navigation function is such that its gradient is a linear combination of the gradient of the objective function and a repelling field from the obstacles. In addition the weights of this linear combination are such that when the agent is away of the obstacles the dominant direction is the one of the gradient of the objective function. Hence the controller behaves away of the obstacles as if it was a pure gradient descent on the objective function. As we get close to the obstacles the repellent field coefficient becomes dominant and the direction of the gradient of the navigation function coincides with the repellent direction. Hence ensuring the avoidance of the obstacles.

II. Problem formulation

We are interested in navigating a punctured space while reaching a target point defined as the minimum of a convex potential function. Formally, let $\mathcal{X} \in \mathbb{R}^n$ be a non empty compact convex set and let $f_0 : \mathcal{X} \to \mathbb{R}_+$ be a convex function whose minimum is the agent's goal. Further consider a set of obstacles $\mathcal{O}_i \subset \mathcal{X}$ with $i = 1 \ldots m$ which are assumed to be open convex sets with nonempty interior and smooth boundary $\partial\mathcal{O}_i$. The free space, representing the set of points accessible to the agent, is then given by the set difference between the space $\mathcal{X}$ and the union of the obstacles $\mathcal{O}_i$,

$$\mathcal{F} = \mathcal{X} \setminus \bigcup_{i=1}^{m} \mathcal{O}_i. \tag{1}$$

The free space in (1) represents a convex set with convex holes; see, e.g., figures 4 and 6. We assume here that the optimal point is in the interior int($\mathcal{F}$) of free space.

Further let $t \in [0, \infty)$ denote a time index and let x^* be the minimum of the objective function, i.e. $x^* := \operatorname{argmin}_{x\in\mathbb{R}^n} f_0(x)$. Then, the navigation problem of interest is to generate a trajectory $x(t)$ that stays in the free space at all times and reaches x^* at least asymptotically,

$$x(t) \in \mathcal{F},\ \forall t \in [0, \infty), \quad \text{and} \quad \lim_{t\to\infty} x(t) = x^*. \tag{2}$$

In the canonical problem of navigating a convex objective defined over a convex set with a fully controllable agent, convergence to the optimal point as in (2) can be assured by defining a trajectory that varies along the negative gradient of the objective function,

$$\dot{x} = -\nabla f_0(x). \tag{3}$$

In a space with convex holes, however, the trajectories arising from the dynamical system defined by (3) satisfy the second goal in (2) but not the first because they are not guaranteed to avoid the obstacles. We aim here to build an alternative function φ such that the trajectory defined by the negative gradient of φ satisfies both conditions. In order to achieve this goal, the function φ must be a navigation function whose formal definition we introduce next [1].

Definition 1 (**Navigation Function**). *Let $\mathcal{F} \subset \mathbb{R}^n$ be a compact connected analytic manifold with boundary. A map $\varphi : \mathcal{F} \to [0, 1]$, is a navigation function in $\mathcal{F}$ if:*

Differentiable. *It is twice continuously differentiable in $\mathcal{F}$.*

Polar at x^*. *It has a unique minimum at x^* which belongs to the interior of the free space, i.e., $x^* \in int(\mathcal{F})$.*

Morse. *It has non degenerate critical points on $\mathcal{F}$.*

Admissible. *All boundary components have the same maximal height, namely $\partial\mathcal{F} = \varphi^{-1}(1)$.*

The properties of navigation functions in Definition 1 are such that the controller $\dot{x} = -\nabla\varphi(x)$ that follows the gradient of φ generates a trajectory that satisfies (2) for almost all initial conditions. To see this observe that, generically, the trajectories arising from gradient flows of a function φ, converge to one of its critical points and that the value of the function evaluated on the trajectory is monotonically decreasing,

$$\varphi(x(t_1)) \geq \varphi(x(t_2)), \quad \text{for any} \quad t_1 < t_2. \tag{4}$$

Admissibility, combined with the observation in (4), ensures that every trajectory whose initial condition is in the free space remains on free space for all future times, thus satisfying the first condition in (2). For the second condition observe that, as per (4), the only trajectory that can have as a limit set a maximum, is a trajectory starting at the maximum itself. This is a set of zero measure if the function satisfies the Morse property. Furthermore, if the function is Morse, the set of initial conditions that have a saddle point as a limit is the stable manifold of the saddle which can be shown to have zero measure as well. It follows that the set of initial conditions for which the trajectories of the system converge to the local minima of φ has measure one. If the function is polar, this minimum is x^* and the second condition in (2) is thereby satisfied. We formally state this result in the next Theorem.

Theorem 1. *Let φ be a navigation function on $\mathcal{F}$ as per Definition 1. Then, the flow given by the gradient control law*

$$\dot{x} = -\nabla\varphi(x), \tag{5}$$

has the following properties:

(i) *$\mathcal{F}$ is a positive invariant set of the flow.*
(ii) *The positive limit set of $\mathcal{F}$ consists of the critical points of φ.*
(iii) *There is a set of measure one, $\tilde{\mathcal{F}} \subset \mathcal{F}$, whose limit set consists of x^*.*

Proof. See Proposition 2.4 [1]. ■

Theorem 1 implies that if φ is a navigation function as defined in 1, the trajectories defined by (5) are such that $x(t) \in \mathcal{F}$ for all $t \in [0, \infty)$ and that the limit of $x(t)$ is the minimum x^* for almost every initial condition. This means that (2) is satisfied for almost all initial conditions. We can therefore recast the original problem (2) as the problem of finding a navigation function φ. Observe that Theorem 1 guarantees that a navigation function can be used to drive a fully controllable agent [cf. (5)]. However, navigation functions can also be used to drive agents with nontrivial dynamics; see Remark 1.

To construct a navigation function φ it is convenient to provide a functional characterization of free space. To that end, consider the space $\mathcal{X}$ and let $\beta_0 : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable concave function such that

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n \,\middle|\, \beta_0(x) \geq 0 \right\}. \tag{6}$$

Since the function β_0 is assumed concave its super level sets are convex. Since the set $\mathcal{X}$ is also convex a function satisfying (6) can always be found. The border $\partial\mathcal{X}$, which is given by the set of points for which $\beta_0(x) = 0$, is called the external boundary of free space. Further consider the m obstacles $\mathcal{O}_i$ and define m twice continuously differentiable convex functions $\beta_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1 \ldots m$. The function β_i is associated with obstacle $\mathcal{O}_i$ and satisfies

$$\mathcal{O}_i = \left\{ x \in \mathbb{R}^n \,\middle|\, \beta_i(x) < 0 \right\}. \tag{7}$$

Functions β_i exist because the sets $\mathcal{O}_i$ are convex and the sublevel sets of convex functions are convex.

Given the definitions of the β_i functions in (6) and (7), the free space $\mathcal{F}$ can be written as the set of points at which all of these functions are nonnegative. For a more succinct characterization, define the function $\beta : \mathbb{R}^n \to \mathbb{R}$ as the product of the $m+1$ functions β_i,

$$\beta(x) = \prod_{i=0}^{m} \beta_i(x). \tag{8}$$

If the obstacles do not intersect, the function $\beta(x)$ is nonnegative if and only if all of the functions $\beta_i(x)$ are nonnegative. This means that $x \in \mathcal{F}$ is equivalent to $\beta(x) \geq 0$ and that we can then define the free space as the set of points for which $\beta(x)$ is nonnegative – when objects are nonintersecting. We state this assumption and definition formally in the following.

Assumption 1 (Objects do not intersect). Let $x \in \mathbb{R}^n$. If for some i we have that $\beta_i(x) \leq 0$, then $\beta_j(x) > 0$ for all $j = 0 \ldots m$ with $j \neq i$.

Definition 2 (Free space). *The free space is the set of points $x \in \mathcal{F} \subset \mathbb{R}^n$ where the function β in* (8) *is nonnegative,*

$$\mathcal{F} = \left\{ x \in \mathbb{R}^n : \beta(x) \geq 0 \right\}. \tag{9}$$

Observe that we have assumed that the optimal point x^* is in the interior of free space. We have also assumed that the objective function f_0 is strongly convex and twice continuously differentiable and that the same is true of the obstacle functions β_i. We state these assumptions formally for future reference.

Assumption 2. The objective function f_0, the obstacle functions β_i and the free space $\mathcal{F}$ are such that:

Optimal point. The minimum $\mathbf{x}^*$ of the objective function f_0 is in the interior of the free space,

$$x^* \in \text{int}(\mathcal{F}). \tag{10}$$

Twice differential strongly convex objective The function f_0 is twice continuously differentiable and strongly convex in $\mathcal{X}$. The eigenvalues of the Hessian $\nabla^2 f_0(x)$ are therefore contained in the interval $[\lambda_{\min}, \lambda_{\max}]$ with $0 < \lambda_{\min}$. In particular, strong convexity implies that for all $x, y \in \mathcal{X}$,

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y - x) + \frac{\lambda_{\min}}{2} \|x - y\|^2, \tag{11}$$

and, equivalently,

$$\left(\nabla f_0(y) - \nabla f_0(x)\right)^T (y - x) \geq \lambda_{\min} \|x - y\|^2. \tag{12}$$

Twice differential strongly convex obstacles The obstacle function β_i is twice continuously differentiable and strongly convex in $\mathcal{X}$. The eigenvalues of the Hessian $\nabla^2 \beta_i(x)$ are therefore contained in the interval $[\mu^i_{\min}, \mu^i_{\max}]$ with $0 < \mu^i \text{min}$.

The goal of this paper is to find a navigation function for the free space $\mathcal{F}$ that has the form of Definition 2 when assumptions 1 and 2 hold. Finding this navigation function is equivalent to attaining the goal in (2) for almost all possible initial conditions. We show that this is possible when the minimum of the objective function takes the value zero. In other cases convergence to a point that can be placed arbitrarily close to x^* is achieved. In the next section we propose a candidate for a navigation function and we derive conditions to guarantee that it is an actual navigation function. We do so after a pertinent remark.

Remark 1 (System with dynamics). If the system is fully controllable, the dynamics in (5) can be imposed and problem (2) be solved by a navigation function. If the system has nontrivial dynamics, a minor modification can be used [29]. Indeed, let $M(x)$ be the inertia matrix of the agent, $g(x, \dot{x})$ and $h(x)$ be fictitious and gravitational forces, and $\tau(x, \dot{x})$ the torque control input. The agent's dynamics can then be written as

$$M(x)\ddot{x} + g(x, \dot{x}) + h(x) = \tau(x, \dot{x}). \tag{13}$$

The model in (13) is of control inputs that generate a torque $\tau(x, \dot{x})$ that acts through the inertia $M(x)$ in the presence of the external forces $g(x, \dot{x})$ and $h(x)$. Let $d(x, \dot{x})$ be a dissipative field – satisfying $\dot{x}^T d(x, \dot{x}) < 0$ – then, by selecting the torque input

$$\tau(x, \dot{x}) = -\nabla\varphi(x) + d(x, \dot{x}), \tag{14}$$

the behavior of the agent converges asymptotically to the gradient controller (5). In particualr, the goal in (2) is achieved.

III. Navigation Function

Following the development in [1] we introduce an order parameter $k > 0$ and define the function φ_k as

$$\varphi_k(x) = \frac{f_0(x)}{\left(f_0^k(x) + \beta(x)\right)^{1/k}}. \tag{15}$$

In this section we state conditions such that for large enough order parameter k, the artificial potential (15) is a navigation function in the sense of Definition 1. These conditions relate the bounds on the eigenvalues of the Hessian of the objective function $\lambda_{\min}$ and $\lambda_{\max}$ as well as the bounds on the eigenvalues of the Hessian of the obstacle functions $\mu^i_{\min}$ and $\mu^i_{\max}$ with the size of the objects and their distance to the minimum of the objective function x^*. The first result concerns the general case where obstacles are defined through general convex functions.

Theorem 2. *Let $\mathcal{F}$ be the free space defined in* (9) *verifying Assumption 1 and let $\varphi_k : \mathcal{F} \to [0,1]$ be the function defined in* (15). *Let $\lambda_{\max}$, $\lambda_{\min}$ and $\mu^i_{\min}$ be the bounds in Assumption 1. Further let the following condition hold for all $i = 1 \ldots m$ and for all x_s in the boundary of $\mathcal{O}_i$*

$$\frac{\lambda_{\max}}{\lambda_{\min}} \frac{\nabla\beta_i(x_s)^T(x_s - x^*)}{\|x_s - x^*\|^2} < \mu^i_{\min}. \tag{16}$$

Then, there exists a constant K such that if $k > K$, the function φ_k in (15) *is a navigation function with minimum at x^* if $f_0(x^*) = 0$ and with minimum arbitrarily close to x^* if $f_0(x^*) \neq 0$.*

Theorem 2 establishes a condition [cf. (16)] on the obstacles and objective function for which φ_k is guaranteed to be a navigation function for sufficiently large order k. The condition has to be satisfied at all the points that lie in the border of an obstacle. The condition in (16) is not difficult to check numerically, but we are more interested in its interpretation that its use in practice. Observe first that, generically, (16) is easier to satisfy when the ratio $\lambda_{\max}/\lambda_{\min}$ is small and when the minimum eigenvalue $\mu^i_{\min}$ is large. The first condition means that we want the objective to be as close to spherical as possible and the second condition that we don't want the obstacle to be too flat. Further note that the left hand side of (16) is negative if $\nabla\beta_i(x_s)$ and $x_s - x^*$ point in opposite directions. This means that the condition can be violated only by points in the border that are "behind" the obstacle as seen from the minimum point. For these points the worst possible situation is when the gradient at the border point x_s is aligned with the line that goes from that point to the minimum x^*. In that case we want the gradient $\nabla\beta_i(x_s)$ and the ratio $(x_s - x^*)/\|x_s - x^*\|^2$ to be small. The gradient $\nabla\beta_i(x_s)$ being small means that we don't want the obstacle to have sharp curvature and the ratio $(x_s - x^*)/\|x_s - x^*\|^2$ being small means that we don't want the destination x^* to be too close to the border. In summary, the simplest navigation problems have objectives and obstacles close to spherical and minima that are not close to the border of the obstacles.

The insights described above notwithstanding, a limitation of Theorem 2 is that it does not provide a simple way to determine if it is possible to build a navigation function with the form in (15) for a given space and objective. In the following section we consider ellipsoidal obstacles and derive a condition that is easy to check and not too restrictive.

A. Ellipsoidal obstacles

Considering the particular case where the obstacles are ellipsoids. Let $A_i \in \mathcal{M}^{n\times n}$, with $i = 1 \ldots m$, be symmetric positive definite matrices. Let x_i and r_i be the center and the length of the largest axis of each one of the obstacles $\mathcal{O}_i$. Then, for each $i = 1 \ldots m$ we define $\beta_i(x)$ to be

$$\beta_i(x) = (x - x_i)^T A_i (x - x_i) - \mu^i_{\min} r_i^2, \tag{17}$$

The obstacle $\mathcal{O}_i$ is defined as those points in $\mathbb{R}^n$ where $\beta_i(x)$ is not positive. In particular for $\beta_i(x) = 0$, the boundary of the obstacle, we have that

$$\frac{1}{\mu^i_{\min}} (x - x_i)^T A_i (x - x_i) = r_i^2, \tag{18}$$

which defines an ellipsoid whose largest axis has length r_i. For the geometry here considered, Theorem 2 takes the following form.

Theorem 3. *Let $\mathcal{F}$ be the free space defined in* (9) *verifying Assumption 1, and let $\varphi_k : \mathcal{F} \to [0,1]$ be the function defined in* (15). *Let $\lambda_{\max}$, $\lambda_{\min}$, $\mu^i_{\max}$ and $\mu^i_{\min}$ be the bounds from Assumption 2. Let β_i take the form of* (17) *and let Let the following inequality hold for all $i = 1..m$*

$$\frac{\lambda_{\max}}{\lambda_{\min}} \frac{\mu^i_{\max}}{\mu^i_{\min}} < 1 + \frac{d_i}{r_i}, \tag{19}$$

where $d_i = \|x_i - x^*\|$. Then there exists a constant K such that if $k > K$, then φ_k is a navigation function with minimum at x^* if $f_0(x^*) = 0$ and with minimum arbitrarily close to x^* if $f_0(x^*) \neq 0$.

Proof. See Appendix C. ■

Condition (19) gives a simple form of evaluating if it is possible to build a navigation function by increasing the parameter k in (15). The more eccentric the obstacles and the level sets of the objective function are, the larger becomes the left hand side of (19). In particular, for a flat obstacle – understood as an ellipses having its minimum eigenvalue equal to zero– the considered condition is impossible to satisfy. Notice that this is consistent with the conclusion obtained from Theorem 2. On the other hand, the density of obstacles plays a role. By increasing the distance between the center of the obstacles and the objective, d_i – or by decreasing the size of the obstacles, r_i – we decrease the density of obstacles in the space. This increases the right hand side of (19), therfeore making it easier to navigate the environment.

Let $v_{\min}$ be the eigenvector associated to the eigenvalue $\lambda_{\min}$. For any situation in which $v_{\min}$ is aligned with the direction $x_i - x^*$, then if condition (19) is violated with equality the artificial potential in (15) fails to be a navigation function. In this sense, we can say that condition (19) is tight. This is no longer the case if these directions are not aligned. We present the next simulation as an illustration of the above discussion. Consider the following example in $\mathbb{R}^2$ with only one circular obstacle of radius 2 and objective function given by

$$f_0(x) = x^T \begin{pmatrix} 1 & 0 \\ 0 & \lambda_{\max} \end{pmatrix} x, \tag{20}$$

For this example, the minimum of the objective function is attained at the origin and the left hand side of (19) takes value $\lambda_{\max}$. For the first two simulations we consider the case in which the direction $x_i - x^*$ is aligned with the direction of the eigenvector associated with the smallest eigenvalue of the objective function. With this purpose we place the center of the obstacle in the horizontal axis, in particular at $(-4, 0)$. The right hand side of (19) takes therefore the value 3. In the simulation depicted in Figure 1, $\lambda_{\max}$ is set to be three, therefore violating condition (19). As it can be observed a local minimum other than x^* appears to the left of the obstacle to which the trajectory converges. Thus, the potential defined in (15) fails to be a navigation function. If $\lambda_{\max} = 2$ as in the simulation depicted in Figure 2 then the convergence to

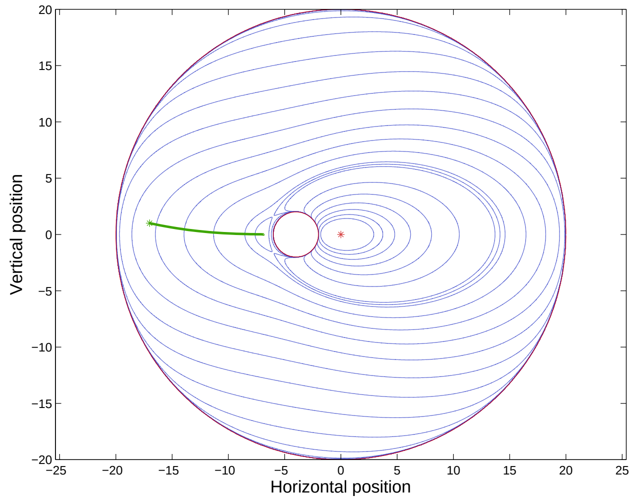

Fig. 1: φ_k fails to be a navigation function when the left and right hand sides of (19) are equal. Observe the presence of a local minimum different than the minimum of f_0 to which the trajectory is attracted. The experiment was performed with $k = 10$.

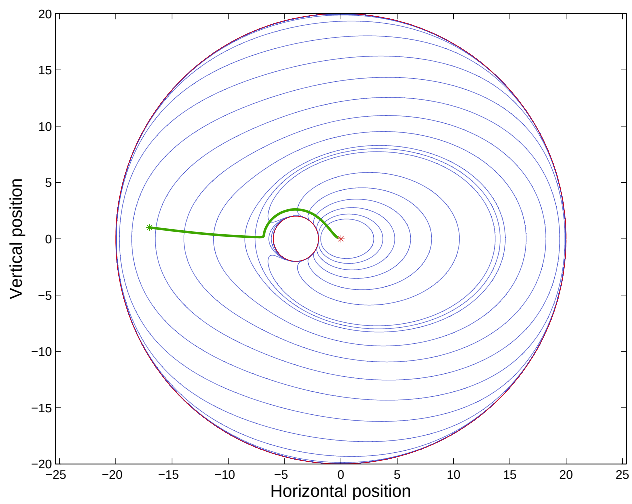

Fig. 2: When (19) is satisfied φ_k is a navigation function and the trajectory arising from the negative gradient flow converges to the minimum of f_0. The local minimum to the left of the obstacle in Figure 1 vanishes. The experiment was performed with $k = 10$.

the origin is achieved. In Figure 3 we observe an example in which the trajectory converges to x^* and condition (19) is violated at the same time. Here, the center of the obstacle is placed at $(0, -4)$, and therefore the direction $x_i - x^*$ is no longer aligned with the direction of the eigenvalue of the Hessian of the objective function associated to the minimum eigenvalue. Hence showing that condition (19) is loose when those directions are not collinear.

Notice that the problem of navigating a spherical world to reach a desired destination x^* [1] can be understood as particular case where the objective function takes the form $\|x - x^*\|^2$ and the obstacles are spheres. In this case φ_k is a navigation function for some large enough k for every valid world (satisfying Assumption 1), irrespectively of the size and placement of the obstacles. This result can be derived

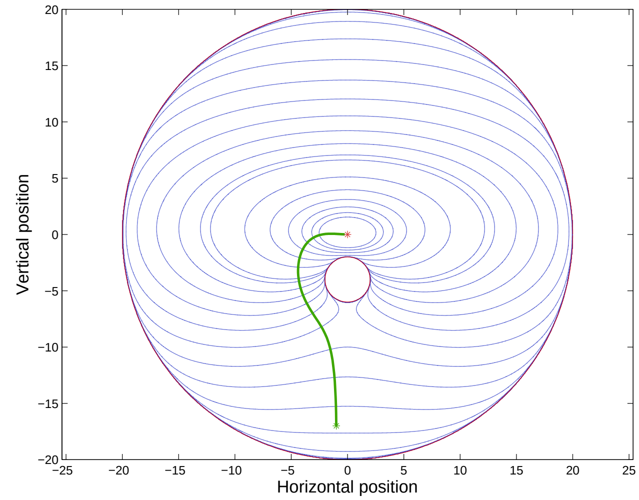

Fig. 3: Condition (19) is violated, however φ_k is a navigation function. We do not observe a local minimum on the side of the obstacle that is opposed to the minimum of f_0 as we do in Figure 1. The latter is because the direction given by the center of the obstacle and x^* is not aligned with the direction corresponding to the maximum eigenvalue of the Hessian of f_0.

as a corollary of Theorem 3 by showing that condition (19) is always satisfied in the setting of [1].

Corollary 1. *Let $\mathcal{F}$ be a compact connected analytic manifold with boundary as defined in* (9) *and let $\varphi_k : \mathcal{F} \to [0, 1]$ be the function defined in* (15)*. Let $\mathcal{F}$ verify Assumption 1 and let $f_0(x) = \|x - x^*\|^2$. Let the obstacles be hyper spheres of centers x_i and radii r_i for all $i = 1..m$. Then there exists a constant K such that if k in* (15) *is larger than K, then φ_k is a navigation function.*

Proof. Since spherical obstacles are a particular case of ellipsoids the hypothesis of Theorem 3 are satisfied. To show that φ_k is a navigation function we need to show that condition (19) is satisfied. For this obstacle geometry we have $\mu^i_{\min} = \mu^i_{\max}$ for all $i = 1 \ldots m$. On the other hand, the Hessian of the function $f_0(x) = \|x - x^*\|^2$ is given by $\nabla^2 f_0(x) = 2I$, where I is the $n \times n$ identity. Thus, all its eigenvalues are equal. This implies that the left hand side of (19) takes the value one. On the other hand, since d_i and r_i are positive quantities the right hand side of (19) is strictly larger than one. Hence the condition is always satisfied and therefore $\varphi_k(x)$ is a navigation function for some large enough k. ■

IV. Proof of Theorem 2

In this section we show that φ_k, defined in (15) is a navigation function if the hypotheses of Theorem 2 are satisfied. To do so, we show that each one of the properties defining a navigation function are satisfied.

A. *Twice Differentiability and Admissibility*

The following lemma shows that the artificial potential (15) is twice differentiable and admissible.

Lemma 1 (**Differentiability and admissibility**). *Let $\mathcal{F}$ be a compact connected analytic manifold with boundary as defined in* (9) *and let $\varphi_k : \mathcal{F} \to [0,1]$ be the function defined in* (15)*. Then, φ_k is admissible and twice continuously differentiable on $\mathcal{F}$.*

Proof. Let us start by showing that φ_k is twice differentiable. Recall that both f_0 and β are twice differentiable functions (c.f Assumption 2). Since the optimum of f_0 is not on the boundary of $\mathcal{F}$ and f_0 is nonnegative (c.f Assumption 2), for all $x \in \mathcal{F}$, we have $f_0^k(x) + \beta(x) > 0$. Therefore $\left(f_0^k(x) + \beta(x)\right)^{-1/k}$ is twice differentiable. Hence φ_k is twice differentiable since it is the product of twice differentiable functions. We show next that φ_k is admissible. For every $x \in \text{int}(\mathcal{F})$ we have that $\beta(x) > 0$, thus $\varphi_k(x) < 1$. On the other hand, if $x \in \partial\mathcal{F}$ we have that $\beta(x) = 0$, hence $\varphi_k(x) = 1$. This means that the pre image of 1 by φ_k is the boundary of the free space. Which completes the proof. ■

B. φ_k is polar on $\mathcal{F}$

In this section we show that the function φ_k defined in (15) is polar on the free space $\mathcal{F}$ defined in (9). Furthermore we show that if $f_0(x^*) = 0$, then its minimum coincides with the minimum of f_0. If this is not the case, then the minimum of $\varphi_k(x)$ can be placed arbitrarily close to x^* by increasing the order parameter k. To do so, we first show that x^* – or the critical point arbitrarily close to x^* – is a non degenerate minimum of the function φ_k on $\mathcal{F}$ and then we show that none of the other critical points of $\varphi_k(x)$ are minima.

Lemma 2. *Let $\mathcal{F}$ be the free space defined in* (9) *verifying Assumption 1 and let $\varphi_k : \mathcal{F} \to [0,1]$ be the function defined in* (15)*. If $f_0(x^*) = 0$, then x^* is a non degenerate minimum of φ_k. If $f_0(x^*) \neq 0$, then for every $\varepsilon > 0$ there exists K such that for $k > K$, φ_k has a non degenerate minimum x_c such that $\|x_c - x^*\| < \varepsilon$.*

Proof. Let us compute the gradient of φ_k and then evaluate it at x^*. For any $x \in \mathcal{F}$ the gradient of φ_k is given by

$$\nabla\varphi_k(x) = \left(f_0^k(x) + \beta(x)\right)^{-1-\frac{1}{k}} \left(\beta(x)\nabla f_0(x) - \frac{f_0(x)\nabla\beta(x)}{k}\right). \tag{21}$$

Let us show that x^* is a non degenerate minimum of φ in the case where $f_0(x^*) = 0$. We defer the second part of the proof to Appendix A. Since x^* is the minimum of f_0 it holds that $\nabla f_0(x^*) = 0$. Hence, $\nabla\varphi_k(x^*) = 0$ and x^* is a critical point of φ_k. Let us now show that it is a non degenerate minimum, for this we are going to show that the Hessian of φ_k evaluated in x^* is a positive definite matrix. Taking the derivative of the gradient and simplifying the expression using the fact that $f_0(x^*) = \nabla f_0(x^*) = 0$ the Hessian of φ_k evaluated at x^* yields

$$\nabla^2\varphi_k(x^*) = \beta(x^*)^{-1/k}\nabla^2 f_0(x^*). \tag{22}$$

Since by Assumption 2 the point x^* is in the interior of the free space we have that $\beta(x^*) > 0$. Furthermore since f_0 is a strongly convex function (c.f Assumption 2) we have that $\nabla^2 f_0 \geq \lambda_{\min} I > 0$ and therefore $\nabla^2\varphi_k(x^*) > 0$. Thus x^* is a non degenerate minimum of φ_k. ■

The minimum of f_0 is also a minimum of the function φ_k as shown by the previous lemma. The next step to prove that x^* is the only minimum of φ_k is to show that the other critical points are not minima. To do so, we first show that there are no critical points in the boundary of the free space. It is convenient to express the gradient of the obstacle function using the following function

$$\bar{\beta}_i(x) = \prod_{j=0, j\neq i}^{m} \beta_j(x). \tag{23}$$

The above function is the product of all the obstacle functions except β_i. Then, for any $i = 0 \ldots m$, the gradient of the obstacle function can be written as

$$\nabla\beta(x) = \beta_i(x)\nabla\bar{\beta}_i(x) + \bar{\beta}_i(x)\nabla\beta_i(x). \tag{24}$$

Now we are in condition of stating the next lemma.

Lemma 3. *Let $\mathcal{F}$ be the compact connected analytic manifold with boundary defined in* (9) *verifying Assumption 1 and let $\varphi_k : \mathcal{F} \to [0,1]$ be the function defined in* (15)*. Then there are not critical points of φ_k in the boundary of the free space.*

Proof. Notice that for any point x in the boundary of the free space we have that $\beta(x) = 0$. Therefore if $x \in \partial\mathcal{F}$, the gradient of φ_k given by (21) reduces to

$$\nabla\varphi_k(x) = -\frac{f_0^{-k}(x)}{k}\nabla\beta(x). \tag{25}$$

Since the minimum of f_0 is not in the boundary of the free space (c.f Assumption 2), it must be the case that $f_0(x) > 0$. It is left to show that $\nabla\beta(x) \neq 0$ for all $x \in \partial\mathcal{F}$. In virtue of Assumption 1 the obstacles do not intersect. Hence if $x \in \partial\mathcal{F}$, it must be the case that for exactly one of the indices $i = 0 \ldots m$ we have that $\beta_i(x) = 0$. Let $i^* = \{i = 0 \ldots m \big| \beta_i(x) = 0\}$. Then (24) reduces to

$$\nabla\beta(x) = \bar{\beta}_{i^*}(x)\nabla\beta_{i^*}(x). \tag{26}$$

Furthermore we have that for all $j \neq i^*$, $\beta_j(x) > 0$ hence $\bar{\beta}(x)_{i^*} > 0$. Since the obstacles are non empty and open sets and in its boundary $\beta_{i^*}(x) = 0$ it must be the case that $\nabla\beta_{i^*}(x) \neq 0$ for any $x \in \partial\mathcal{O}_{i^*}$. Which shows that $\nabla\beta(x) \neq 0$ and therefore, there are no critical points in the boundary of the free space. ■

While there are no critical points in the boundary of the obstacles, we show in Lemma 4 that all the critical points of φ_k – except for the one near to x^* – can be placed arbitrarily close to the obstacles by selecting an appropriate value for the parameter k of the function φ_k. This result combined with lemmas 2 and 3 is used to show that the φ_k is polar in Lemma 5.

Lemma 4. *Let $\mathcal{F}$ be the free space defined in* (9) *verifying 1 and let $\varphi_k : \mathcal{F} \to [0,1]$ be the function defined in* (15)*. If x_s is a critical point of φ_k away from x^*, then for any $\varepsilon > 0$ there exists a K such that, if $k > K$ then $\beta(x_s) < \varepsilon$. Furthermore* *the critical points cannot be close to the external boundary of the free space.*

Proof. Notice that for any point in the interior of $\mathcal{F}$ it holds that $f_0^k(x_s) + \beta(x_s) > 0$. For x_s to be a critical point it must satisfy $\nabla\varphi_k(x_s) = 0$. Thus, from (21) we conclude x_s is a critical point if and only if

$$k\beta(x_s)\nabla f_0(x_s) = f_0(x_s)\nabla\beta(x_s). \tag{27}$$

Notice that since $f_0\nabla\beta(x_s)$ is a continuous function and the free space is a bounded set the norm of the right hand side of (27) is bounded. Further notice that $\|\nabla f_0(x_s)\|$ is lower bounded by a positive quantity since we are considering critical points of φ_k away from x^*. This implies that we can upper bound $\beta(x_s)$ by

$$\beta(x_s) \leq \frac{L}{k}, \tag{28}$$

where L is an upper bound for $f_0(x_s)\|\nabla\beta(x_s)\|/\|\nabla f_0(x_s)\|$. For every $\varepsilon > 0$ we can select k large enough such that $\frac{L}{k} < \varepsilon$. Therefore critical points away from the optimum of f_0 can be placed arbitrarily close to the obstacles.

Let us show next that the critical points cannot be close to the external boundary of the free space. Since x_s is a critical point it satisfies (27), hence the gradient of f_0 is collinear with the gradient of β. Write the gradient of β as

$$\nabla\beta(x_s) = \bar{\beta}_0(x_s)\nabla\beta_0(x_s) + \beta_0(x_s)\nabla\bar{\beta}_0(x_s) \tag{29}$$

Suppose that x_s could be placed arbitrarily close to the external boundary. Then, for large enough k, $\beta_0(x_s)$ can be made arbitrarily smalll and therefore $\nabla f_0(s_s)$ and $\nabla\beta_0(x_s)$ are collinear. Since β_0 is concave in $\mathcal{X}$, its gradient – and the gradient of f_0 – points inwards the $\mathcal{X}$. However, this contradicts the fact that the objective function is convex, because the angle between $(x_s - x^*)$ and $\nabla f_0(x_s)$ must be larger than $\pi/2$. Hence x_s cannot be placed arbitrarily close to the external boundary. ■

Notice that since the obstacles do not intersect we can ensure that for large k the critical point will be close to only one of the obstacles. Which translates in the fact that for large enough k, $\beta_i(x_s) < \varepsilon$ for exactly one of the indices $i = 1 \ldots m$. Let us use i^* to denote that particular index. Then write the gradient of β as

$$\nabla\beta(x) = \beta_{i^*}(x)\nabla\bar{\beta}_{i^*}(x) + \bar{\beta}_{i^*}(x)\nabla\beta_{i^*}(x). \tag{30}$$

The first term of the above equation can be made arbitrarily small, resulting in the gradient of β being nearly collinear with the gradient of $\nabla\beta_{i^*}$. In order to show that the critical points of φ_k cannot be minima we will select a test direction to evaluate its Hessian. Since the function φ_k attains its maximum value in the boundary of the obstacles we can guess that the Hessian evaluated in a direction that points towards the obstacle $\mathcal{O}_{i^*}$ will be positive at x_s. In fact in Lemma 6 we show that the previous statement is true. The test direction on which we evaluate the Hessian of φ_k to show that it cannot be a minimum are then those perpendicular to $\nabla\beta(x_s)$. Lemma 5 shows that under some conditions, the Hessian on these directions is negative.

Lemma 5. *et $\mathcal{F}$ be the free space defined in* (9) *verifying Assumption 1 and let $\varphi_k : \mathcal{F} \to [0, 1]$ be the function defined in* (15)*. Let $\lambda_{\max}$, $\lambda_{\min}$ and $\mu^i_{\min}$ the bounds in Assumption 1. Further let* (16) *hold for all $i = 1 \ldots m$ and for all x_s in the boundary of $\mathcal{O}_i$. Then, there exists K such that if $k > K$, φ_k is polar.*

Proof. See Appendix B. ■

The previous lemma, completes the first part of the proof of Theorem 2. It only remains to show that the critical points of φ are non degenerate. We do this in the next section.

C. Non degeneracy of the critical points

In the previous section, in particular in Lemmas 5 we show that the Hessian of φ_k at a critical point is negative in the direction $\nabla\beta(x_s)^\perp$ under conditions related to the geometry of the problem and the condition number of the objective function. Notice that the subspace perpendicular to $\nabla\beta(x_s)$ is of dimension $n - 1$. Therefore to show that the critical points are non degenerate we just need to verify that the Hessian is not zero in the direction $\nabla\beta(x_s)$. We formalize this in the following lemma.

Lemma 6. *Let $\mathcal{F}$ be the compact connected analytic manifold with boundary defined in* (9) *verifying Assumption 1 and let $\varphi_k : \mathcal{F} \to [0, 1]$ be the function defined in* (15)*. Then, there exists K such that if $k > K$ the critical points of the function φ_k are non degenerated.*

Proof. In Lemma 2 we showed that the minimum of f_0 was a non degenerated minimum of φ_k. Hence we need to restrict our attention to the critical points that are different from x^*. Furthermore , in Lemma 5 we showed that the Hessian of φ_k evaluated at these points in the direction $\nabla\beta(x_s)^\perp$ was negative. Since for each critical point the direction $\nabla\beta(x_s)^\perp$ form a subspace of dimension $n - 1$ it only remains to check that the Hessian evaluated in the direction $\nabla\beta(x_s)$ is nonzero. Let v be a unit vector collinear with $\nabla\beta(x_s)$. Using (41) – the expression for the Hessian at a critical point x_s – we have that the sign of $\nabla\beta(x_s)^T\nabla^2\varphi_k\nabla\beta(x_s)$ is given by the sign of

$$v^T\bigg(k\beta(x_s)\nabla^2 f_0(x_s) + (k-1)\nabla\beta(x_s)\nabla f_0^T(x_s) - f_0(x_s)\nabla^2\beta(x_s)\bigg)v. \tag{31}$$

Notice that the term $k\beta(x_s)v^T\nabla^2 f_0(x_s)v$ is always positive since f_0 is convex. The last term of the above sum can be negative, but it is lower bounded since both $f_0(x_s)$ and $\|\nabla^2\beta(x_s)\|$ are bounded. We prove next that the second term is always positive and can be made arbitrarily large, which implies that the sign of the Hessian can always be made positive. Recall that because x_s is a critical point of φ_k it must be the case that $\nabla\beta(x_s)$ and $\nabla f_0(x_s)$ are collinear (c.f. (27)). Hence we have that

$$(k-1)v^T\nabla\beta(x_s)\nabla f_0^T(x_s)v = (k-1)\|\nabla\beta(x_s)\|\|\nabla f_0(x_s)\| \tag{32}$$

Since the critical points considered here are away from x^* we have that that $\|\nabla f_0(x_s)\|$ is lower bounded. In addition, the norm of the gradient of $\beta(x_s)$ is lower bounded as we show next. Consider (24) to write the gradient of the obstacle function as

$$\nabla\beta(x) = \beta_i(x)\nabla\bar{\beta}_i(x) + \bar{\beta}_i(x)\nabla\beta_i(x). \quad (33)$$

While the first term can be made arbitrarily small, the norm of the second one is lower bounded. The latter was argued in proof of Lemma 3. This implies that the product $\|\nabla\beta(x_s)\|\,\|\nabla f_0(x_s)\|$ is lower bounded. Therefore by selecting k large enough we make sure that the Hessian evaluated at the direction $\nabla\beta(x_s)$ is positive. Hence we showed that the critical points are not degenerate. ■

The non degeneracy of the critical points of the potential defined in (15) completes the proof of Theorem 2.

V. Numerical experiments

We evaluate the performance of the navigation function (15) in different scenarios. To do so, we consider a discrete approximation of the gradient flow (5)

$$x_{t+1} = x_t - \varepsilon_t\nabla\varphi_k(x_t). \quad (34)$$

Where x_0 – the initial position – is selected at random on the free space and ε_t is a diminishing step size so oscillations of the iterates near the optimal point are reduced. In Section V-A we consider a free space where the obstacles considered are ellipsoids –the obstacle functions $\beta_i(x)$ for $i = 1 \ldots m$ take the form (17). In particular we study the effect of diminishing the distance between the obstacles while keeping the length of its mayor axis constant. In this section we build the free space such that condition (19) is satisfied. As already shown through a numerical experiment in Section III the previous condition is tight for particular configurations, yet the experiment depicted in Figure 3 shows that navigation is still possible if (19) is violated. Because of that observation we study this situation in Section V-B. In V-C we consider egg shaped obstacles as an example of convex obstacles different than the ellipsoids. The numerical section concludes in Section V-D where we consider a system with the dynamics of (13).

A. *Elliptical obstacles in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$

In this section we consider m elliptical obstacles in $\mathbb{R}^n$, where $\beta_i(x)$ is of the form (18), with $n = 2$ and $n = 3$. We set the number of obstacle to be $m = 2^n$. We define the external boundary to be a spherical shell of center x_0 and radius r_0. The center of each ellipsoid is placed in a different orthant. To do so, we set each center to be in the position $d\,(\pm 1, \pm 1, \ldots, \pm 1)$ and then we add a random variation drawn uniformly from $[-\Delta, \Delta]^n$, where $0 < \Delta < d$. The maximum axis of the ellipse –r_i – is drawn uniformly from $[r_0/10, r_0/5]$. We build orthogonal matrices A_i for $i = 1 \ldots m$ where its eigenvalues are randomly picked through a uniform distribution over $[1, 2]$. We check that the obstacles selected through the previous process do not intersect. If they do, we re draw all previous

d	k	max final dist	min initial dist	collisions
10	2	4.45×10^{-2}	10.06	0
9	2	17.25	10.01	0
9	5	4.45×10^{-2}	10.01	0
6	5	21.61	10.01	0
6	7	4.74×10^{-2}	10.02	0
5	7	22.29	10.027	0
5	10	4.73×10^{-2}	10.05	0
3	10	14.28	10.12	0
3	15	4.65×10^{-2}	10.80	0

TABLE I: Results for the experimental setting described in Section V-A. Observe that the smaller the value of d – the closer the obstacles are between them – the environment becomes harder to navigate, i.e. k must be increased to converge to the minimum of f_0.

parameters. For the objective function we consider a quadratic cost given by

$$f_0(x) = (x - x^*)^T\, Q\, (x - x^*)\,, \quad (35)$$

where $x^* = \operatorname{argmin} f_0(x)$ and $Q \in \mathcal{M}^{n\times n}$ is a positive symmetric matrix. x^* is drawn uniformly over $[-r_0/2, r_0/2]^n$ and we verify that it is in the free space. Then, we compute the condition number for Q such that (16) is satisfied. Let N_{cond} be the strictest of these conditions. The eigenvalues of Q are selected randomly from $[1, N_{cond} - 1]$, hence ensuring that (16) is satisfied. Finally the initial position is also selected randomly over $[-r_0, r_0]^n$ and it is checked that it is on the freespace. For this experiments we set $r_0 = 20$ and $\Delta = 1$. We run 100 simulations varying the parameter d – controlling the distance between the obstacles– and k. With this information we build Table I, where we report the number of collisions, the maximal distance of the last iterate to the minimum of f_0 and the minimal initial distance to the minimum of f_0. As we can conclude from Table I, the artificial potential (15) provides collision free paths. Notice that the smaller the distance between the obstacles the hardest it is to navigate the environment and k needs to be increased to achieve the goal. For instance we observe that setting $k = 5$ it sufficient to navigate an environment where d is set to be 9, yet it is not enough to navigate an environment where $d = 6$. In Figure 4 we can observe the level curves of the function φ_k and the trajectory arising from (5) when d is set to be 6 and $k = 7$. Observe that the length of the path of the agent is larger than the distance between the initial position and the minimum. We perform a statistical study reporting in Table II the mean and the variance of the ratio between these two quantities. We only consider those values of d and k that always achieve convergence (c.f Table I). Observe that when the distance d is reduced while keeping k constant the ratio increases. On the contrary if d is maintained constant and k is increased the ratio becomes smaller, meaning that the trajectory approaches the optimal trajectory. In Figure 5 we simulate one instance of an elliptical world in $\mathbb{R}^3$, with $d = 10$ and $k = 25$. For four initial conditions we observe that the trajectories succeed to achieve the minimum of f_0.

B. *Violation of condition* (19)

In this section we generate objective functions such that condition (19) is violated. To do so, we generate the obstacles

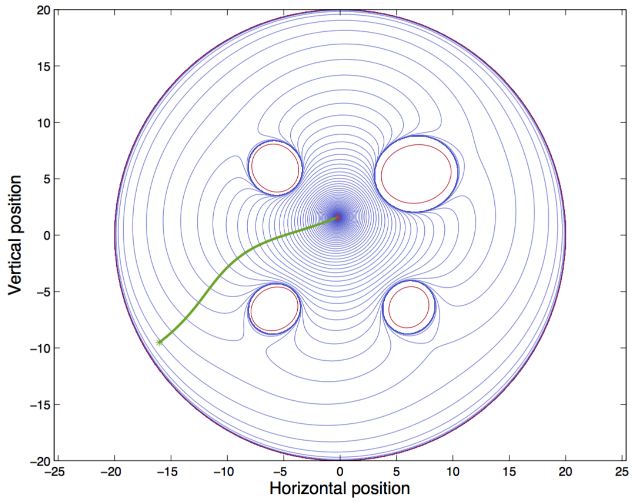

Fig. 4: Trajectory arising from the system (5) in the case of $d = 6$ and $k = 7$. As per Theorem 3 the trajectory converges to the minimum of the objective function while avoiding the obstacles.

d	k	μ_r	σ_r^2
10	2	1.07	6.53×10^{-3}
10	15	1.01	6.95×10^{-5}
9	5	1.03	2.10×10^{-3}
9	15	1.01	7.74×10^{-4}
6	7	1.19	1.01×10^{-2}
6	15	1.03	1.59×10^{-3}
5	10	1.06	6.14×10^{-3}
5	15	1.05	2.57×10^{-3}
3	15	1.06	3.60×10^{-3}

TABLE II: Mean and variance of the ratio between the path length and the initial distance to the minimum. For each scenario 100 simulations where made. Observe that the smallest the value of d the larger the ratio becomes. On the other hand when we increase k for a given value of d the ratio diminishes.

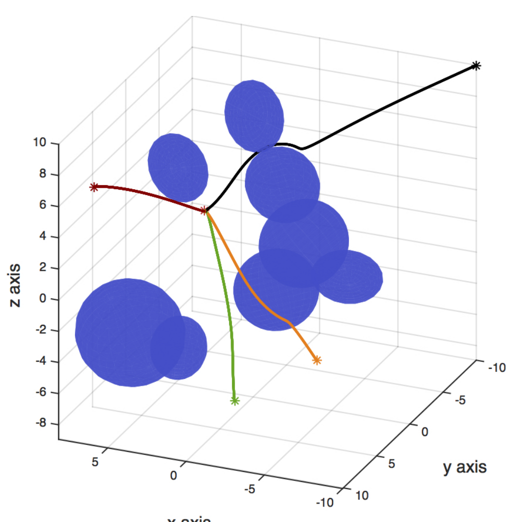

Fig. 5: Trajectories for different initial conditions in an elliptical world in $\mathbb{R}^3$. As per Theorem 3 the trajectory converges to the minimum of the objective function while avoiding the obstacles. In this example we have $d = 10$ and $k = 25$.

d	k	Succes
10	2	99%
9	5	95%
6	7	81%
5	10	82%
3	15	82%

TABLE III: Percentage of successful simulations when the condition guaranteeing that φ_k is a navigation function is violated. We observe that as the distance between obstacles becomes smaller the failure percentage increases.

as in Section V-A and the objective function is such that all the eigenvalues of the Hessian are set to be one, except for the maximum which is set to be $\max_{i=1\ldots m} N_{cond} + 1$, hence assuring that condition (19) is violated for all the obstacles. In this simulation Theorem 3 does not ensures that φ_k is a navigation function so it is expected that the trajectory fails to converge in some simulations. We run 100 simulations for different values of d and k. In particular for each value of d we select k such that the simulation in Section V-A succeeds to achieve the minimum of the objective function. In Table III we report the percentage of successful simulations. Notice that when the distance between the obstacles is decreased the probability of converging to a local minimum different than x^* increases.

C. Egg shaped obstacles

In this section we consider a new class of obstacles: egg shaped obstacles. We draw the center of the each obstacle,x_i, from a uniform distribution over $[-d/2, d/2] \times [-d/2, d/2]$. The distance between the "tip" and the "bottom" of the egg, r_i, is drawn uniformly over $[r_0/10; r_0/5]$ and with probability 0.5 the egg is horizontal,

$$\beta_i(x) = \|x - x_i\|^4 - 2r_i \left(x^{(1)} - x_i^{(1)}\right)^3, \tag{36}$$

where the superscript (1) refers to first component of a vector. With probability 0.5 the egg is placed verticaly

$$\beta_i(x) = \|x - x_i\|^4 - 2r_i \left(x^{(2)} - x_i^{(2)}\right)^3. \tag{37}$$

Notice that the functions β_i as defined above are not convex on $\mathbb{R}^2$, however their Hessians are positive definite outside the obstacles. To be formal it is needed to define an extension of the function inside the obstacles in order to say that the function describing the obstacle is convex. This extension is not needed in practice because our interest resides on how $\beta_i(x)$ behaves outside the obstacle. In Figure 6 we observe the level sets of the navigation function and a trajectory arising from (34) when we set $k = 25$, $r_0 = 20$ and $d = 10$. In this example the hypotheses of Theorem 2 are satisfied, hence the function φ_k is a navigation function and trajectories arising from the gradient flow (5) converge to the optimum of f_0 without running into the free space boundary.

D. A system with dynamics

In this section we consider a system with the following simplified version of the dynamics (13)

$$\ddot{x} = \tau, \tag{38}$$

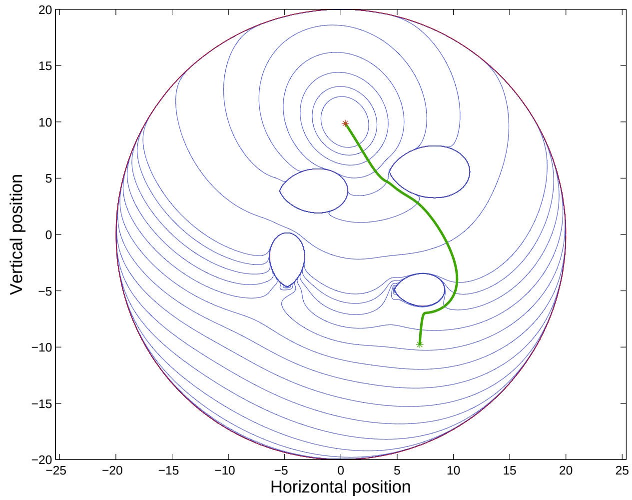

Fig. 6: Navigation function in an Egg shaped world. As predicted by Theorem 2 the trajectory arising from (34) converges to the minimum of the objective function f_0 while avoiding the obstacles.

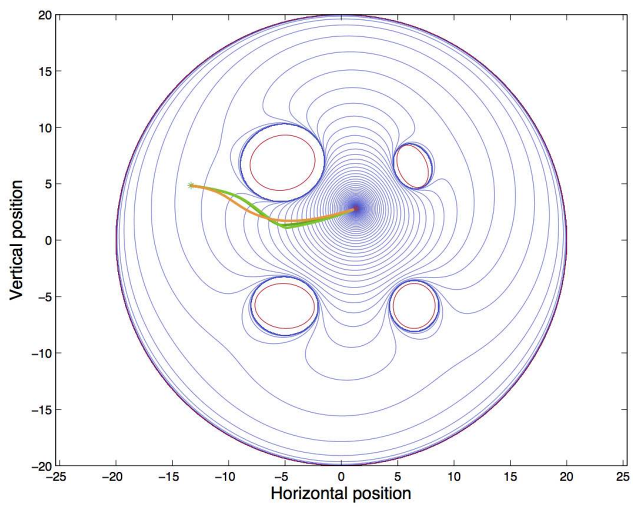

Fig. 7: In orange we observe the trajectory arising from the system without dynamics (c.f. (5)). In green we observe trajectories arising from the system (38) when we the control law (39) is applied. The trajectory in dark green has a larger damping constant than the trajectory in light green and therefore it is closer to the trajectory of the system without dynamics.

and the following control law

$$\tau = -\nabla\varphi_k(x) - K\dot{x}. \tag{39}$$

In Figure 7 we observe the behavior of the system (38) when the control law (39) is used (green trajectories) against the behavior of the gradient flow system (5) (orange trajectory). Thee light green line correspond to a system where the damping constant $K = 4\times 10^3$ and the dark green correspond to a damping constant of 5×10^3. As we can observe the larger the damping constant the closest the trajectory is to the one of the system without dynamics.

VI. Conclusions

We considered a set with convex holes in which an agent must navigate to the minimum of a convex function. This function is unknown and only local information about it was used, in particular its gradient and its value at the current location. We defined an artificial potential function and we showed that under some conditions of the free space geometry and the condition number of the objective function, this function was a navigation function. Then a controller that moves along the direction of the negative gradient of this function ensures convergence to the minimum of the objective function while avoiding the obstacles. Numerical experiments support the theoretical results.

Appendix

A. Proof of Lemma 2

The same argument used in Lemma 4 can be used, to ensure that every critical point must satisfy (27). Consider a critical point away from the boundary, i.e for some C we have $\beta(x_c) > C$. Since the right hand side of (27) is a continuous function and the free space is a bounded set, the right hand side of (27) is bounded by some constant M. Then the norm of $\nabla f_0(x_c)$ can be upper bounded by

$$\|\nabla f_0(x_c)\| \leq \frac{M}{kC}. \tag{40}$$

Therefore by selecting k large enough the norm of the gradient of f_0 can be made arbitrarily small, which shows that for any ε there exists a K such that if $k > K$ then there is a critical point arbitrarily close to x^*. Let us show next that this critical point is a non degenerate minimum. To do so let us compute Hessian of φ_k on x_c. Differentiate (21) and taking into account that for every critical point (27) must be satisfied, the Hessian yields

$$\begin{aligned}\nabla^2\varphi_k(x_c) = \frac{1}{k}\left(f_0^k(x_c)+\beta(x_c)\right)^{-1-\frac{1}{k}} \times \Big(& k\beta(x_c)\nabla^2 f_0(x_c) \\ & + (k-1)\nabla\beta(x_c)\nabla f_0^T(x_c) - f_0(x_c)\nabla^2\beta(x_c)\Big).\end{aligned} \tag{41}$$

Since we are considering a critical point away from the boundary of the free space we have that $f_0^k(x_s) + \beta(x_s) > 0$. Then the Hessian is positive definite if and only if

$$\begin{aligned}k\beta(x_c)\nabla^2 f_0(x_c) + (k-1)\nabla\beta(x_c)\nabla f_0^T(x_c) \\ - f_0(x_c)\nabla^2\beta(x_c) > 0.\end{aligned} \tag{42}$$

Rewrite (27) as $\nabla f_0(x_c) = f_0(x_c)\nabla\beta(x_c)\left(k\beta(x_c)\right)^{-1}$ and substitute this expression in (42). Thus, the sign of the Hessian of φ_k is given by the sign of

$$\begin{aligned}k\beta(x_c)\nabla^2 f_0(x_c) + \frac{(k-1)}{k}\frac{f_0(x_c)}{\beta(x_c)}\nabla\beta(x_c)\nabla\beta^T(x_c) \\ - f_0(x_c)\nabla^2\beta(x_c).\end{aligned} \tag{43}$$

Notice that the second term in the above equation is a positive semi definite matrix. Finally by selecting k large enough we can make the first term dominate over the third one. Since f_0 is strongly convex, its Hessian is positive definite and therefore for large enough k we have that x_c is a non degenerate minimum.

B. Proof of Lemma 5

It was proved that x^* –or a point that can be placed arbitrarily close to x^*– is a minimum of φ_k (Lemma 2). It remains to show that the rest of the critical points cannot be local minima of φ_k. Recall that these are in the interior of the free space (Lemma 3) and that these can be placed arbitrarily close to the obstacles (Lemma 4). Let x_s be the critical point placed close to $\mathcal{O}_i$ and define the following test direction

$$v = \left\{ u \in \mathbb{R}^n \middle| u^T \nabla \beta(x_s) = 0, \|u\| = 1 \right\}. \tag{44}$$

If we prove that $v^T \nabla^2 \varphi_k(x_s) v < 0$ then x_s is not a local minimum. As already established in the proof of Lemma 2, the Hessian of a critical point of the function φ_k takes the form (41). Evaluating the Hessian in the direction v we conclude that $v^T \nabla^2 \varphi_k(x_s) v$ is negative if and only if

$$k\beta(x_s) v^T \nabla^2 f_0(x_s) v - f_0(x_s) v^T \nabla^2 \beta(x_s) v < 0. \tag{45}$$

We next show that (45) holds to complete the proof. Since x^* is the minimum of the objective function, (12) takes the following form when it is evaluated for x_s and x^*

$$\lambda_{\min} \|x_s - x^*\|^2 \leq \nabla f_0^T(x_s)(x_s - x^*). \tag{46}$$

Then, multiply both sides of (27) by $(x_s - x^*)$ to write

$$k\beta(x_s) \nabla f_0^T(x_s)(x_s - x^*) = f_0(x_s) \nabla \beta(x_s)^T (x_s - x^*), \tag{47}$$

and use the bound from (46) to further upper bound the product $k\beta(x_s)$

$$k\beta(x_s) \leq f_0(x_s) \frac{\nabla \beta(x_s)^T (x_s - x^*)}{\lambda_{\min} \|x_s - x^*\|^2}. \tag{48}$$

Substitute $\nabla \beta(x_s)$ in the above equation by (24)

$$\begin{aligned} k\beta(x_s) \leq & \frac{f_0(x_s)}{\lambda_{\min} \|x_s - x^*\|^2} \bar{\beta}_i(x_s) \nabla \beta_i(x_s)^T (x_s - x^*) \\ & + \frac{f_0(x_s)}{\lambda_{\min} \|x_s - x^*\|^2} \beta_i(x_s) \nabla \bar{\beta}_i(x_s)^T (x_s - x^*). \end{aligned} \tag{49}$$

We argue next that the second term of (49) is bounded by a constant. Since x_s is close to the objects and therefore away of the optimum x^*, the distance $\|x_s - x^*\|$ is lower bounded. In addition we are considering a bounded set, therefore the other terms are bounded because all the functions involved are continuous. Let B be the constant bounding the terms multiplying $\beta_i(x_s)$, we have then

$$\frac{f_0(x_s)}{\lambda_{\min} \|x_s - x^*\|^2} \nabla \bar{\beta}_i(x_s)^T (x_s - x^*) \leq B. \tag{50}$$

Now, let us focus on the second term of (45), in particular the Hessian of $\beta(x_s)$ can be computed by differentiating (24)

$$\begin{aligned} \nabla^2 \beta(x_s) = & \beta_i(x_s) \nabla^2 \bar{\beta}_i(x_s) + \bar{\beta}_i(x_s) \nabla^2 \beta_i(x_s) \\ & + 2 \nabla \beta_i(x_s) \nabla^T \bar{\beta}_i(x_s). \end{aligned} \tag{51}$$

Combine (24) and (27) to express the gradient of $\nabla \beta_i(x_s)$ as

$$\nabla \beta_i(x_s) = k\beta_i(x_s) \frac{\nabla f_0(x_s)}{f_0(x_s)} - \beta_i(x_s) \frac{\nabla \bar{\beta}_i(x_s)}{\bar{\beta}_i(x_s)}. \tag{52}$$

Notice that the above equation is well defined since $f_0(x_s) > 0$ and because x_s is close to the object $\mathcal{O}_i$ and therefore $\bar{\beta}_i(x_s) > C$, for some positive C. Recall from (27) that at the critical point $\nabla \beta(x_s)$ and $\nabla f_0(x_s)$ are collinear, thus $v^T \nabla f_0(x_s) = 0$. Using this observation and the expression for $\nabla \beta_i(x_s)$ in (52), write the product $v^T \nabla \beta_i(x_s)$ as

$$v^T \nabla \beta_i(x_s) = -\beta_i(x_s) v^T \frac{\nabla \bar{\beta}_i(x_s)}{\bar{\beta}_i(x_s)}. \tag{53}$$

Combine (51) and (53) to evaluate the Hessian of $\beta(x_s)$ along the direction v

$$\begin{aligned} v^T \nabla^2 \beta(x_s) v = & v^T \nabla^2 \beta_i(x_s) v \bar{\beta}_i(x_s) \\ & + \beta_i(x_s) \left(v^T \nabla^2 \bar{\beta}_i(x_s) v - 2 \frac{\|v^T \nabla \bar{\beta}_i(x_s)\|^2}{\bar{\beta}_i(x_s)} \right). \end{aligned} \tag{54}$$

In the above equation the function multiplying $\beta_i(x_s)$ is upper bounded. Then, there exists a constant B' that allows us to upper bound the second of (45) term by

$$-f_0(x_s) v^T \nabla^2 \beta(x_s) v \leq -v^T \beta_i(x_s) v \bar{\beta}_i(x_s) f_0(x_s) + \beta_i(x_s) B'. \tag{55}$$

Use the bounds (49), (50) and (55) to further upper bound the left hand side of (45) by

$$\begin{aligned} & k\beta(x_s) v^T \nabla^2 f_0(x_s) v - f_0(x_s) v^T \nabla^2 \beta(x_s) v \\ & \leq v^T \nabla^2 f_0(x_s) v \frac{f_0(x_s) \bar{\beta}(x_s)}{\lambda_{\min} \|x_s - x^*\|^2} \nabla \beta_i(x_s)^T (x_s - x^*) \\ & -v^T \nabla^2 \beta_i(x_s) v f_0(x_s) \bar{\beta}_i(x_s) + \beta_i(x_s) (B + B'). \end{aligned} \tag{56}$$

Notice that in virtue of Lemma 4, for every ε we can select k such that

$$\beta_i(x_s) (B + B') < \varepsilon. \tag{57}$$

With this observation now we can upper bound the right hand side of (56) by

$$\begin{aligned} & k\beta(x_s) v^T \nabla^2 f_0(x_s) v - f_0(x_s) v^T \nabla^2 \beta(x_s) v \\ & \leq v^T \nabla^2 f_0(x_s) v \frac{f_0(x_s) \bar{\beta}(x_s)}{\lambda_{\min} \|x_s - x^*\|^2} \nabla \beta_i(x_s)^T (x_s - x^*) \\ & -v^T \nabla^2 \beta_i(x_s) v f_0(x_s) \bar{\beta}_i(x_s) + \varepsilon. \end{aligned} \tag{58}$$

Notice that since $\beta_j(x_s) > 0$ for all $j \neq i$ we can divide the right hand side of the expression by $\bar{\beta}_i(x_s)$ to study its sign. In addition we can divide the expression by $f_0(x_s)$ since $x_s \neq x^*$. Define $\varepsilon' = \varepsilon \left(2\bar{\beta}_i(x_s) f_0(x_s)\right)^{-1}$ then we have that $v^T \nabla^2 \varphi_k(x_s) v < 0$ if and only if

$$v^T \nabla^2 f_0(x_s) v \frac{\nabla \beta_i(x_s)^T (x_s - x^*)}{\lambda_{\min} \|x_s - x^*\|^2} - v^T \nabla^2 \beta_i(x_s) v + \varepsilon' < 0, \tag{59}$$

Notice that $v^T \nabla^2 f_0(x_s) v \leq \lambda_{\max}$ and $v^T \nabla^2 \beta_i(x_s) v \geq \mu_{\min}^i$, hence if

$$\frac{\lambda_{\max}}{\lambda_{\min}} \frac{\nabla \beta_i(x_s)^T (x_s - x^*)}{\|x_s - x^*\|^2} - \mu_{\min}^i + \varepsilon' < 0, \tag{60}$$

then $v^T \nabla^2 \varphi(x_s) v < 0$. To complete the proof observe that ε' can be made arbitrarily small by increasing k.

C. Proof of Theorem 3

In the particular case where the functions β_i take the form (17), the condition (16) of the general Theorem 2 translates into

$$\frac{\lambda_{\max}}{\lambda_{\min}}\frac{(x_s-x_i)^T A_i(x_s-x^*)}{\|x_s-x^*\|^2}-\mu^i_{\min}<0. \tag{61}$$

Since A_i is positive definite, there exists $A_i^{1/2}$ such that

$$A_i=\left(A_i^{1/2}\right)^T A_i^{1/2}. \tag{62}$$

Consider the change of variables $z=A_i^{1/2}x$, and write

$$\frac{(x_s-x_i)^T A_i(x_s-x^*)}{\|x_s-x^*\|^2}=\frac{(z_s-z_i)^T(z_s-z^*)}{\|A_i^{-1/2}\left(z_s-z^*\right)\|^2}. \tag{63}$$

Denote by $\mu^i_{\max}$ the maximum eigenvalue of the matrix A_i. Then we have that

$$\frac{1}{\mu^i_{\max}}\|\left(z_s-z^*\right)\|^2\leq\|A_i^{-1/2}\left(z_s-z^*\right)\|^2. \tag{64}$$

Use the above inequality to upper bound the left hand side of (61) as follows

$$\begin{aligned}\frac{\lambda_{\max}}{\lambda_{\min}}&\frac{(x_s-x_i)^T A_i(x_s-x^*)}{\|x_s-x^*\|^2}-\mu^i_{\min}\\ &\leq\frac{\lambda_{\max}}{\lambda_{\min}}\frac{(z_s-z_i)^T(z_s-z^*)}{\|z_s-z^*\|^2}\mu^i_{\max}-\mu^i_{\min}.\end{aligned} \tag{65}$$

The change of coordinates transforms the elliptical obstacle in a sphere of radius $r_i(\mu^i_{\min})^{1/2}$ since the function β_i takes the following form for the variable z

$$\beta_i(z)=\|z-z_i\|^2-r_i^2\mu^i_{\min}. \tag{66}$$

Since the obstacle is after considering the change of coordinate a circle we define for convenience the radial direction $\hat{e}_r$, whit $\|\hat{e}_r\|=1$. Let θ be the angle between $\hat{e}_r$ and the direction z_i-z^*. Further define $\tilde{r}$ to be the distance between the critical point z_s and z_i.. Further notice that if the $|\theta|\leq\pi/2$ then

$$\frac{(x_s-x_i)^T(x_s-x^*)}{\|x_s-x^*\|^2}\leq 0, \tag{67}$$

and in that case the right hand side of (65) is negative which completes the proof of the lemma. However if $|\theta|>\pi/2$ then the term under consideration is positive. In particular the larger the norm of $\tilde{r}$ the larger the value. Hence define $\tilde{r}_{\max}=r_i(\mu^i_{\min})^{1/2}+\varepsilon$, and the following bound holds

$$\frac{(z_s-z_i)^T(z_s-z^*)}{\|z_s-z^*\|^2}\leq\frac{\tilde{r}_{\max}(\tilde{r}_{\max}-d_i\cos\theta)}{\tilde{d_i}^2+\tilde{r}_{\max}^2-2\tilde{d}_i\tilde{r}_{\max}\cos\theta}, \tag{68}$$

where $\tilde{d}_i$ is the distance between z_s and z^*. Differentiating the right hand side of the above equation with respect to θ we conclude that its critical points are multiples of π. Notice that for multiples of π of the form $2k\pi$, with $k\in\mathbb{Z}$ will correspond to negative values and and for multiples of π of the form $(2k+1)\pi$ with $k\in\mathbb{Z}$, we have that

$$RHS(2k\pi+1)=\frac{\tilde{r}_{\max}(\tilde{r}_{\max}+\tilde{d}_i)}{\left(\tilde{d}_i+\tilde{r}_{\max}\right)^2}=\frac{\tilde{r}_{\max}}{\tilde{d}_i+\tilde{r}_{\max}} \tag{69}$$

Combine the previous bound with (65) to bound the expression in (61) by

$$\begin{aligned}\frac{\lambda_{\max}}{\lambda_{\min}}&\frac{(x_s-x_i)^T A_i(x_s-x^*)}{\|x_s-x^*\|^2}\mu^i_{\max}-\mu^i_{\min}\\ &\leq\frac{\lambda_{\max}}{\lambda_{\min}}\frac{\tilde{r}_{\max}}{\tilde{d}_i+\tilde{r}_{\max}}\mu^i_{\max}-\mu^i_{\min}.\end{aligned} \tag{70}$$

Notice than a lower bound for that distance is given by $\tilde{d}_i\geq\mu^i_{\min}d_i$. Notice that since z_s can be placed arbitrarily close to the boundary of the obstacle $\mathcal{O}_i$ we have that $\tilde{r}\leq r_i(\mu^i_{\min})^{1/2}+\varepsilon$ To finish the proof notice that

$$\frac{\tilde{r}_{\max}}{\tilde{d}_i+\tilde{r}_{\max}}=\frac{r_i+\frac{\varepsilon}{\mu^i_{\min}}}{d_i+r_i+\frac{\varepsilon}{\mu^i_{\min}}}, \tag{71}$$

hence since ε can be made arbitrarily small by increasing k we have that if

$$\frac{\lambda_{\max}}{\lambda_{\min}}\frac{\mu^i_{\max}}{\mu^i_{\min}}<1+\frac{d_i}{r_i}. \tag{72}$$

Thus proving that condition (16) takes the form stated in the theorem.

References

[1] D. E. Koditschek and E. Rimon, "Robot navigation functions on manifolds with boundary," *Advances in Applied Mathematics*, vol. 11, no. 4, pp. 412–442, 1990.

[2] E. Rimon and D. E. Koditschek, "Exact robot navigation using artificial potential functions," *Robotics and Automation, IEEE Transactions on*, vol. 8, no. 5, pp. 501–518, 1992.

[3] O. Khatib, "Real-time obstacle avoidance for manipulators and mobile robots," *Int. J. Rob. Res.*, vol. 5, pp. 90–98, Apr. 1986.

[4] T. Lozano-Perez, J. L. Jones, E. Mazer, P. O'Donnell, E. W. Grimson, P. Tournassoud, A. Lanusse, *et al.*, "Handey: A robot system that recognizes, plans, and manipulates," in *Robotics and Automation. Proceedings. 1987 IEEE International Conference on*, vol. 4, pp. 843–849, IEEE, 1987.

[5] J. Barraquand, B. Langlois, and J.-C. Latombe, "Numerical potential field techniques for robot path planning," *Systems, Man and Cybernetics, IEEE Transactions on*, vol. 22, no. 2, pp. 224–241, 1992.

[6] P. Khosla and R. Volpe, "Superquadric artificial potentials for obstacle avoidance and approach," in *Robotics and Automation, 1988. Proceedings., 1988 IEEE International Conference on*, pp. 1778–1784, IEEE, 1988.

[7] C. W. Warren, "Global path planning using artificial potential fields," in *Robotics and Automation, 1989. Proceedings., 1989 IEEE International Conference on*, pp. 316–321, IEEE, 1989.

[8] M. Takegaki and S. Arimoto, "A new feedback method for dynamic control of manipulators," *Journal of Dynamic Systems, Measurement, and Control*, vol. 103, no. 2, pp. 119–125, 1981.

[9] G. Lionis, X. Papageorgiou, and K. J. Kyriakopoulos, "Locally computable navigation functions for sphere worlds," in *Robotics and Automation, 2007 IEEE International Conference on*, pp. 1998–2003, IEEE, 2007.

[10] G. Lionis, X. Papageorgiou, and K. J. Kyriakopoulos, "Towards locally computable polynomial navigation functions for convex obstacle workspaces," in *Robotics and Automation, 2008. ICRA 2008. IEEE International Conference on*, pp. 3725–3730, IEEE, 2008.

[11] G. Roussos and K. J. Kyriakopoulos, "Completely decentralised navigation of multiple unicycle agents with prioritisation and fault tolerance," in *Decision and Control (CDC), 2010 49th IEEE Conference on*, pp. 1372–1377, IEEE, 2010.

[12] I. Filippidis and K. J. Kyriakopoulos, "Adjustable navigation functions for unknown sphere worlds," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, pp. 4276–4281, IEEE, 2011.

[13] L. L. Whitcomb, D. E. Koditschek, and J. B. Cabrera, "Toward the automatic control of robot assembly tasks via potential functions: The case of 2-d sphere assemblies," in *Robotics and Automation, 1992. Proceedings., 1992 IEEE International Conference on*, pp. 2186–2191, IEEE, 1992.

[14] C. S. Karagöz, H. I. Bozma, and D. E. Koditschek, “Feedback-based event-driven parts moving,” *Robotics, IEEE Transactions on*, vol. 20, no. 6, pp. 1012–1018, 2004.
[15] C. S. Karagoz, H. I. Bozma, and D. E. Koditschek, “Coordinated navigation of multiple independent disk-shaped robots,” *Robotics, IEEE Transactions on*, vol. 30, no. 6, pp. 1289–1304, 2014.
[16] J. T. Schwartz and M. Sharir, “On the piano movers problem. ii. general techniques for computing topological properties of real algebraic manifolds,” *Advances in applied Mathematics*, vol. 4, no. 3, pp. 298–351, 1983.
[17] J. Canny, *The complexity of robot motion planning*. MIT press, 1988.
[18] B. Chazelle, “Convex partitions of polyhedra: a lower bound and worst-case optimal algorithm,” *SIAM Journal on Computing*, vol. 13, no. 3, pp. 488–507, 1984.
[19] H. Choset and P. Pignon, “Coverage path planning: The boustrophedon cellular decomposition,” in *Field and Service Robotics*, pp. 203–209, Springer, 1998.
[20] S. M. LaValle and J. J. Kuffner Jr, “Rapidly-exploring random trees: Progress and prospects,” 2000.
[21] H. Choset, E. Acar, A. A. Rizzi, and J. Luntz, “Exact cellular decompositions in terms of critical points of morse functions,” in *Robotics and Automation, 2000. Proceedings. ICRA’00. IEEE International Conference on*, vol. 3, pp. 2270–2277, IEEE, 2000.
[22] E. Freund, “Fast nonlinear control with arbitrary pole-placement for industrial robots and manipulators,” *The International Journal of Robotics Research*, vol. 1, no. 1, pp. 65–78, 1982.
[23] J. Y. Luh, M. W. Walker, and R. P. Paul, “Resolved-acceleration control of mechanical manipulators,” *Automatic Control, IEEE Transactions on*, vol. 25, no. 3, pp. 468–474, 1980.
[24] M. W. Hirsch, S. Smale, and R. L. Devaney, *Differential equations, dynamical systems, and an introduction to chaos*, vol. 60. Academic press, 2004.
[25] K. B. Ariyur and M. Krstic, *Real-time optimization by extremum-seeking control*. John Wiley & Sons, 2003.
[26] C. Zhang, D. Arnold, N. Ghods, A. Siranosian, and M. Krstic, “Source seeking with non-holonomic unicycle without position measurement and with tuning of forward velocity,” *Systems & control letters*, vol. 56, no. 3, pp. 245–252, 2007.
[27] E. Burian, D. Yoerger, A. Bradley, and H. Singh, “Gradient search with autonomous underwater vehicles using scalar measurements,” in *Autonomous Underwater Vehicle Technology, 1996. AUV’96., Proceedings of the 1996 Symposium on*, pp. 86–98, IEEE, 1996.
[28] A. M. Johnson, M. T. Hale, G. Haynes, and D. E. Koditschek, “Autonomous legged hill and stairwell ascent,” in *Safety, Security, and Rescue Robotics (SSRR), 2011 IEEE International Symposium on*, pp. 134–142, IEEE, 2011.
[29] D. E. Koditschek, “The control of natural motion in mechanical systems,” *Journal of dynamic systems, measurement, and control*, vol. 113, no. 4, pp. 547–551, 1991.