Stochastic Soft Backpressure Algorithms for Routing and Scheduling in Wireless Ad-hoc Networks

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***Abstract*— We develop a generalization of the backpressure (BP) algorithm to find routes and schedules in wireless ad-hoc networks. Different from BP that schedules links that maximize queue differentials, the proposed stochastic soft (SS)BP algorithm randomizes schedules across links and flows with significant queue differentials. We show that SSBP shares the fundamental property of BP, namely, that if given arrival rates can be supported by some routing-scheduling policy, they can be supported by SSBP. Simulations illustrate SSBP's improvement in delay performance.**

I. Introduction

This paper studies joint routing and scheduling in wireless ad-hoc networks where nodes collaborate in delivering information for each other. Packets arrive at source nodes for delivery to given destinations. Since direct links between source and destination are unlikely, packets are routed through neighboring nodes. At every time slot, terminals have to select flows to service and next hops for the corresponding packets. The objective is to design a mechanism to determine routes and schedules to guarantee delivery of information.

A joint solution to this routing and scheduling problem is offered by the backpressure (BP) algorithm [1]. Nodes compute the difference between their queue lengths and the queues of neighboring nodes for every flow. The flow and neighbor that maximize the queue difference is scheduled for transmission. BP can be proved to be an optimal policy in that if given arrival rates can be supported by some routing-scheduling policy, they can be supported by BP. However, maximizing queue differentials is sensitive to random variations. While inconsequential for queue stability, this may increase delays significantly.

Such sensitivity to random events is a drawback in wireless networks since unreliability is a defining property of wireless channels. In wired networks, link reliabilities are very close to 1. In a wireless network however, different reliability values may and indeed happen in practice, as testified by experimental measurements [2] and the usefulness of routing protocols based on link reliability metrics; see e.g., [3]–[5].

Our goal is to design stochastic soft (SS)BP algorithms by randomizing schedules across links and flows with significant queue differentials – as opposed to scheduling maximum queue differential only. Using a supermartingale argument we show that such soft policies share the fundamental property of BP, namely, that if given arrival rates can be supported by some routing-scheduling policy, they are supported by SSBP. Simulations illustrate SSBP's improvement in delay performance.

II. Routing and scheduling in wireless networks

Consider a wireless network composed of J nodes $\{N_j\}_{j=1}^J$ supporting K information flows $\{F^k\}_{k=1}^K$. The destination of flow F^k is $N_{\mathrm{dest}(k)}$. Define the indicator variable $\mathbb{A}_i^k(t) \in \{0,1\}$ to denote the arrival of a packet belonging to flow F^k at node N_i. The arrival process is assumed stationary with expected value $\mathbb{E}\left[\mathbb{A}_i^k(t)\right] = a_i^k$. Nodes maintain queues Q_i^k for each flow, being $q_i^k(t)$ the number of packets belonging to flow F^k queued at node N_i. Packets are assumed of fixed length and information is measured in number of packets. Generalization to packets of varying length is straightforward but complicates notation.

Following the work in [3]–[5] we capture channel reliability by considering that successful packet reception occurs randomly. Let the indicator variable $\mathbb{R}_{ij}(t) \in \{0,1\}$ denote the event that a packet transmitted by node N_i is correctly decoded by node N_j. The set of terminals that can decode N_i's transmissions, i.e., those with indices $n(i) := \{j : \Pr\{\mathbb{R}_{ij}(t)\} > 0\}$ are said to compose the neighborhood of N_i. At time slot t, node N_i determines a flow F^k to service and a suitable next hop N_j for packets of this flow. The indicator variable $\mathbb{T}_{ij}^k(t) \in \{0,1\}$ marks this event. We further constrain nodes to transmit, at most, one packet per time slot. Therefore, at most one $\mathbb{T}_{ij}^k(t)$ variable can be nonzero for given i. As is customary, e.g. [6], we assume that if $\mathbb{T}_{ij}^k(t) = 1$ but the corresponding queue length is $q_i^k(t) = 0$, node N_i transmits a dummy packet. A dummy packet is also transmitted if $\mathbb{T}_{ij}^k(t) = 0$ for all j, k. Dummy packets render decoding events $\mathbb{R}_{ij}(t)$ independent of schedules $\mathbb{T}_{ij}^k(t) = 1$ and are added for tractability purposes. We denote as $r_{ij} := \mathbb{E}\left[\mathbb{R}_{ij}(t)\right]$ the reliability of the link from N_i to N_j.

The number of packets in the queue Q_i^k increases whenever a packet is accepted from upper layers, i.e. when $\mathbb{A}_i^k(t) = 1$; or; when a flow F^k packet is transmitted by a neighboring node, i.e., $\mathbb{T}_{ji}^k(t) = 1$ and successfully decoded, i.e., $\mathbb{R}_{ji}(t) = 1$. The number of packets ${q^{\mathrm{in}}}_i^k(t)$ added to the queue $Q_i^k(t)$ at time t is therefore

$$ {q^{\mathrm{in}}}_i^k(t) = \mathbb{A}_i^k(t) + \sum_{j\in n(i)} \mathbb{T}_{ji}^k(t)\mathbb{R}_{ji}(t)\mathbb{I}\left\{q_j^k(t) \neq 0\right\} \qquad (1) $$

where we added the requirement $\mathbb{I}\{q_{ij}(t) \neq 0\}$ to signify that dummy packets are transmitted but not added to the receiving node's queue.

Similarly, a packet leaves Q_i^k when node N_i schedules flow F^k in any of its links, i.e., $\mathbb{T}_{ij}^k(t) = 1$ and the corresponding neighbor successfully decodes the transmitted packet, i.e., $\mathbb{R}_{ij}(t) = 1$. Thus, the number of packets ${q^{\mathrm{out}}}_i^k$ subtracted from Q_i^k at time t is

$$ {q^{\mathrm{out}}}_i^k(t) = \sum_{j\in n(i)} \mathbb{T}_{ij}^k(t)\mathbb{R}_{ij}(t)\mathbb{I}\left\{q_i^k(t) \neq 0\right\}. \qquad (2) $$

where as in (1) we include $\mathbb{I}\left\{q_i^k(t) \neq 0\right\}$ because flow packets are transmitted only if Q_i^k is not empty. The number of packets at queue Q_i^k evolves according to

$$ q_i^k(t+1) = q_i^k(t) + {q^{\mathrm{in}}}_i^k - {q^{\mathrm{out}}}_i^k \qquad (3) $$

with ${q^{\mathrm{in}}}_i^k(t)$ given by (1) and ${q^{\mathrm{out}}}_i^k(t)$ given by (2). Indicator variables in (1) obviously satisfy $\mathbb{I}\left\{q_j^k(t) \neq 0\right\} \leq 1$. When $q_i^k(t) \neq 0$, the indicator variable for packet transmission in (2) is $\mathbb{I}\left\{q_i^k(t) \neq 0\right\} = 1$. From these two observations it follows that when $q_i^k(t) \neq 0$ it is possible to upper bound $q_i^k(t+1)$ as

$$ q_i^k(t+1) \leq q_i^k(t) + \mathbb{A}_i^k(t) + \sum_{j\in n(i)} \mathbb{T}_{ji}^k(t)\mathbb{R}_{ji}(t) - \mathbb{T}_{ij}^k(t)\mathbb{R}_{ij}(t) \qquad (4) $$

Our goal is to determine transmission stochastic processes $\{\mathbb{T}_{ij}^k(\mathbb{N})\}_{i,j,k}$ to guarantee that all queue lengths remain bounded. Because queue lengths depend on random arrivals and packet decodings this can only be guaranteed probabilistically. Upon defining the vector $\mathbf{q}(t)$ grouping all queue lengths we say that $\{\mathbb{T}_{ij}^k(\mathbb{N})\}_{i,j,k}$ guarantees stability if

$$ \lim_{Q\to\infty} \Pr\left\{\max_t \|\mathbf{q}(t)\| \leq Q\right\} = 1. \qquad (5) $$

I.e., with probability 1, no queue becomes arbitrarily large. The condition in (5) follows if there exists Q_0 such that for arbitrary time T_0,

$$ \Pr\left\{\min_{t\geq T_0} \|\mathbf{q}(t)\| \leq Q_0 \,\middle|\, \mathbf{q}(T_0)\right\} = 1. \qquad (6) $$

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I.e., if for any current queue state $\mathbf{q}(T_0)$ all queues almost surely become small at some future time. We will use (6) in subsequent proofs.

Processes $\{\mathbb{T}_{ij}^k(\mathbb{N})\}_{i,j,k}$ to ensure queue stability can be derived by finding variables $\{t_{ij}^k\}_{i,j,k}$ in the set

$$\mathcal{T} := \left[\{t_{ij}^k\}_{i,j,k} : \sum_{j \in n(i)} \left(t_{ij}^k r_{ij} - t_{ji}^k r_{ji} \right) - a_i^k > 0, \ \sum_{j,k} T_{ij}^k \leq 1 \right] \quad (7)$$

and selecting $\mathbb{T}_{ij}^k(t)$ independently across time with $\mathbb{E}\left[\mathbb{T}_{ij}^k(t)\right] = t_{ij}^k$ for all times t. Indeed, with schedules $\mathbb{T}_{ij}^k(t)$ having this property and $\{t_{ij}^k\}_{i,j,k} \in \mathcal{T}$ it follows from the update bound in (4) that $q_i^k(t)$ obeys a supermartingale equation when $q_i^k(t) \neq 0$. Validity of (6) then follows from the supermartingale convergence theorem [8, Theorem E7.4]. Assuming that for given arrival rates $\{a_i^k\}_{i,k}$ there exist variables $\{t_{ij}^k\}_{i,j,k}$ satisfying (7) we will develop an algorithm to determine $\mathbb{T}_{ij}^k(t)$ so that queues are stable in the sense of (5).

Start considering functions $f\left(t_{ij}^k\right)$ convex and non-increasing but otherwise arbitrary and define the following optimization problem

$$\begin{aligned} P = \max \sum_{i,j,k} f\left(t_{ij}^k\right) & \quad (8) \\ \text{s.t.} \sum_{j \in n(i)} \left(t_{ij}^k r_{ij} - t_{ji}^k r_{ji} \right) - a_i^k \geq 0, \quad \sum_{j,k} T_{ij}^k \leq 1. & \end{aligned}$$

Further consider multipliers λ_i^k and define the problem Lagrangian

$$\mathcal{L}(\mathbf{t}, \boldsymbol{\lambda}) := \sum_{i,j,k} f\left(t_{ij}^k\right) + \sum_{i,k} \lambda_i^k \left[\sum_{j \in n(i)} \left(t_{ij}^k r_{ij} - t_{ji}^k r_{ji} \right) - a_i^k \right] \quad (9)$$

with the constraint $\sum_{j,k} T_{ij}^k \leq 1$ left implicit. Vectors $\mathbf{t}$ and $\boldsymbol{\lambda}$ group all transmission rates $\{t_{ij}^k\}_{i,j,k}$ and all dual variables $\{\lambda_i^k\}_{i,k}$ respectively. Define now the dual function and dual problem as

$$D = \min_{\boldsymbol{\lambda} \geq 0} g(\boldsymbol{\lambda}) = \min_{\boldsymbol{\lambda} \geq 0} \max_{\sum_{j,k} T_{ij}^k \leq 1} \mathcal{L}(\mathbf{t}, \boldsymbol{\lambda}). \quad (10)$$

Since the dual function is convex it can be minimized by a descent algorithm. For doing this we compute a subgradient $\mathbf{s}(\boldsymbol{\lambda}(t))$. For given $\boldsymbol{\lambda}(t)$ consider the Lagrangian maximizers

$$\mathbf{t}(t) := \underset{\sum_{j,k} T_{ij}^k \leq 1}{\operatorname{argmax}} \mathcal{L}(\mathbf{t}, \boldsymbol{\lambda}(t)). \quad (11)$$

As it usually happens, the Lagrangian in (9) can be reordered to allow a separable maximization. Indeed, rewrite (9) as

$$\mathcal{L}(\mathbf{t}, \boldsymbol{\lambda}) := \sum_{i,j,k} f\left(t_{ij}^k\right) + t_{ij}^k \left(\lambda_i^k r_{ij} - \lambda_j^k r_{ji} \right) - \sum_{i,k} \lambda_i^k a_i^k \quad (12)$$

and notice that t_{ij}^k appears in only one term of the summation. The maximization in (11) can then be simplified to (with $\mathbf{t}_i(t)$ grouping variables $\{t_{ij}^k\}_{j,k}$)

$$\mathbf{t}_i(t) := \underset{\mathbf{t}_i^T \mathbf{1} \leq 1}{\operatorname{argmax}} f(\mathbf{t}_i) + \sum_{j,k} t_{ij}^k \left(\lambda_i^k r_{ij} - \lambda_j^k r_{ji} \right). \quad (13)$$

Define now the subgradient components by evaluating the constraints in (8) for $\mathbf{t} = \mathbf{t}(t)$

$$s_i^k(\boldsymbol{\lambda}(t)) := \sum_{j \in n(i)} \left(t_{ij}^k(t) r_{ij} - t_{ji}(t)^k r_{ji} \right) - a_i^k. \quad (14)$$

The subgradient $\mathbf{s}(t)$ with components $s_i^k(t)$ has the important property of being a descent direction for the dual function $g(\boldsymbol{\lambda}(t))$. In particular we have [7, Section 3.2]

$$\mathbf{s}^T(\boldsymbol{\lambda}(t)) \left(\boldsymbol{\lambda}(t) - \boldsymbol{\lambda}^* \right) \geq g(\boldsymbol{\lambda}(t)) - D > 0. \quad (15)$$

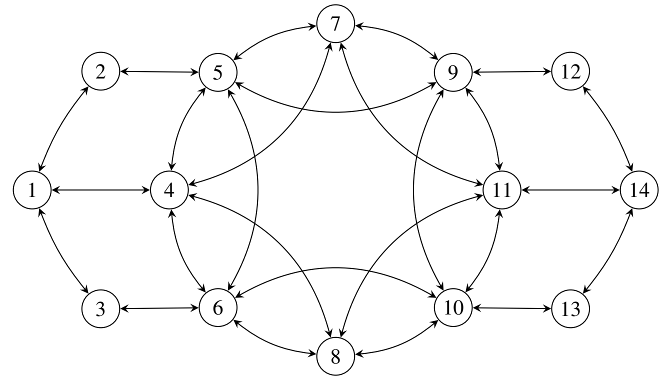

Fig. 1. Connectivity graph of an example wireless network.

It follows from (15) that the angle between $\mathbf{s}^T(\boldsymbol{\lambda}(t))$ and $\boldsymbol{\lambda}(t) - \boldsymbol{\lambda}^*$ is smaller than $\pi/2$. It is in this sense that $\mathbf{s}^T(\boldsymbol{\lambda}(t))$ is a decent direction for the dual function at $g(\boldsymbol{\lambda}(t))$. We can then propose the descent update

$$\lambda_i^k(t+1) = \left[\lambda_i^k(t) - \sum_{j \in n(i)} \left(t_{ij}^k(t) r_{ij} - t_{ji}(t)^k r_{ji} \right) - a_i^k \right]^+ \quad (16)$$

where we used the componentwise expression of $\mathbf{s}(\boldsymbol{\lambda}(t))$ in (14). Dual subgradient descent consists of iterative application of (16) and (13).

Notice the similarity of (16) and (4) that has been first observed in [6]. Ignore for the time being that (16) is an equality and (4) an inequality and interpret $t_{ij}^k(t)$ as the expected value of $\mathbb{T}_{ij}^k(t)$, i.e., $t_{ij}^k(t) = \mathbb{E}\left[\mathbb{T}_{ij}^k(t)\right]$. Since $a_i^k = \mathbb{E}\left[\mathbb{A}_i^k(t)\right]$ and $r_{ij} = \mathbb{E}\left[\mathbb{R}_{ij}(t)\right]$ for the dominant system, we can interpret (16) as an averaged version of (4). This similarity is used in the next section to develop stochastic BP algorithms.

III. Stochastic backpressure algorithms

Let node N_i know its own queue lengths $q_i^k(t)$ and the queue lengths of neighboring terminals $q_j^k(t)$, for $j \in n(i)$. Given this information, N_i computes variables $\mathbf{t}_i(t)$ as

$$\mathbf{t}_i(t) := \underset{\mathbf{t}_i^T \mathbf{1} \leq 1}{\operatorname{argmax}} f(\mathbf{t}_i) + \sum_{j,k} t_{ij}^k \left(q_i^k(t) r_{ij} - q_j^k(t) r_{ji} \right). \quad (17)$$

Flow F^k is scheduled in the link i, j with probability $t_{ij}^k(t)$. Notice that as a consequence $t_{ij}^k(t) = \mathbb{E}\left[\mathbb{T}_{ij}^k(t)\right]$.

This section proves that this routing-scheduling algorithm stabilizes queues in the sense of (5). We start with the following lemma.

Lemma 1 *Consider queues evolving as per* (3) *with* ${q^{\text{in}}}_i^k(t)$ *as in* (1) *and* ${q^{\text{out}}}_i^k(t)$ *as in* (2). *Assume* $\mathbb{E}\left[\mathbb{A}_i^k\right] = a_i^k$ *and that events* $\mathbb{R}_{ij}(t)$ *and* $\mathbb{T}_{ij}^k(t)$ *are independent with* $\mathbb{E}\left[\mathbb{R}_{ij}(t)\right] = r_{ij}$ *and* $\mathbb{E}\left[\mathbb{T}_{ij}^k(t)\right] = t_{ij}^k(t)$. *Update the latter averages* $t_{ij}^k(t)$ *as per* (17) *with functions* $f(t_{ij}^k)$ *non-increasing. Then,*

$$\mathbb{E}\left[\left\| \mathbf{q}(t+1) - \boldsymbol{\lambda}^* \right\|^2 \mid \mathbf{q}(t) \right] \leq \left\| \mathbf{q}(t) - \boldsymbol{\lambda}^* \right\|^2 + \hat{S}^2 - 2[g(\mathbf{q}(t)) - D]. \quad (18)$$

Proof : Start noting that because $f(t_{ij}^k)$ is non-decreasing, whenever $q_i^k(t) = 0$, the transmission probability $t_{ij}^k = 0$, for all $j \in n(i)$ [cf. (17)]. Indeed, let k_0 be a flow for which $q_i^{k_0}(t) = 0$. The summand involving $t_{ij}^{k_0}$ in (17) becomes

$$t_{ij}^{k_0} \left(q_i^{k_0}(t) r_{ij} - q_j^{k_0}(t) r_{ji} \right) = -t_{ij}^{k_0} \left(q_j^{k_0}(t) r_{ji} \right). \quad (19)$$

Thus $t_{ij}^{k_0}$ appears as factor in a negative linear term and as argument in a non-increasing function. Consequently, the maximization in (17) yields as optimal argument $t_{ij}^{k_0} = 0$.

Recall now that t_{ij}^k is the probability of the event $\mathbb{T}_{ij}^k(t)$, it follows then that $\mathbb{T}_{ij}^k(t) = 0$ if $q_i^k(t) = 0$, or equivalently if $\mathbb{I}\left\{q_i^k(t) \neq 0\right\} = 0$.

Algorithm 1 Stochastic soft backpressure algorithm (SSBP)

1: Receive queue length information $q_j^k(t)$ from neighbors $j \in n(i)$.
2: Compute transmit rates as [cf. (37)]
$t_{ij}^k(t) = \frac{1}{2}\left[q_i^k(t)r_{ij} - q_j^k(t)r_{ji} - w_i(t)\right]^+$
3: Select j, k with probability $t_{ij}^k(t)$.
4: Transmit to N_j, packets for flow F^k.
5: Broadcast queue length information q_i^k .

This is a good property, because it says that flow F^k is not scheduled by N_i if there are no packets to be transmitted. For the purpose of this proof it implies that the expressions for packets added and subtracted from the queue Q_i^k simplify to ${q^{\text{in}}}_i^k(t) = \mathbb{A}_i^k(t) + \sum_{j\in n(i)} \mathbb{T}_{ji}^k(t)\mathbb{R}_{ji}(t)$ [cf. (1)] and ${q^{\text{out}}}_i^k(t) = \sum_{j\in n(i)} \mathbb{T}_{ij}^k(t)\mathbb{R}_{ij}(t)$ [cf. (2)]. Thus the queue update in (3) becomes

$$\begin{aligned} q_i^k(t+1) =& q_i^k(t) + \mathbb{A}_i^k(t) + \sum_{j\in n(i)} \mathbb{T}_{ji}^k(t)\mathbb{R}_{ji}(t) - \mathbb{T}_{ij}^k(t)\mathbb{R}_{ij}(t) \\ :=& q_i^k(t) + \mathbb{S}_i^k(t) \end{aligned} \tag{20}$$

where we defined the queue update $\mathbb{S}_i^k(t) := q_i^k(t+1) - q_i^k(t)$. Denote as $\mathbb{S}(t)$ the vector of queue updates so that we can write $\mathbf{q}(t+1) = \mathbf{q}(t) + \mathbb{S}(t)$.

Consider now the expected value of $\mathbb{S}_i^k(t)$

$$\begin{aligned} \mathbb{E}\left[\mathbb{S}_i^k(t)\,\middle|\,\mathbf{q}(t)\right] &= \mathbb{E}\left[\mathbb{A}_i^k(t) + \sum_{j\in n(i)} \mathbb{T}_{ji}^k(t)\mathbb{R}_{ji}(t) - \mathbb{T}_{ij}^k(t)\mathbb{R}_{ij}(t)\,\middle|\,\mathbf{q}(t)\right] \\ &= a_i^k(t) + \sum_{j\in n(i)} \left(t_{ji}(t)^k r_{ji} - t_{ij}^k(t) r_{ij}\right) \\ &= -s_i^k(\mathbf{q}(t)) \end{aligned} \tag{21}$$

where the first equality follows from the hypotheses $\mathbb{E}\left[\mathbb{A}_i^k\right] = a_i^k$, $\mathbb{E}\left[\mathbb{R}_{ij}(t)\right] = r_{ij}$ and $\mathbb{E}\left[\mathbb{T}_{ij}^k(t)\right] = t_{ij}^k(t)$. The second equality follows from the definition of $\mathbf{t}_i(t)$ in (17) and the subgradient's definition in (14). This is the key observation for the proof.

To complete the derivation use the vector version of the relation in (20) to express the squared distance

$$\begin{aligned} \left\|\mathbf{q}(t+1) - \boldsymbol{\lambda}^*\right\|^2 &= \left\|\mathbf{q}(t) + \mathbb{S}(t) - \boldsymbol{\lambda}^*\right\|^2 \\ &= \left\|\mathbf{q}(t) - \boldsymbol{\lambda}^*\right\|^2 + \left\|\mathbb{S}(t)\right\|^2 + 2\mathbb{S}^T(t)\left(\mathbf{q}(t) - \boldsymbol{\lambda}^*\right) \end{aligned} \tag{22}$$

Condition both sides of (22) on $\mathbf{q}(t)$ and take expected values to obtain

$$\begin{aligned} \mathbb{E}\left[\left\|\mathbf{q}(t+1) - \boldsymbol{\lambda}^*\right\|^2 \,\middle|\, \mathbf{q}(t)\right] &= \left\|\mathbf{q}(t) - \boldsymbol{\lambda}^*\right\|^2 \\ &+ \mathbb{E}\left[\left\|\mathbb{S}(t)\right\|^2 \,\middle|\, \mathbf{q}(t)\right] + 2\mathbb{E}\left[\mathbb{S}^T(t)\,\middle|\,\mathbf{q}(t)\right]\left[\mathbf{q}(t) - \boldsymbol{\lambda}^*\right] \end{aligned} \tag{23}$$

Substitute the second term $\mathbb{E}\left[\left\|\mathbb{S}(t)\right\|^2 \,\middle|\, \mathbf{q}(t)\right]$ for its bound $\hat{S}^2$ and the expected value $\mathbb{E}\left[\mathbb{S}^T(t)\,\middle|\,\mathbf{q}(t)\right]$ for its expression in (21) to obtain

$$\mathbb{E}\left[\left\|\mathbf{q}(t+1) - \boldsymbol{\lambda}^*\right\|^2 \middle| \mathbf{q}(t)\right] = \left\|\mathbf{q}(t) - \boldsymbol{\lambda}^*\right\|^2 + \hat{S}^2 - 2\mathbf{s}^T(\boldsymbol{\lambda}(t))\left(\mathbf{q}(t) - \boldsymbol{\lambda}^*\right) \tag{24}$$

Substituting the subgradient property in (15) into (24) yields (18). ∎

The term $g[\mathbf{q}(t)] - D$ in (18) is always positive. Therefore, if we could neglect the term $\hat{S}^2$, it would follow from Lemma 1 that $\left\|\mathbf{q}(t) - \boldsymbol{\lambda}^*\right\|^2$ is a supermartingale. As it turns out, it is likely that $\hat{S}^2$ can be neglected. If queue lengths $\mathbf{q}(t)$ become large, $g[\mathbf{q}(t)]$ also does and eventually $g[\mathbf{q}(t)] - D$ exceeds $\hat{S}^2$. Thus, for large queue lengths $\left\|\mathbf{q}(t) - \boldsymbol{\lambda}^*\right\|^2$ behaves like a supermartingale. This observation is used to prove the following theorem.

Theorem 1 *Consider queues evolving in accordance with* (1)-(3) *with transmission probabilities* $\mathbf{t}_i(t)$ *given by* (17). *Assume the same hypotheses of Lemma 1. Let queue lengths* $\mathbf{q}(T_0)$ *at time* T_0 *be given and define* $g_{\text{b}}\left[t\,\middle|\,\mathbf{q}(T_0)\right] := \min_{u\in[T_0,t]} g(\mathbf{q}(u))$. *Such best value almost surely converges to within* $\hat{S}^2/2$ *of* D, *i.e.,*

$$\lim_{t\to\infty} g_{\text{b}}\left[t\,\middle|\,\mathbf{q}(T_0)\right] - D \le \hat{S}^2/2 \quad \text{a.s.} \tag{25}$$

Proof : For simplicity of exposition let $T_0 = 0$ and denote $g_{\text{b}}(t) = g_{\text{b}}\left[t\,\middle|\,\mathbf{q}(T_0)\right] - D$. Let also $g(t) := g(\mathbf{q}(t))$. Start defining the process $\alpha(t)$ to track $\left\|\mathbf{q}(t) - \boldsymbol{\lambda}^*\right\|^2$ until the gap $g(t) - D$ falls below $\hat{S}^2/2$ for the first time, i.e.,

$$\alpha(t) := \left\|\mathbf{q}(t) - \boldsymbol{\lambda}^*\right\|^2 \mathbb{I}\left\{g_{\text{b}}(t) - D > \hat{S}^2/2\right\}. \tag{26}$$

Similarly, define the process

$$\beta(t) := \left[2[g(t) - D] - \hat{S}^2\right]\mathbb{I}\left\{g_{\text{b}}(t) - D > \hat{S}^2/2\right\} \tag{27}$$

that follows $2[g(t) - D] - \hat{S}^2$ until $g(t) - D$ falls below $\hat{S}^2/2$ for the first time. Let also, $\mathcal{A}(0:t)$ be a sequence of nested σ-algebras measuring $\alpha(t)$, $\beta(t)$ and $\mathbf{q}(t)$. It will be shown that $\alpha(t)$ and $\beta(t)$ satisfy $\mathbb{E}\left[\alpha(t)\,\middle|\,\mathcal{A}(0:t)\right] \le \alpha(t) - \beta(t)$, wherefore they comply with the hypotheses of the supermartingale convergence theorem [8, Theorem E7.4] with respect to the sequence of σ-algebras $\mathcal{A}(0:t)$.

To prove this, start separating the latter expectation in the cases when $\alpha(t) = 0$ and $\alpha(t) \ne 0$ to write

$$\begin{aligned} \mathbb{E}\left[\alpha(t)\,\middle|\,\mathcal{A}(0:t)\right] =& \mathbb{E}\left[\alpha(t)\,\middle|\,\mathbf{q}(t), \alpha(t) = 0\right]\Pr\{\alpha(t) = 0\} \\ &+ \mathbb{E}\left[\alpha(t)\,\middle|\,\mathbf{q}(t), \alpha(t) \ne 0\right]\Pr\{\alpha(t) \ne 0\}. \end{aligned} \tag{28}$$

Start considering the case when $\alpha(t) = 0$. The definitions in (26) and (27) dictate that if $\alpha(t) = 0$, then it must be $\beta(t) = 0$ and $\alpha(t+1) = 0$. The following equality is therefore evident because all terms are zero

$$\mathbb{E}\left[\alpha(t+1)\,\middle|\,\mathbf{q}(t), \alpha(t) = 0\right] = \alpha(t) - \beta(t). \tag{29}$$

When $\alpha(t) \ne 0$ the values of $\alpha(t)$ and $\beta(t)$ are completely determined by $\mathbf{q}(t)$. Therefore, conditioning on $\mathcal{A}(0:t)$ is equivalent to conditioning on $\mathbf{q}(t)$. The conditional expected value of $s(t+1)$ then satisfies

$$\begin{aligned} &\mathbb{E}\left[\alpha(t+1)\,\middle|\,\mathbf{q}(t), \alpha(t) \ne 0\right] \\ &= \mathbb{E}\left[\left\|\mathbf{q}(t+1) - \boldsymbol{\lambda}^*\right\|^2 \mathbb{I}\left\{g(t+1) - D \ge \hat{S}^2/2\right\}\,\middle|\,\mathbf{q}(t)\right] && (30) \\ &\le \mathbb{E}\left[\left\|\mathbf{q}(t+1) - \boldsymbol{\lambda}^*\right\|^2\,\middle|\,\mathbf{q}(t)\right] && (31) \\ &\le \left\|\mathbf{q}(t) - \boldsymbol{\lambda}^*\right\|^2 + \hat{S}^2 - 2[g(t) - D] && (32) \\ &= \alpha(t) - \beta(t). && (33) \end{aligned}$$

The equality in (30) follows from the definition of $\alpha(t+1)$ in (26) and noting that because $\alpha(t) \ne 0$ $g_{\text{b}}(t+1) - D \le \hat{S}^2/2$ if and only if $g(t+1) - D \le \hat{S}^2/2$. The first inequality in (31) is true because the indicator term is not larger than 1. The inequality in (32) follows from Lemma 1. The last equality (33) simply uses the definitions of $\alpha(t)$ and $\beta(t)$ in (26) and (27) respectively.

Substituting (29) and (33) into (28), it finally follows

$$\begin{aligned} \mathbb{E}\left[\alpha(t)\,\middle|\,\mathcal{A}(0:t)\right] &\le \left[\alpha(t) - \beta(t)\right]\left[\Pr\{\alpha(t) = 0\} + \Pr\{\alpha(t) \ne 0\}\right] \\ &= \alpha(t) - \beta(t). \end{aligned} \tag{34}$$

Given (34) and the fact that processes $\alpha(t)$ and $\beta(t)$ are non-negative by definition if follows from the supermartingale convergence theorem that: (i) $\alpha(t)$ converges w.p.1.; and (ii) the sum $\sum_{t=1}^{\infty} \beta(t) < \infty$ is almost surely finite [8, Theorem E7.4]. Writing the latter consequence in terms of the explicit value of $\beta(t)$ in (27) yields

$$\sum_{t=1}^{\infty} \beta(t) = \sum_{t=1}^{\infty} \left[2[g(t) - D] - \hat{S}^2\right]\mathbb{I}\left\{g_{\text{b}}(t) - D \ge \hat{S}^2/2\right\} \le \infty \quad \text{w.p.1.} \tag{35}$$

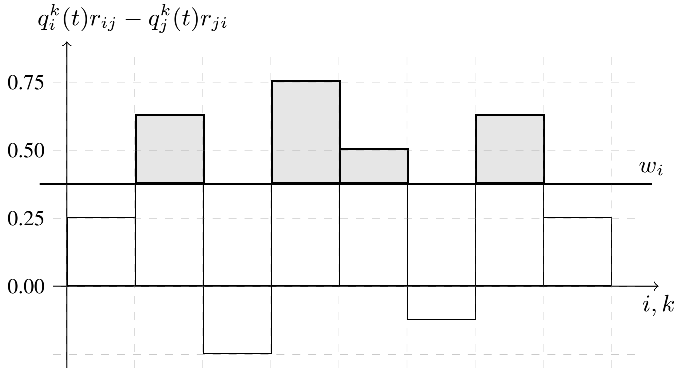

Fig. 2. Soft BP algorithm performs inverse waterfilling in the weighted queue backlogs $q_i^k(t)r_{ij} - q_j^k(t)r_{ji}$.

In particular, almost sure convergence of the sum in (35) implies that

$$\liminf_{t\to\infty} \left[2[g(t) - D] - \hat{S}^2\right] \mathbb{I}\left\{g_{\mathrm{b}}(t) - D \geq \hat{S}^2/2\right\} = 0 \quad \text{w.p.1.} \tag{36}$$

The latter is true if either $g_{\mathrm{b}}(t) - D \leq \hat{S}^2/2$ for sufficiently large t so that the indicator function is null or if $\liminf_{t\to\infty} \left[2[g(t) - D] - \hat{S}^2\right] = 0$. From any of these two events, (25) follows. ■

According to Theorem 1 it holds that for arbitrary $\delta > 0$ $g[\mathbf{q}(t)] - D$ almost surely falls below $\hat{S}^2/2 + \delta$ at least once as t grows. Comparing this observation with the queue stability condition in (6) we see that the only difference is that the latter requires $\|\mathbf{q}(t)\|$ falling below a certain Q_0. This mismatch is simple to solve because when $g[\mathbf{q}(t)]$ comes close to D, $\mathbf{q}(t)$ comes close to $\boldsymbol{\lambda}^*$.

Corollary 1 *If there exist strictly feasible transmission rates* $\{t_{ij0}^k\} \in \mathcal{T}$ *such that* $\sum_{j\in n(i)} \left(t_{ij0}^k r_{ij} - t_{ji0}^k r_{ji}\right) - a_i^k \leq C < 0$ *for all* i, k *all queues are stable.*

Proof: If strictly feasible rates t_{ij0}^k exist, as required by hypothesis, then a finite value of the dual function $g(\boldsymbol{\lambda})$ implies a finite argument $\boldsymbol{\lambda}$. In particular if $g[\mathbf{q}(t)] \leq D + \hat{S}^2/2$ there exists Q_0 such that $\|\mathbf{q}(t)\| \leq Q_0$. Since $g[\mathbf{q}(t)] \leq D + \hat{S}^2/2$ almost surely as per Theorem 1, $\|\mathbf{q}(t)\| \leq Q_0$ with probability 1. Queue stability follows from (6). ■

IV. Soft stochastic backpressure algorithm

Functions $f(t_{ij}^k)$ in (8) can be selected arbitrarily. The SSBP algorithm is obtained by making $f(t_{ij}^k) = -{t_{ij}^k}^2$. The rationale for using this quadratic function is because descent algorithms in quadratic convex optimization problems exhibit fast convergence rates. Further note that with $f(t_{ij}^k) = -{t_{ij}^k}^2$ the maximization in (17) is a simple quadratic program that can be solved analytically yielding transmission probabilities

$$t_{ij}^k(t) = \frac{1}{2}\left[q_i^k(t)r_{ij} - q_j^k(t)r_{ji} - w_i(t)\right]^+ \tag{37}$$

where $w_i(t) \geq 0$ is selected to ensure that $\sum_{j,k} t_{ij}^k(t) \leq 1$.

The expression for transmission probabilities in (37) dictates that SSBP performs reverse waterfilling in the weighted queue backlogs $q_i^k(t)r_{ij} - q_j^k(t)r_{ji}$; see Fig. 2. Water level $w_i = 0$ is used if $\sum_{j,k}\left[q_i^k(t)r_{ij} - q_j^k(t)r_{ji}\right]^+ \leq 1$, a situation that we expect to happen in lightly loaded nodes. Otherwise, $w_i(t) > 0$ is selected to make $\sum_{j,k}\left[q_i^k(t)r_{ij} - q_j^k(t)r_{ji} - w_i(t)\right]^+ = 1$.

Operation of SSBP is summarized in Algorithm 1. The core of the algorithm is the computation of probabilities $t_{ij}^k(t)$ as per (37) and the subsequent determination of the scheduled pair j, k (Steps 2-4). Exchange of queue length information necessary to compute $t_{ij}^k(t)$ completes the algorithm (Steps 1 and 5).

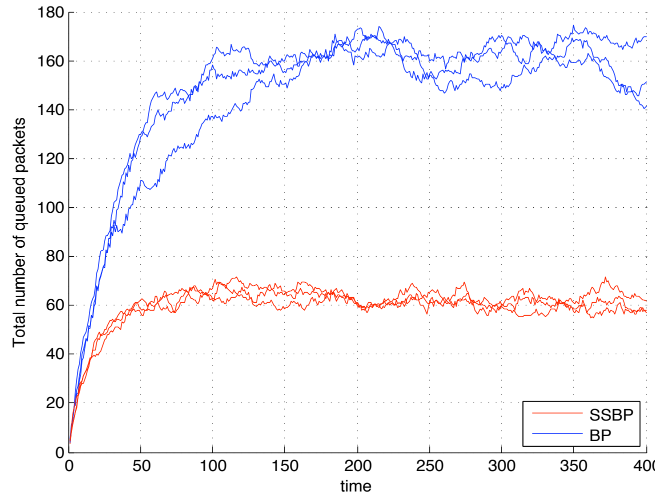

Fig. 3. Total number of packets in the network for backpressure (BP) and stochastic soft (SS)BP.

A. Simulations

To illustrate the advantage of SSBP with respect to BP in terms of delay we simulate both algorithms for the network in Fig. 1. Link reliabilities are selected as $r_{ij} = 0.5$ for links $N_4 \leftrightarrow N_7$, $N_5 \leftrightarrow N_9$, $N_7 \leftrightarrow N_{11}$, $N_9 \leftrightarrow N_{10}$, $N_{11} \leftrightarrow N_8$, $N_{10} \leftrightarrow N_6$, $N_8 \leftrightarrow N_4$ and $N_6 \leftrightarrow N_5$ and $r_{ij} = 0.75$ for the remaining links. Nodes N_1 and N_{14} are considered as flow destinations with all other nodes generating packets at a rate of $a_i^k = 0.15$ packets for delivery to each of them.

The total number of packets queued in the network $\sum_{i,k} q_i^k(t)$ is shown in Fig. 3. It can be seen that SSBP results in a substantial reduction in the number of queued packets. Recall that average delay is roughly proportional to average number of queued packets.

V. Conclusions

We developed stochastic soft backpressure algorithms (SSBP) for wireless ad-hoc networks. Schedules are randomized across links and flows according to time varying scheduling probabilities that are computed by reverse waterfilling on queue differentials. Simulations illustrate that SSBP exhibits improved delay performance with respect to conventional backpressure. This is due to smaller sensibility to random events typical of unreliable wireless channels.

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