ERGODIC STOCHASTIC OPTIMIZATION ALGORITHMS FOR WIRELESS COMMUNICATION AND NETWORKING

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ABSTRACT

This paper introduces ergodic stochastic optimization (ESO) algorithms to solve resource allocation problems that involve a random state and where optimality criteria are expressed in terms of long term averages. A policy that observes the state and decides on a resource allocation is proposed and shown to almost surely satisfy problem constraints and optimality criteria. Salient features of the ESO algorithm are that it does not require access to the state's probability distribution, that it can handle non-convex constraints in the resource allocation variables, and that convergence to optimal operating points holds almost surely. ESO is applied to determine operating points of an orthogonal frequency division multiplexing broadcast channel that maximize a given rate utility.

***Index Terms*—** Wireless communications, wireless networks, optimization, adaptive algorithms, resource allocation

1. INTRODUCTION

This paper develops ergodic stochastic optimization (ESO) algorithms to solve problems that involve a time varying random state $\mathbf{h}(t)$ with probability distribution function (pdf) $m(\mathbf{h})$, a resource allocation function $\mathbf{p}(t)$ and an ergodic limit variable $\mathbf{x} := \lim_{t\to\infty}(1/t)\sum_{u=1}^{t}\mathbf{x}(u)$. The goal is to design an adaptive algorithm that observes $\mathbf{h}(t)$ to determine $\mathbf{p}(t)$ and $\mathbf{x}(t)$ without knowledge of the state's distribution $m(\mathbf{h})$ in order to satisfy given problem constraints and optimality criteria. Problem constraints restrict instantaneous values $\mathbf{p}(t)$ and $\mathbf{x}(t)$ as well as ergodic limits $\mathbf{x}$. Optimality criteria depend only on the ergodic average $\mathbf{x}$.

Problems with these characteristics are common in signal processing, notably in optimal wireless communications and networking. In this case $\mathbf{h}(t)$ denotes time varying fading channels, $\mathbf{p}(t)$ instantaneous power allocations and $\mathbf{x}(t)$ includes communication rates and other problem-specific variables. If the operation time scale is much larger than the communication time scale, perceived quality of service is reasonably captured as a function of ergodic limits $\mathbf{x}$. Particular examples where this type of optimization problem might arise are orthogonal frequency division multiplexing (OFDM), e.g. [1], beamforming, e.g., [2] cognitive radio, e.g., [3] and wireless networking, e.g., [4].

The proposed ESO algorithm uses *stochastic* subgradient descent on the dual function. Subgradient descent was developed for minimizing non-differential convex functions and is commonly used to minimize dual functions which are always convex and often non-differentiable. When the function to be minimized involves a random component it is possible to devise a stochastic counterpart, e.g., [5]. Stochastic and deterministic subgradient exhibit similar convergence properties. Of importance here, is the fact that while iterates do not necessarily converge to the solution of the optimization problem, optimal variables can be recovered from the time average of iterates.

Implementation of dual subgradient descent yields, as a byproduct, a sequence of primal iterates. Do these primal iterates approximate optimal primal variables? Not always. When using deterministic or stochastic subgradient descent on the dual function this is true only when the problem Lagrangian is strictly concave with respect to primals. This condition is not satisfied if some variables appear only in linear constraints and linear terms of the optimization objective. Although this restriction might seem minor, non strictly concave Lagrangians do appear frequently – e.g., wireless networking problems are typically not strictly concave with respect to routing variables. To overcome this limitation in deterministic subgradient descent, the use of ergodic averages of primal iterates has been proposed and shown to approximate optimal primal variables [6]. While this much is know in a deterministic setting, convergence results for primal variables in stochastic subgradient descent are mostly restricted to convergence in mean for problems with strict convexity [5]. This paper shows that ergodic limits of primal iterates obtained from the implementation of a stochastic subgradient descent algorithm converge almost surely to the solution of the given optimization problem. With respect to existing work on stochastic subgradient descent the contributions of this work are: (i) we allow for non strictly concave Lagrangians; (ii) we allow for non-convex constraints in the resource allocation variables; and (iii) we prove almost sure convergence of the ergodic averages of primal iterates – as opposed to convergence in mean. These properties are important in signal processing applications

Section 2 introduces the optimization problem whose solution determines optimal resource allocations and ergodic limits. Problem constraints are assumed convex in the ergodic limits but not necessarily so with respect to the resource allocation. The problem's objective is a concave function of ergodic limits only. The ESO algorithm and the main result of the paper, regarding convergence of resource allocation and ergodic limit sequences, is also introduced here in Theorem 1. It is claimed that: (i) resource allocation and ergodic limit sequences almost surely satisfy problem constraints in an ergodic sense; and (ii) the ergodic limit sequence is almost surely close to optimal. The ESO algorithm does not require access to the state's pdf, can handle non-convex constraints in the resource allocation variables, and guarantees almost sure convergence to optimal operating points. To exemplify these characteristics the ESO algorithm is applied in Section 3 to determine the optimal operating point of an OFDM broadcast channel. The example serves as illustration of how ESO can be used to solve a non-convex optimization problem with thousands of variables with reasonable computational cost. See also [7] for the application of ESO to general wireless networking problems.

2. ERGODIC STOCHASTIC OPTIMIZATION ALGORITHM

Consider a random state $\mathbf{h} \in \mathcal{H}$ with probability distribution $m(\mathbf{h})$, a resource allocation $\mathbf{p}(\mathbf{h})$ corresponding to state realization $\mathbf{h}$ and having pdf $m\big(\mathbf{p}(\mathbf{h})\big)$ and an ergodic limit variable $\mathbf{x}$. The goal is to determine resource allocations and ergodic limits that are optimal in the sense of solving the optimization problem

$$
\begin{aligned}
P = \max\ & f_0(\mathbf{x}) \\
\text{s.t. } & \mathbf{x} \leq \mathbb{E}_{\mathbf{h}}\Big[\mathbb{E}_{m(\mathbf{p}(\mathbf{h}))}\Big(\mathbf{f}_1\big(\mathbf{p}(\mathbf{h});\mathbf{h}\big)\Big)\Big], \\
& \mathbf{f}_2(\mathbf{x}) \geq \mathbf{0},\ \mathbf{x}\in\mathcal{X},\ \big\{m\big(\mathbf{p}(\mathbf{h})\big) : \mathbf{p}(\mathbf{h}) \in \mathcal{P}(\mathbf{h})\big\}_{\mathbf{h}\in\mathcal{H}}.
\end{aligned} \tag{1}
$$

The optimization in (1) is with respect to ergodic limits $\mathbf{x}$ and pdfs $m\big(\mathbf{p}(\mathbf{h})\big)$ for all $\mathbf{h} \in \mathcal{H}$. The expected value is taken with respect to the pdfs $m(\mathbf{h})$ of the state $\mathbf{h}$ and $m\big(\mathbf{p}(\mathbf{h})\big)$ of the resource allocation $\mathbf{p}(\mathbf{h})$. Since $m(\mathbf{h})$ is fixed we denote expected value with respect to $m(\mathbf{h})$ as $\mathbb{E}_{\mathbf{h}}(\cdot)$. The pdfs $m\big(\mathbf{p}(\mathbf{h})\big)$, however, are part of the optimization space. To make this clear we denote expected value with respect to $m\big(\mathbf{p}(\mathbf{h})\big)$ as $\mathbb{E}_{m(\mathbf{p}(\mathbf{h}))}(\cdot)$. Functions $f_0(\mathbf{x})$ and $\mathbf{f}_2(\mathbf{x})$ in (1) are concave with respect to their argument $\mathbf{x}$. The family of functions $\mathbf{f}_1\big(\mathbf{p}(\mathbf{h});\mathbf{h}\big)$ is parameterized by $\mathbf{h}$ and is not necessarily concave with respect to $\mathbf{p}(\mathbf{h})$. The sole requirement for the functions $\mathbf{f}_1\big(\mathbf{p}(\mathbf{h});\mathbf{h}\big)$ is that they be finite for finite argument, i.e., for every bounded vector of resources $\mathbf{p}(\mathbf{h}) < \infty$, function $\mathbf{f}_1\big(\mathbf{p}(\mathbf{h});\mathbf{h}\big) \leq \infty$ is also bounded. The set $\mathcal{X}$ to which the ergodic limits $\mathbf{x}$ are constrained is compact and convex. The set $\mathcal{P}(\mathbf{h})$ constraining resource allocation *values* $\mathbf{p}(\mathbf{h})$ is compact but not necessarily convex.

The problem in (1) originates in optimal resource allocation problems with infinite time horizons allowing performance characterization through ergodic limits. System operation is affected by a random state with realizations $\mathbf{h}(t)$. In response to the observed state $\mathbf{h}(t)$, a resource allocation variable $\mathbf{p}(t) \in \mathcal{P}(\mathbf{h}(t))$ measuring how many units of a certain resource are allocated at time t is determined. Allocation of $\mathbf{p}(t)$ units of resource when the random state is $\mathbf{h}(t)$, results in production of $\mathbf{f}_1\big(\mathbf{p}(t);\mathbf{h}(t)\big)$ units of goods. In the same time slot t consumption is determined by $\mathbf{x}(t) \in \mathcal{X}$ variables. Consumption cannot exceed production, but if long time horizons are of interest, instead of imposing such restriction for every t it suffices to constraint the ergodic limits, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \mathbf{x}(u) \leq \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \mathbf{f}_1\big(\mathbf{p}(u);\mathbf{h}(u)\big). \tag{2}$$

The first constraint in (1) follows upon defining the time ergodic limit $\mathbf{x} := \lim_{t\to\infty}(1/t)\sum_{u=1}^{t}\mathbf{x}(u)$ and using the law of large numbers in the limit on the right hand side of (2). Notice that equal state realizations $\mathbf{h}(t_1) = \mathbf{h}(t_2)$ can be associated with different resource allocations $\mathbf{p}(t_1) \neq \mathbf{p}(t_2)$. That is why the expected value in (1) is taken with respect to the pdfs $m(\mathbf{h})$ and $m\big(\mathbf{p}(\mathbf{h})\big)$ and the optimization is with respect to probability distributions $m\big(\mathbf{p}(\mathbf{h})\big) \in \mathcal{P}(\mathbf{h})$ not values $\mathbf{p}(\mathbf{h}) \in \mathcal{P}(\mathbf{h})$. The constraint $\mathbf{f}_2(\mathbf{x}) \geq \mathbf{0}$ imposes further restrictions on the ergodic average $\mathbf{x}$.

Observe that while there is an infinite number of variables in the primal domain, there is a finite number of inequality constraints. Thus, the dual problem contains a finite number of variables hinting that the problem is likely more tractable in the dual space. Define then dual variables $\boldsymbol{\lambda}_1 \geq \mathbf{0}$ associated with the constraint $\mathbf{x} \leq \mathbb{E}_{\mathbf{h}}\big[\mathbb{E}_{m(\mathbf{p}(\mathbf{h}))}\big(\mathbf{f}_1\big(\mathbf{p}(\mathbf{h});\mathbf{h}\big)\big)\big]$ and $\boldsymbol{\lambda}_2 \geq \mathbf{0}$ associated with $\mathbf{f}_2(\mathbf{x}) \geq \mathbf{0}$.The Lagrangian for the optimization problem in (1) is then written as

$$\begin{aligned}
&\mathcal{L}[\boldsymbol{\lambda},\mathbf{x},m\big(\mathbf{p}(\mathbf{h})\big)] \qquad (3)\\
&= f_0(\mathbf{x}) + \boldsymbol{\lambda}_1^T\Big[\mathbb{E}_{\mathbf{h}}\Big[\mathbb{E}_{m(\mathbf{p}(\mathbf{h}))}\Big(\boldsymbol{\lambda}_1^T\mathbf{f}_1\big(\mathbf{p}(\mathbf{h});\mathbf{h}\big)\Big)\Big] - \mathbf{x}\Big] + \boldsymbol{\lambda}_2^T\mathbf{f}_2(\mathbf{x})\\
&= f_0(\mathbf{x}) - \boldsymbol{\lambda}_1^T\mathbf{x} + \boldsymbol{\lambda}_2^T\mathbf{f}_2(\mathbf{x}) + \mathbb{E}_{\mathbf{h}}\Big[\mathbb{E}_{m(\mathbf{p}(\mathbf{h}))}\Big(\boldsymbol{\lambda}_1^T\mathbf{f}_1\big(\mathbf{p}(\mathbf{h});\mathbf{h}\big)\Big)\Big],
\end{aligned}$$

where we defined the aggregate dual variable $\boldsymbol{\lambda} := [\boldsymbol{\lambda}_1^T,\boldsymbol{\lambda}_2^T]^T$ and reordered terms to obtain the second equality. The dual function is defined as the maximum of the Lagrangian over the set of feasible ergodic limits $\mathbf{x} \in \mathcal{X}$ and probability distributions $m\big(\mathbf{p}(\mathbf{h})\big)$ in the set of feasible powers $\mathbf{p}(\mathbf{h}) \in \mathcal{P}(\mathbf{h})$, i.e.,

$$\begin{aligned}
g(\boldsymbol{\lambda}) := &\max\ \mathcal{L}[\boldsymbol{\lambda},\mathbf{x},m\big(\mathbf{p}(\mathbf{h})\big)]\\
&\text{s.t.}\quad \mathbf{x} \in \mathcal{X},\ \big\{m\big(\mathbf{p}(\mathbf{h})\big) : \mathbf{p}(\mathbf{h}) \in \mathcal{P}(\mathbf{h})\big\}_{\mathbf{h}\in\mathcal{H}}.
\end{aligned} \tag{4}$$

The dual problem is defined as the minimization of the dual function over all positive dual variables, i.e., $D = \min_{\boldsymbol{\lambda}\geq\mathbf{0}} g(\boldsymbol{\lambda})$.

Introduce now a discrete time index t and consider a state stochastic process $\mathbf{H}(\mathbb{N})$ with realizations $\mathbf{h}(\mathbb{N})$ having values $\mathbf{h}(t)$ identically and independently distributed (i.i.d.) according to $m(\mathbf{h})$. The ESO algorithm consists of iterative application of the following steps.

(S1) Primal iteration. Given multipliers $\boldsymbol{\lambda}(t)$ find primal variables $\mathbf{x}(t) \in \mathcal{X}$ and $\mathbf{p}(t) \in \mathcal{P}(\mathbf{h}(t))$ such that

$$\mathbf{x}(t) = \mathbf{x}(\boldsymbol{\lambda}(t)) = \operatorname*{argmax}_{\mathbf{x}\in\mathcal{X}} f_0(\mathbf{x}) - \boldsymbol{\lambda}_1^T(t)\mathbf{x} + \boldsymbol{\lambda}_2^T(t)\mathbf{f}_2(\mathbf{x}), \tag{5}$$

$$\mathbf{p}(t) = \mathbf{p}(\mathbf{h}(t),\boldsymbol{\lambda}(t)) = \operatorname*{argmax}_{\mathbf{p}(\mathbf{h}(t))\in\mathcal{P}(\mathbf{h}(t))} \boldsymbol{\lambda}_1^T(t)\mathbf{f}_1\big(\mathbf{p}(\mathbf{h}(t));\mathbf{h}(t)\big). \tag{6}$$

(S2) Dual stochastic subgradients. Define the dual function stochastic subgradient $\hat{\mathbf{s}}(t) = \hat{\mathbf{s}}(\mathbf{h}(t),\boldsymbol{\lambda}(t)) = [\hat{\mathbf{s}}_1^T(t),\hat{\mathbf{s}}_2^T(t)]^T$ with components

$$\hat{\mathbf{s}}_1(t) := \mathbf{f}_1\big(\mathbf{p}(t);\mathbf{h}(t)\big) - \mathbf{x}(t), \qquad \hat{\mathbf{s}}_2(t) := \mathbf{f}_2(\mathbf{x}(t)). \tag{7}$$

(S3) Dual iteration. Update dual variables along direction $-\hat{\mathbf{s}}(t)$ with step size ϵ (the operator $[\cdot]^+$ denotes projection to the positive orthant)

$$\boldsymbol{\lambda}(t+1) = \left[\begin{array}{c} \boldsymbol{\lambda}_1(t) - \epsilon\Big(\mathbf{f}_1\big(\mathbf{p}(t);\mathbf{h}(t)\big) - \mathbf{x}(t)\Big)\\ \boldsymbol{\lambda}_2(t) - \epsilon\mathbf{f}_2(\mathbf{x}) \end{array}\right]^+. \tag{8}$$

Solving (1) entails finding optimal value P and optimal arguments $\mathbf{x}^*$ and $m^*\big(\mathbf{p}(\mathbf{h})\big)$ such that constraints in (1) are satisfied and $P = f_0(\mathbf{x}^*)$. Heeding the connection with ergodic constraints we adopt a different definition of solution. Our goal is not to find $\mathbf{x}^*$ and $m^*\big(\mathbf{p}(\mathbf{h})\big)$ but to show that sequences $\mathbf{x}(\mathbb{N})$ and $\mathbf{p}(\mathbb{N})$ generated by the ESO algorithm (S1)-(S3) satisfy (2) with the ergodic limit $\mathbf{x}$ of the $\mathbf{x}(\mathbb{N})$ sequence further satisfying $\mathbf{f}_2(\mathbf{x}) \geq \mathbf{0}$ and $P \approx f_0(\mathbf{x})$. Because the algorithm is stochastic, these results will be established in probability. A formal statement is presented next[1].

Theorem 1 *Consider the optimization problem in* (1) *and sequences* $\mathbf{x}(\mathbb{N})$ *and* $\mathbf{p}(\mathbb{N})$ *generated by the ESO algorithm defined by* (5)-(8). *Let* $\hat{S}^2 \geq \mathbb{E}\left[\|\hat{\mathbf{s}}(t)\|^2 \,\middle|\, \boldsymbol{\lambda}(t)\right]$ *be a bound on the second moment of the norm of* $\hat{\mathbf{s}}(t)$ *and assume that there exist strictly feasible* $\mathbf{x}_0 \in \mathcal{X}$ *and* $m_0\big(\mathbf{p}(\mathbf{h})\big)$ *such that* $\mathbb{E}_{\mathbf{h}}\big[\mathbb{E}_{m_0(\mathbf{p}(\mathbf{h}))}\big(\mathbf{f}_1\big(\mathbf{p}(\mathbf{h});\mathbf{h}\big)\big)\big] - \mathbf{x}_0 > C$ *and* $\mathbf{f}_2(\mathbf{x}_0) > C$ *for some strictly positive constant* $C > 0$. *Then*

Almost sure feasibility. *Sequences* $\mathbf{x}(\mathbb{N})$ *and* $\mathbf{p}(\mathbb{N})$ *are almost surely feasible, i.e.,*

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \mathbf{x}(u) \leq \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \mathbf{f}_1\big(\mathbf{p}(u);\mathbf{h}(u)\big) \qquad \text{a.s.}, \tag{9}$$

$$\mathbf{f}_2\left[\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \mathbf{x}(u)\right] \geq \mathbf{0} \qquad \text{a.s.} \tag{10}$$

Almost sure near optimality. *The ergodic average of* $f_0(\mathbf{x}(u))$ *almost surely converges to a value with optimality gap smaller than* $\epsilon\hat{S}^2/2$*, i.e,*

$$P - \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} f_0(\mathbf{x}(u)) \leq \frac{\epsilon\hat{S}^2}{2}. \tag{11}$$

The limits in (9)-(11) might be different for different state sequences $\mathbf{h}(\mathbb{N})$. The claims in Theorem 1 are on the probability distributions of these limits. The ergodic limit $\mathbf{x} := (1/t)\sum_{u=1}^{t}\mathbf{x}(u)$ satisfies the constraints in (1) and the objective function evaluated at $\mathbf{x}$ is within $\epsilon\hat{S}^2/2$ of optimal. Since $\mathcal{X}$ and $\mathcal{P}(\mathbf{h})$ are compact sets it follows that the bound $\hat{S}^2$ is finite. Therefore, reducing ϵ it is possible to make $\mathbf{f}_0(\mathbf{x})$ arbitrarily close to P and as a consequence $\mathbf{x}$ arbitrarily close to some optimal argument $\mathbf{x}^*$. The optimal resource allocation distribution $m^*\big(\mathbf{p}(\mathbf{h})\big)$, however, is not computed by the ESO algorithm. Rather, (9) implies that, asymptotically, the ESO algorithm is drawing resource allocation realizations $\mathbf{p}(t)$ from a resource allocation distribution that is close to the optimal $m^*\big(\mathbf{p}(\mathbf{h})\big)$. This is not a drawback in practice because realizations $\mathbf{p}(t)$ are sufficient for implementation. In that sense, (5)-(8) yield an optimal resource allocation policy, i.e., allocate $\mathbf{p}(t)$ units at time t, that supports optimal consumption $\mathbf{x}$ in an ergodic sense.

The feasibility claim of Theorem 1 assures that sequences $\mathbf{p}(\mathbb{N})$ and $\mathbf{x}(\mathbb{N})$ satisfy problem constraints with probability 1 [cf. (9) and (10)]. This is a stronger claim when compared with the optimality result that establishes a typically small but not null performance gap [cf. (11)]. It is also worth remarking that this is true independently of the step size ϵ. While we think of ϵ as a small number, and it is indeed desirable to select small ϵ, this is not necessary to ensure feasibility of $\mathbf{p}(\mathbb{N})$ and $\mathbf{x}(\mathbb{N})$. The strength of Result (i) of Theorem 1 is important from a practical standpoint. A small optimality gap is acceptable in general, but a small violation of problem constraints results in a set of variables incompatible with the physical constraints of the system.

Remark 1 The problem formulation in (1) makes what seems an arbitrary distinction between constraints $\mathbf{f}_2(\mathbf{x}) \geq \mathbf{0}$ and $\mathbf{x} \in \mathcal{X}$. While one is expressed as a function inequality and the other as a set inclusion both are convex contraints in the ergodic limit $\mathbf{x}$. Despite this similarity they are intended to model different constraint modalities. The constraint $\mathbf{f}_2(\mathbf{x}) \geq \mathbf{0}$ is incorporated into the Lagrangian in (3) and becomes a maximization objective in the primal ESO iteration in (5). As a consequence, it is satisfied in an ergodic sense. Ergodic limits of

[1]Proofs or results in this paper are available in [8]

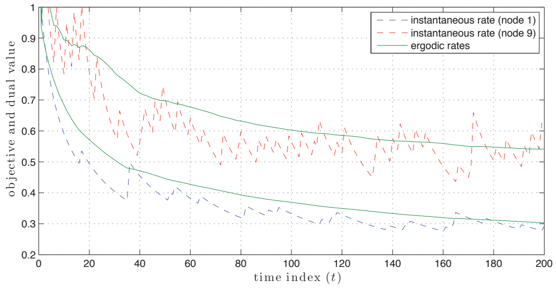

Fig. 1. Evolution of capacities $c_i(t)$ for representative nodes 1 with average channels $\mathbb{E}\left[h_{1f}(t)\right] = 1$ and 9 with $\mathbb{E}\left[h_{9f}(t)\right] = 3$. Ergodic averages $\bar{c}_i(t) = (1/t)\sum_{u=1}^{t} c_i(u)$ also shown.

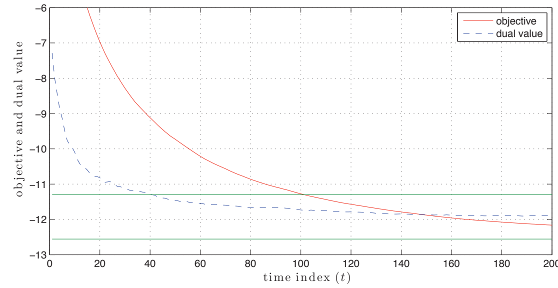

Fig. 2. Objective value $\sum_{i=1}^{J} U_i(\bar{c}_i(u))$ and dual function's value $g(\boldsymbol{\lambda}(t))$. Lines for optimal objective and 90% of optimal objective also shown.

$\mathbf{x}(\mathbb{N})$ sequences satisfy $\mathbf{f}_2(\mathbf{x}) \geq \mathbf{0}$ but individual variables $\mathbf{x}(t)$ might or might not satisfy $\mathbf{f}_2(\mathbf{x}(t)) \geq \mathbf{0}$. Constraint $\mathbf{x} \in \mathcal{X}$ is not incorporated into $\mathcal{L}[\boldsymbol{\lambda}, \mathbf{x}, m(\mathbf{p}(\mathbf{h}))]$ and is an implicit constraint in the primal ESO iteration in (5). It is thus satisfied for all times, i.e., $\mathbf{x}(t) \in \mathcal{X}$, for all t. This is an important distinction in applications, e.g., transmitted power in wireless communications must comply with ergodic and instantaneous power constraints.

Remark 2 The ESO algorithm (S1)-(S3) is related to subgradient descent on the dual function. The primal iteration of subgradient descent consists of finding arguments that maximize the Lagrangian $\mathcal{L}\left[\boldsymbol{\lambda}(t), \mathbf{x}, m(\mathbf{p}(\mathbf{h}))\right]$ [cf. step (S1)]. The constraints are then evaluated at these maximizing arguments to compute a dual function subgradient [cf. step (S2)] and the dual variables descend opposite the subgradient direction [cf. step (S3)]. To appreciate similarities and differences let us develop dual subgradient descent equations for the problem in (1). To compute a subgradient for the dual function we find primal arguments $\mathbf{x}(t)$ and $\left\{m(\mathbf{p}(\mathbf{h}), t)\right\}_{\mathbf{h}\in\mathcal{H}}$ that maximize the Lagrangian, i.e,

$$\begin{aligned} \mathbf{x}(t), \left\{m(\mathbf{p}(\mathbf{h}), t)\right\}_{\mathbf{h}\in\mathcal{H}} & \\ := \operatorname{argmax} & \ \mathcal{L}\left[\boldsymbol{\lambda}(t), \mathbf{x}, m(\mathbf{p}(\mathbf{h}))\right] \\ \text{s.t.} & \quad \mathbf{x} \in \mathcal{X},\ \left\{m(\mathbf{p}(\mathbf{h})) : \mathbf{p}(\mathbf{h}) \in \mathcal{P}(\mathbf{h})\right\}_{\mathbf{h}\in\mathcal{H}}. \end{aligned} \tag{12}$$

The Lagrangian $\mathcal{L}\left[\boldsymbol{\lambda}(t), \mathbf{x}, m(\mathbf{p}(\mathbf{h}))\right]$ exhibits a separable structure. Variables $\mathbf{x}$ and $\mathbf{p}(\mathbf{h})$ appear in different summands and the maximization of $\mathbb{E}_{m(\mathbf{p}(\mathbf{h}))}\left(\boldsymbol{\lambda}_1^T \mathbf{f}_1(\mathbf{p}(\mathbf{h}); \mathbf{h})\right)$ can be reduced to separate maximizations with respect to each individual distribution $m(\mathbf{p}(\mathbf{h}))$. Therefore, the maximizers in (12) can be computed separately as

$$\mathbf{x}(t) = \operatorname*{argmax}_{\mathbf{x}\in\mathcal{X}} f_0(\mathbf{x}) - \boldsymbol{\lambda}_1^T(t)\mathbf{x} + \boldsymbol{\lambda}_2^T(t)\mathbf{f}_2(\mathbf{x}), \tag{13}$$

$$m(\mathbf{p}(\mathbf{h}), t) = \operatorname*{argmax}_{m(\mathbf{p}(\mathbf{h})):\mathbf{p}(\mathbf{h})\in\mathcal{P}(\mathbf{h})} \mathbb{E}_{m(\mathbf{p}(\mathbf{h}))}\left(\boldsymbol{\lambda}_1^T(t)\mathbf{f}_1(\mathbf{p}(\mathbf{h}); \mathbf{h})\right) \tag{14}$$

A subgradient $\mathbf{s}(t) = \mathbf{s}(\boldsymbol{\lambda}(t)) = [\mathbf{s}_1^T(t), \mathbf{s}_2^T(t)]^T$ of the dual function can now obtained by evaluating the constraints at the maximizing arguments $\mathbf{x}(t)$ and $\left\{m(\mathbf{p}(\mathbf{h}), t)\right\}_{\mathbf{h}\in\mathcal{H}}$. Components $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t)$ are therefore given by

$$\mathbf{s}_1(t) := \mathbb{E}_{\mathbf{h}}\left[\mathbb{E}_{m(\mathbf{p}(\mathbf{h}))}\left(\boldsymbol{\lambda}_1^T \mathbf{f}_1(\mathbf{p}(\mathbf{h}); \mathbf{h})\right)\right] - \mathbf{x}(t),\ \mathbf{s}_2(t) := \mathbf{f}_2(\mathbf{x}(t)). \tag{15}$$

The dual update has the same functional form of (8) with $\mathbf{s}(t)$ replacing $\hat{\mathbf{s}}(t)$. The subgradient descent algorithm for (1) consists of iterative application of (13), (15) and (8) with $\mathbf{s}(t)$ replacing $\hat{\mathbf{s}}(t)$.

Stochastic subgradients $\hat{\mathbf{s}}(t)$ are easier to compute. To determine $\mathbf{s}_1(t)$ it is necessary to solve the maximization in (14) for a large number of states $\mathbf{h}$ in order to obtain a good approximation of the expected value in (15). To compute $\hat{\mathbf{s}}(t)$ only one such maximization, for $\mathbf{h} = \mathbf{h}(t)$ is required. Further note that the maximization in (14) is with respect to probability distributions $m(\mathbf{p}(\mathbf{h}))$ while the maximization in (6) is with respect to values $\mathbf{p}(\mathbf{h}(t))$. Also, to implement subgradient descent the state pdf $m(\mathbf{h})$ is needed to compute the expected value in (15). The stochastic version only needs access to current state realizations $\mathbf{h}(t)$.

3. WIRELESS BROADCAST CHANNEL

Consider a wireless broadcast channel using OFDM. An access point (AP) administers frequency tones $\mathcal{F}$ and average power budget P_0 to serve J terminals. The goal is to design an algorithm that allocates power and frequency to maximize a given ergodic rate utility metric. At time t the AP observes fading channels $h_{if}(t)$ for all frequencies $f \in \mathcal{F}$ and nodes i. Depending on the fading channels' values it decides on frequency allocation $a_{if}(t) \in \{0, 1\}$ and power allocation $p_{if}(t) \in [0, p_{\max}]$. Variable $a_{if}(t) = 1$ if and only if frequency f is allocated to node i at time t. If $a_{if}(t) = 1$, the power allocated for such communication is $p_{if}(t)$. Since no more than one communication can utilize a given frequency, for given f and t at most one $a_{if}(t)$ can be different from zero. To capture this constraint define the vector $\mathbf{a}_f(t) = [a_{1f}(t), \ldots, a_{Jf}(t)]^T$ and require $\mathbf{a}_f(t) \in \mathcal{A}$ with the set $\mathcal{A}$ defined as $\mathcal{A} := \left\{\mathbf{a} = [a_1, \ldots a_J]^T : a_j \in \{0, 1\},\ \mathbf{a}^T\mathbf{1} \leq 1\right\}$.

With channel $h_{if}(t)$ and power allocation $p_{if}(t)$, information delivered to i is $C_{if}\left(h_{if}(t), p_{if}(t)\right)$. The map $C_{if}\left(h_{if}(t), p_{if}(t)\right)$ from channels and powers to transmission rates depends on the type of modulation and codes used. E.g., adaptive modulation and coding (AMC) relies on a set of L communication modes. The l-th mode supports a rate α_l and is used when the signal to noise ratio (SNR) is between β_{l-1} and β_l. Normalizing channels $h_{if}(t)$ so that noise power is $\sigma_{if}(t) = 1$ the map $C_{if}\left(h_{if}(t), p_{if}(t)\right)$ for AMC can be written as

$$C_{if}\left(h_{if}(t), p_{if}(t)\right) = \sum_{l=1}^{L} \alpha_l \mathbb{I}\left(\beta_l \leq h_{if}(t)p_{if}(t) < \beta_{l+1}\right). \tag{16}$$

While $C_{if}\left(h_{if}(t), p_{if}(t)\right)$ units of information are *delivered* by the AP, $c_{ij}(t)$ units of information are *accepted* for delivery and queued to await transmission. To guarantee delivery of packets it suffices to ensure stability of information queues by requiring

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} c_i(t) \leq \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \left[\sum_{f\in\mathcal{F}} a_{if}(t) C_{if}\left(h_{if}(t), p_{if}(t)\right)\right], \tag{17}$$

Similarly, the amount of power consumed at time t is the sum of powers used on all frequencies for communication with all terminals, i.e., $\sum_{i=1}^{J}\sum_{f\in\mathcal{F}} a_{if}(t)p_{if}(t) = \sum_{i,f} a_{if}(t)p_{if}(t)$. This cannot exceed allocated power P_0 thus yielding the constraint

$$P_0 \geq \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \left[\sum_{i,f} a_{if}(u)p_{if}(u)\right]. \tag{18}$$

If the time scale of communication is much smaller than the time scale of operation, perceived quality of service is determined by the ergodic limit $c_i := \lim_{t\to\infty}(1/t)\sum_{u=1}^{t} c_i(u)$. Assigning value $U_i(c_i)$ to ergodic rate c_i the goal is to determine sequences of frequency allocations $a_{if}(\mathbb{N})$, powers $p_{if}(\mathbb{N})$ and rates $c_i(\mathbb{N})$ such that: (i) the ergodic limits of $c_i(\mathbb{N})$ sequences maximize a sum utility $\sum_{i=1}^{j} U_i(c_i)$; (ii) constraints in (17) and (18) are satisfied; and (iii) instantaneous frequency allocations are feasible, i.e., $\mathbf{a}_f(t) \in \mathcal{A}$.

This is the type of problem solved by the ESO algorithm. Let $\mathbf{h}$ aggregate channel variables h_{if} for all i and f and define frequency and power allocations $a_{if}(\mathbf{h})$ and $p_{if}(\mathbf{h})$. Also, let $\mathbf{a}(\mathbf{h})$ aggregate all $a_{if}(\mathbf{h})$ and $\mathbf{p}(\mathbf{h})$ represent all $p_{if}(\mathbf{h})$. Define then the problem

$$\max \sum_{i=1}^{J} U_i(c_i) \tag{19}$$
$$\text{s.t. } c_i \leq \mathbb{E}\bigg[\sum_{f\in\mathcal{F}} a_{if}(\mathbf{h}) C_{if}\big(h_{if}, p_{if}(\mathbf{h})\big)\bigg],$$
$$P_0 \geq \mathbb{E}\bigg[\sum_{i,f} a_{if}(\mathbf{h}) p_{if}(\mathbf{h})\bigg],\ \mathbf{a}_f(\mathbf{h}) \in \mathcal{A},\ 0 \leq p_{if}(\mathbf{h}) \leq p_{\max}.$$

where expected values $\mathbb{E}(\cdot)$ are over the pdfs $m(\mathbf{h})$ and $m(\mathbf{a}(\mathbf{h}), \mathbf{p}(\mathbf{h}))$. The optimization in (19) is with respect to ergodic limits c_i and probability distributions $m\big(\mathbf{a}(\mathbf{h}), \mathbf{p}(\mathbf{h})\big)$ restricted to $\mathbf{a}_f(\mathbf{h}) \in \mathcal{A}$ and $0 \leq p_{if}(\mathbf{h}) \leq p_{\max}$. This optimization problem is of the form in (1). Introducing multipliers λ_i associated with the capacity constraints and μ with the power constraint, ESO's primal iteration is [cf. (5) and (6)]

$$c_i(t) = \ \operatorname{argmax} \ \ U_i(c_i) - \lambda_i c_i \tag{20}$$
$$a_{if}(t), p_{if}(t) = \operatorname*{argmax}_{0\leq p_{if}, \mathbf{a}_f(t)\in\mathcal{A}} a_{if}(t)\Big(\lambda_i C_{if}\big(h_{if}(t), p_{if}\big) - \mu p_{if}\Big). \tag{21}$$

Since for fixed f at most one variable $a_{if}(t)$ can be 1 in (21), determination of $a_{if}(t)$ and $p_{if}(t)$ can be separated further. Indeed, compute

$$p_{if}(t) = \operatorname*{argmax}_{0\leq p_{if}} \ \lambda_i C_{if}\big(h_{if}(t), p_{if}\big) - \mu p_{if} \tag{22}$$
$$i_f(t) = \operatorname*{argmax}_{i} \ \lambda_i C_{if}\big(h_{if}(t), p_{if}(t)\big) - \mu p_{if}(t), \tag{23}$$

and set $a_{i_f(t)f}(t) = 1$ and $a_{if}(t) = 0$ for all other $i \neq i_f(t)$. While the maximization in (22) involves the non concave function $C_{if}\big(h_{if}(t), p_{if}\big)$, it is nonetheless simple to solve. The ESO algorithm is completed with the dual iteration [cf. (7) and (8)]

$$\lambda_i(t+1) = \bigg[\lambda_i(t) + \epsilon\bigg[\sum_{f\in\mathcal{F}} a_{if}(t) C_{if}\big(h_{if}(t), p_{if}(t)\big) - c_i(t)\bigg]\bigg]^+$$
$$\mu(t+1) = \bigg[\mu(t) + \epsilon\bigg[P_0 - \sum_{i=1}^{j}\sum_{f\in\mathcal{F}} a_{if}(t) p_{if}(t)\bigg]\bigg]^+. \tag{24}$$

As per Theorem 1 iterative application of (20) and (22)-(24) yields sequences $c_i(\mathbb{N})$, $a_{if}(\mathbb{N})$ and $p_{if}(\mathbb{N})$ such that: (i) the sum utility for the ergodic limits of $c_i(\mathbb{N})$ is almost surely within a small constant of optimal; (ii) the constraints in (17) and (18) are almost surely satisfied; and (iii) instantaneous frequency values of $a_{if}(t)$ are feasible. The stated goal is then satisfied with probability 1.

3.1. Numerical results

The ESO algorithm (20) and (22)-(24) for optimal resource allocation in an OFDM broadcast channel is simulated for a system with $J = 16$ nodes using 3 frequency tones for communication. Three AMC modes corresponding to capacities 1, 2 and 3 bits/s/Hz are used with transitions at SINR 1, 3 and 7. Fading channels are generated as i.i.d. Rayleigh with average powers 1 for the first four nodes, i.e., $j = 1$–4, and 2, 3 and 4 for subsequent groups of 4 nodes. Noise power is 1 for all frequencies and average power is $P_0 = 3$. Rate of packet acceptance is constrained to be $0 \leq c_i(t) \leq 2$ bits/s/Hz. The optimality criteria is proportional fair scheduling, i.e., $U_i(c_i) = \log(c_i)$ for all i. Steps size is $\epsilon = 0.1$.

Fig. 1 shows evolution of capacities $c_i(t)$ for representative nodes 1 with average channels $\mathbb{E}[h_{1f}(t)] = 1$ and 9 with $\mathbb{E}[h_{9f}(t)] = 3$. The ergodic average $\bar{c}_i(t) = (1/t)\sum_{u=1}^{t} c_i(u)$ is also shown. Capacities $c_i(u)$ do not converge, but ergodic rates $\bar{c}_i(t)$ do converge. Convergence of the algorithm is ratified by Figs. 2 and 3. Fig. 2 shows evolution of the objective $\sum_{i=1}^{J} U_i(\bar{c}_i(u))$ and the dual function value $g(t)$. Notice that the objective value is decreasing towards the maximum objective. This is not a contradiction, because variables $\bar{c}_i(t)$ are infeasible but approach feasibility as t grows. The dual function's value is an upper bound on the maximum utility and it can be observed to approach the objective as t grows. Eventually, the objective value becomes smaller than the dual value as expected. Fig. 3 corroborates satisfaction of the constraints in (17) by showing the time evolution of the minimum and maximum amount by which the capacity constraint (17) is violated.

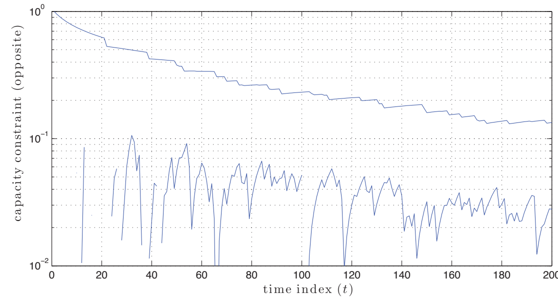

Fig. 3. Feasibility as time grows is corroborated for the capacity constraints in (17). Minimum and maximum constraint violation shown.

4. CONCLUSIONS

We have developed ESO algorithms for optimal resource allocation in problems with long time horizons allowing optimality criteria defined in terms of ergodic limits. A resource $\mathbf{p}(t)$ is allocated at time t, in response to a random state realization $\mathbf{h}(t)$. State and resource allocation constrain the production of a certain good $\mathbf{x}(t)$ that we seek to optimize. We proposed an algorithm using stochastic subgradient descent in the dual function and showed that with probability 1 problem constraints are satisfied and close to optimal production achieved. ESO algorithms do not require access to the state probability distribution and while they assume convexity of objective functions and constraints with respect to $\mathbf{x}(t)$, they do not require convexity with respect to $\mathbf{p}(t)$. Application to find the optimal operating point of an OFDM broadcast channel was considered as an example of a large scale non-convex optimization problem that can be solved with reasonable computational cost. We have also considered applications to general wireless networking problems in [7]. Applications to different problems, e.g., cognitive radio, beamforming and multiple input multiple access channels are left for further research.

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