STOCHASTIC LEARNING ALGORITHMS FOR OPTIMAL DESIGN OF WIRELESS NETWORKS

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ABSTRACT

We introduce algorithms to optimize wireless networks in the presence of fading. Central to the problem considered is the need to learn the fading's probability distribution while determining optimal operating points. A stochastic subgradient descent algorithm in the dual domain is developed to accomplish this task. Even though the optimization problems considered are not convex, convergence of the proposed algorithms is claimed. Numerical results using adaptive modulation over an interference limited physical layer corroborate theoretical results.

1. INTRODUCTION

Optimization of wireless networks incurs a large computational cost because the associated optimization problems are generally not convex. Further complications arise from the adjustment of power allocations to fading channels. Power allocations not only add but dominate the dimensionality of the optimization space, in fact leading to a variational problem with an infinite number of variables. Furthermore, the optimal operating point depends on the probability distribution functions (pdf) of the fading channels. Models for these pdfs exist but are rough approximations of pdfs actually observed in practice. It is then necessary to learn the fading's pdf while determining the optimal operating point. The stochastic learning algorithms (SLA) developed here overcome lack of convexity, infinite dimensionality, and learning of the fading's pdf by using *stochastic* subgradient descent on the dual function.

This paper intends to contribute to the work on optimal cross-layer design of wireless networks; see e.g., [1] and references therein. A significant part of the effort in this direction relies on the use of *deterministic* subgradient descent on the dual domain. This is because the associated Lagrangian function exhibits a separable structure that simplifies optimization and enables distributed implementations, e.g., [2]. However, accounting for fading in subgradient descent algorithms leads to computationally costly solutions. A comprehensive alternative to optimal wireless networking in the presence of fading is the work on stochastic network optimization [3, 4] that extends the back-pressure algorithm to systems with a finite number of random states. In stochastic network optimization, allocation of resources is determined by the difference between queue lengths of neighboring terminals. Interestingly, it is possible to identify queue lengths as dual variables of an optimization problem and interpret stochastic network optimization as a stochastic subgradient descent algorithm [4]. The work here builds on this result by proposing an analyzing general stochastic subgradient descent algorithms to solve wireless networking problems.

Stochastic subgradient descent algorithms descend along random directions whose expected value is a subgradient of the function being minimized, e.g., [5]. As such, they are related to (determinstic) subgradient descent [6] from which their convergence properties can be inferred using stochastic approximation tools. A subtlety when using deterministic or stochastic subgradient descent on the dual function is that convergence of dual variables guarantees convergence of primal variables only when the problem Lagrangian is strictly concave in the primal variables. This condition is frequently not satisfied in networking problems as many constraints are linear. In these cases the use of ergodic averages of primal variables permits recovering solutions that are asymptotically feasible and close to optimal [7]. In stochastic subgradient descent, however, convergence of primal variables is mostly restricted to problems with strict convexity [8]. In the wireless networking problems studied here, non-convex constraints further complicate convergence prospects in the primal domain.

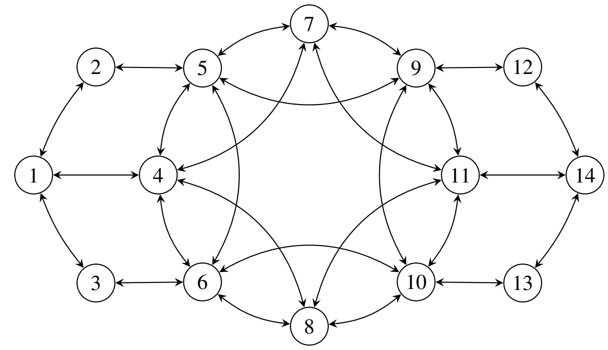

Fig. 1. Connectivity graph of an example wireless network.

These challenges notwithstanding it is shown in this paper that dual stochastic subgradient descent algorithms solve optimal wireless networking problems. The paper starts introducing network variables, power allocations and optimality criteria (Section 2). Feasible operating points require satisfaction of constraints that are introduced here. As is customary, see e.g. [1] these constraints are defined in an ergodic sense, i.e., in terms of long term averages, not instantaneous realizations. Having defined an optimization problem whose solution defines the desired operating point, we describe the use of stochastic subgradient descent algorithms in the dual function (Section 3). This algorithm is interpreted as a method to generate a sequence of network variables and power allocations. Optimality claims introduced in the paper are with respect to this sequence, not its individual values. The main contribution of the paper is stated next in Theorem 1. We claim that: (i) Sequences generated by SLA yield an ergodic feasible operating point with probability 1. (ii) The ergodic average of network variables can be made arbitrarily close to optimal by proper selection of a descent step size. Properties of SLA are then discussed (Section 4). We close presenting numerical results for a wireless network using adaptive modulation and coding over an interference limited physical layer (Section5). This example illustrates the computational feasibility of using SLA in medium sized networks.

2. OPTIMAL WIRELESS NETWORKS

Consider a wireless network composed of J terminals T_j for $j = 1 \dots J$. The network supports K application level flows with the destination of the k-th flow denoted as $T^k = T_{\text{dest}(k)}$. Terminal and flow indexing are different to, e.g., allow different priorities for flows to the same destination. Terminal T_i can communicate only with terminals T_j in its neighborhood that we denote as $n(i)$, see Fig. 1. Actual connections depend on traffic requirements and other parameters defining the network state. The neighborhood $n(i)$ is to be interpreted as dictating that communication between T_i and T_j is not possible if $j \notin n(i)$.

Information flow in the network is determined by three different types of variables: admission control variables $a_i^k(t)$, flow routing variables $r_{ij}^k(t)$ and link capacities $c_{ij}(t)$. The admission control variable $a_i^k(t)$ determines the amount of k-flow information accepted at T_i for delivery to T^k at time t. The value of the routing variable $r_{ij}^k(t)$ corresponds to the units of information sent from T_i to T_j on behalf of the

k-th flow at time t. The amount of information that the physical layer *accepts* for delivery from T_i to T_j at time t is the link capacity $c_{ij}(t)$. For future reference, let $a_i^k(\mathbb{N})$, $r_{ij}^k(\mathbb{N})$ and $c_{ij}(\mathbb{N})$ respectively denote the stochastic processes with values $a_i^k(t)$, $r_{ij}^k(t)$ and $c_{ij}(t)$ at time t.

Network variables are dependent of each other to ensure bounded queue lengths. For given flow k and terminal T_i the total information received up to time t is the sum of information $\sum_{u=1}^t a_i^k(u)$ accepted into the network at T_i and that received from neighbors $\sum_{u=1}^t \sum_{j\in n(i)} r_{ji}^k(u)$. The total information leaving T_i is the sum of information sent to neighbors $\sum_{u=1}^t \sum_{j\in n(i)} r_{ij}^k(u)$. To avoid accumulation of information at T_i the information delivered must exceed information received. To satisfy this constraint in the long run it suffices to require

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} a_i^k(u) \leq \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t}\sum_{j\in n(i)} \left[r_{ij}^k(u) - r_{ji}^k(u)\right]. \quad (1)$$

The constraint in (1) is required for all k and $i \neq \text{dest}(k)$ since packets are not queued at destination.

Similarly, the amount of information accepted for delivery on the link from T_i to T_j up to time t is $\sum_{u=1}^t c_{ij}(u)$. This limits the amount of information that can be delivered from T_i to T_j on behalf of all flows, i.e., $\sum_{u=1}^t \sum_k r_{ij}^k(u)$ leading to the constraint

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t}\sum_{k} r_{ij}^k(u) \leq \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} c_{ij}(u) \quad \text{for all } i, j \in n(i). \quad (2)$$

Upon defining ergodic limits $a_i^k := \lim_{t\to\infty}(1/t)\sum_{u=1}^t a_i^k(u)$, for admission control variables, $r_{ij}^k := \lim_{t\to\infty}(1/t)\sum_{u=1}^t r_{ij}^k(u)$ for routing variables and $c_{ij} := \lim_{t\to\infty}(1/t)\sum_{u=1}^t c_{ij}(u)$ for link capacities, the inequalities in (1) and (2) can be written as

$$a_i^k \leq \sum_{j\in n(i)} r_{ij}^k - r_{ji}^k, \quad \sum_k r_{ij}^k \leq c_{ij}, \quad (3)$$

where the first inequality holds for all k and $i \neq \text{dest}(k)$ and the second one for all i and $j \in n(i)$.

Variables $c_{ij}(t)$ determine the amount of information *accepted* by the physical layer. This quantity is bounded by the amount of information the physical layer *delivers* at time t as determined by instantaneous values of fading and associated bandwidth and power allocations.

To describe the latter, let communication between terminals occur over a set of frequency bands $f \in \mathcal{F}$. Define $h_{ij}^f(t)$ as the time varying fading channel gain from T_i to T_j on frequency $f \in \mathcal{F}$ at time t. In response to observed fading channels, terminal T_i allocates power $p_{ij}^f(t)$ for communication to T_j on frequency f. Define vectors $\mathbf{h}^f(t)$ and $\mathbf{p}^f(t)$ grouping channels $h_{ij}^f(t)$ and power allocations $p_{ij}^f(t)$ for all links and given frequency f as well as vectors $\mathbf{h}(t)$ and $\mathbf{p}(t)$ with channels and power allocations for all links and frequencies. Let $\mathbf{p}(\mathbb{N})$ denote the stochastic process with values $\mathbf{p}(t)$. Channels $\mathbf{h}(t_1)$ and $\mathbf{h}(t_2)$ at different times $t_1 \neq t_2$ are independent.

Power allocations $\mathbf{p}(t)$ and channels $\mathbf{h}(t)$ determine the amount of information that can be sent from T_i to T_j on frequency f through a function $C_{ij}^f(\mathbf{h}(t), \mathbf{p}(t))$ – see Section 5. Independently of the particular form of $C_{ij}^f(\mathbf{h}(t), \mathbf{p}(t))$, total *offered* capacity from T_i to T_j until time t is $\sum_{u=1}^t \sum_{f\in\mathcal{F}} C_{ij}^f[\mathbf{h}(u), \mathbf{p}(u)]$. To avoid accumulation of packets at the physical layer, the latter has to exceed total *used* capacity $\sum_{u=1}^t c_{ij}(u)$ for what it suffices to enforce

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} c_{ij}(u) \leq \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t}\left[\sum_{f\in\mathcal{F}} C_{ij}^f[\mathbf{h}(u), \mathbf{p}(u)]\right]. \quad (4)$$

Functions $C_{ij}^f(\mathbf{h}(t), \mathbf{p}(t))$ in (4) are assumed to be finite for finite argument. This requirement is lax enough to allow, e.g., discontinuous functions $C_{ij}^f[\mathbf{h}(t), \mathbf{p}(t)]$.

Powers $p_{ij}^f(t)$ allocated at time t draw against T_i's power budget. The total power consumed by T_i at time t is the sum of the powers used to communicate with each neighbor on each frequency band, i.e., $p_i(t) = \sum_{f\in\mathcal{F}}\sum_{j\in n(i)} p_{ij}^f(t)$. The ergodic version of this equality is

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} p_i(u) = \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t}\left[\sum_{f\in\mathcal{F}}\sum_{j\in n(i)} p_{ij}^f(u)\right]. \quad (5)$$

The ergodic limits in the right hand sides of (4) and (5) can be replaced by expected values. For doing that let $m_{\mathbf{h}}(\mathbf{h})$ be the pdf of $\mathbf{h}$. With each channel realization $\mathbf{h}$ associate a power allocation measure $m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})]$. When channel $\mathbf{h}(t)$ is observed at time t, power allocation $\mathbf{p}(t)$ is randomly drawn from $m_{\mathbf{p}(\mathbf{h}(t))}[\mathbf{p}(\mathbf{h}(t))]$. Then, link capacities and power inequalities in (4) and (5) can be written as

$$c_{ij} \leq \mathbb{E}\left[\sum_{f\in\mathcal{F}} C_{ij}^f[\mathbf{h}, \mathbf{p}(\mathbf{h})]\right], \quad p_i = \mathbb{E}\left[\sum_{f\in\mathcal{F}}\sum_{j\in n(i)} p_{ij}^f(\mathbf{h})\right], \quad (6)$$

were we defined the ergodic limit for the power consumed at T_i as $p_i := \lim_{t\to\infty}(1/t)\sum_{u=1}^t p_i(u)$. The expected value operators $\mathbb{E}[\cdot]$ in (6) average over the pdf $m_{\mathbf{h}}(\mathbf{h})$ of the channel $\mathbf{h}$ and the pdf $m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})]$ of the power allocation $\mathbf{p}(\mathbf{h})$.

Define now a vector $\mathbf{x}(t)$ grouping variables $a_i^k(t)$, $r_{ij}^k(t)$, $c_{ij}(t)$ and $p_i(t)$. Define also a set $\mathcal{X}$ such that $a_{\min} \leq a_i^k \leq a_{\max}$, $r_{\min} \leq r_{ij}^k \leq r_{\max}$, $c_{\min} \leq c_{ij} \leq c_{\max}$ and $p_{\min} \leq p_i \leq p_{\max}$ and require $\mathbf{x}(t) \in \mathcal{X}$. As a consequence of the latter the ergodic limit $\mathbf{x} := \lim_{t\to\infty}(1/t)\sum_{u=1}^t \mathbf{x}(u)$ is also constrained to $\mathbf{x} \in \mathcal{X}$. Similarly, define spectral masks $\mathcal{P}(\mathbf{h}) \subseteq \{p_{ij}^f(\mathbf{h}) : 0 \leq p_{ij}^f(\mathbf{h}) \leq p_{\text{mask}}\}$ and require $\mathbf{p}(t) \in \mathcal{P}(\mathbf{h}(t))$. The power spectral mask constraints the power allocations pdfs $m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})]$ to those taking values on $\mathcal{P}(\mathbf{h})$, i.e., $m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})] : \mathbf{p}(\mathbf{h}) \in \mathcal{P}(\mathbf{h})$. Sets $\mathcal{P}(\mathbf{h})$ are compact but not necessarily convex.

Any set of ergodic limits a_i^k, r_{ij}^k, c_{ij} and p_i and probability measures $m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})]$ that satisfy the constraints in (3) and (6) is feasible. Among all these feasible operating points our goal is to select one that is optimal is some sense as described next.

2.1. Optimal operating point

Introduce convex utility $U_i^k(a_i^k)$ to measure the value of average transmission rate a_i^k and concave cost $V_i(p_i)$ measuring the cost of average power consumption p_i. The optimal wireless network is defined as the solution of the optimization problem [cf. (3) and (6)]

$$\begin{aligned}
\max \quad & \sum_{i,k} U_i^k(a_i^k) - \sum_i V_i(p_i) \qquad (7)\\
\text{s.t.} \quad & a_i^k \leq \sum_{j\in n(i)} r_{ij}^k - r_{ji}^k, \ \sum_k r_{ij}^k \leq c_{ij},\\
& p_i \geq \mathbb{E}\left[\sum_{f\in\mathcal{F}}\sum_{j\in n(i)} p_{ij}^f(\mathbf{h})\right], \ c_{ij} \leq \mathbb{E}\left[\sum_{f\in\mathcal{F}} C_{ij}^f[\mathbf{h}, \mathbf{p}(\mathbf{h})]\right],\\
& \mathbf{x} \in \mathcal{X}, \ \Big\{m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})] : \mathbf{p}(\mathbf{h}) \in \mathcal{P}(\mathbf{h})\Big\}_{\mathbf{h}}
\end{aligned}$$

where we relaxed the power constraint in (6), which can be done without loss of optimality. The optimization variables in (7) are the probability measures $m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})]$ and the ergodic limits $\mathbf{x}$, which groups a_i^k, r_{ij}^k, c_{ij} and p_i. The fading's pdf $m_{\mathbf{h}}(\mathbf{h})$, however, is fixed.

The goal of this paper is to develop methods to solve the optimization problem in (7) to determine optimal operating points of wireless networks. The three challenges that need to be overcome are: (i) The optimization problem is not convex because $C_{ij}^f[\mathbf{h}, \mathbf{p}(\mathbf{h})]$ is not concave with respect to $\mathbf{p}(\mathbf{h})$. (ii) For each channel realization $\mathbf{h}$, a pdf $m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})]$ needs to be determined. Therefore, the number of problem variables is infinite. (iii) Optimal operating points depend on the channel's pdf $m_{\mathbf{h}}(\mathbf{h})$. This pdf is not known, rather, the fading's probability distribution is learnt online from channel observations $\mathbf{h}(t)$. An algorithm addressing issues (i)-(iii) is introduced in the next section.

3. STOCHASTIC DUAL SUBGRADIENT DESCENT

To simplify upcoming discussions rewrite (7) in generic form as

$$\begin{aligned} P = \max \ & f_0(\mathbf{x}) \\ \text{s.t. } & \mathbf{x} \leq \mathbb{E}\left[\mathbf{f}_1(\mathbf{p}(\mathbf{h}); \mathbf{h})\right], \ \mathbf{f}_2(\mathbf{x}) \geq \mathbf{0}, \\ & \mathbf{x} \in \mathcal{X}, \ \left\{m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})] : \mathbf{p}(\mathbf{h}) \in \mathcal{P}(\mathbf{h})\right\}_{\mathbf{h}}. \end{aligned} \tag{8}$$

Comparing (7) with (8) it follows that functions $f_0(\mathbf{x})$ and $\mathbf{f}_2(\mathbf{x})$ in (8) are concave with respect to their argument $\mathbf{x}$. The family of functions $\mathbf{f}_1[\mathbf{p}(\mathbf{h}); \mathbf{h}]$ is parameterized by the random state $\mathbf{h}$ and, different from $f_0(\mathbf{x})$ and $\mathbf{f}_2(\mathbf{x})$, is not necessarily concave with respect to the resource allocation $\mathbf{p}(\mathbf{h})$. The set $\mathcal{X}$ to which the ergodic limits $\mathbf{x}$ are constrained is compact and convex, while the set $\mathcal{P}(\mathbf{h})$ constraining resource allocation *values* $\mathbf{p}(\mathbf{h})$ is compact but not necessarily convex. Recall that the set $\mathcal{P}(\mathbf{h})$ constrains the resource allocation $\mathbf{p}(\mathbf{h})$ on a per-state basis, i.e., there exists a set $\mathcal{P}(\mathbf{h})$ for each random state realization $\mathbf{h}$.

Observe that there are an infinite number of variables in the primal domain but a finite number of inequality constraints. Thus, the dual problem contains a finite number of variables hinting that the problem is likely more tractable in the dual space. Define then dual variables $\boldsymbol{\lambda}_1 \geq \mathbf{0}$ associated with the constraint $\mathbf{x} \leq \mathbb{E}\left[\mathbf{f}_1[\mathbf{p}(\mathbf{h}); \mathbf{h}]\right]$ and $\boldsymbol{\lambda}_2 \geq \mathbf{0}$ associated with $\mathbf{f}_2(\mathbf{x}) \geq \mathbf{0}$. Using these definitions the Lagrangian for the optimization problem in (8) is written as

$$\begin{aligned} &\mathcal{L}[\boldsymbol{\lambda}, \mathbf{x}, \mathbf{p}(\mathbf{h})] \qquad (9) \\ &= f_0(\mathbf{x}) + \boldsymbol{\lambda}_1^T \Big[\mathbb{E}\left[\mathbf{f}_1[\mathbf{p}(\mathbf{h}); \mathbf{h}]\right] - \mathbf{x}\Big] + \boldsymbol{\lambda}_2^T \mathbf{f}_2(\mathbf{x}) \\ &= f_0(\mathbf{x}) - \boldsymbol{\lambda}_1^T \mathbf{x} + \boldsymbol{\lambda}_2^T \mathbf{f}_2(\mathbf{x}) + \mathbb{E}_{\mathbf{h}}\Big[\mathbb{E}_{\mathbf{p}(\mathbf{h})}\Big(\boldsymbol{\lambda}_1^T \mathbf{f}_1[\mathbf{p}(\mathbf{h}); \mathbf{h}]\Big)\Big] \end{aligned}$$

where we defined the aggregate dual variable $\boldsymbol{\lambda} := [\boldsymbol{\lambda}_1^T, \boldsymbol{\lambda}_2^T]^T$. To obtain the second equality we wrote $\mathbb{E}(\cdot) = \mathbb{E}_{\mathbf{h}}\left[\mathbb{E}_{\mathbf{p}(\mathbf{h})}(\cdot)\right]$ and reordered terms. The dual function is then defined as the maximum of the Lagrangian over the set of feasible ergodic limits $\mathbf{x} \in \mathcal{X}$ and probability distributions $m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})]$ in the set of feasible powers $\mathbf{p}(\mathbf{h}) \in \mathcal{P}(\mathbf{h})$,

$$\begin{aligned} g(\boldsymbol{\lambda}) := \max \ & \mathcal{L}[\boldsymbol{\lambda}, \mathbf{x}, \mathbf{p}(\mathbf{h})] \\ \text{s.t.} \ & \mathbf{x} \in \mathcal{X}, \ \left\{m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})] : \mathbf{p}(\mathbf{h}) \in \mathcal{P}(\mathbf{h})\right\}_{\mathbf{h}} \end{aligned} \tag{10}$$

Introduce now a discrete time index t and consider the channel stochastic process $\mathbf{h}(\mathbb{N})$ with realizations $\mathbf{h}(t)$ identically and independently distributed (i.i.d.) according to $m_{\mathbf{h}}(\mathbf{h})$. The stochastic subgradient descent algorithm on the dual function starts with given multipliers $\boldsymbol{\lambda}(t)$ to find feasible variables $\mathbf{x}(t) \in \mathcal{X}$ and $\mathbf{p}(t) \in \mathcal{P}(\mathbf{h}(t))$ such that

$$\mathbf{x}(t) = \mathbf{x}(\boldsymbol{\lambda}(t)) = \operatorname*{argmax}_{\mathbf{x} \in \mathcal{X}} f_0(\mathbf{x}) - \boldsymbol{\lambda}_1^T(t)\mathbf{x} + \boldsymbol{\lambda}_2^T(t)\mathbf{f}_2(\mathbf{x}), \tag{11}$$

$$\mathbf{p}(t) = \mathbf{p}(\mathbf{h}(t), \boldsymbol{\lambda}(t)) = \operatorname*{argmax}_{\mathbf{p}(\mathbf{h}(t)) \in \mathcal{P}(\mathbf{h}(t))} \boldsymbol{\lambda}_1^T(t)\mathbf{f}_1[\mathbf{p}(\mathbf{h}(t)); \mathbf{h}(t)]. \tag{12}$$

In (11), $\mathbf{x}(t)$ maximizes the part of $\mathcal{L}[\boldsymbol{\lambda}(t), \mathbf{x}, \mathbf{p}(\mathbf{h})]$ that depends on $\mathbf{x}$. But in (12) the maximization is with respect to power allocations $\mathbf{p}(\mathbf{h}(t)) \in \mathcal{P}(\mathbf{h}(t))$, not pdfs $m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})]$ as in (10). Also, no maximization is attempted with respect to $\mathbf{p}(\mathbf{h})$ for any $\mathbf{h} \neq \mathbf{h}(t)$.

Based on $\mathbf{x}(t)$ and $\mathbf{p}(t)$ we define the dual function stochastic subgradient $\hat{\mathbf{s}}(t) = \hat{\mathbf{s}}(\mathbf{h}(t), \boldsymbol{\lambda}(t)) = [\hat{\mathbf{s}}_1^T(t), \hat{\mathbf{s}}_2^T(t)]^T$ with components

$$\hat{\mathbf{s}}_1(t) := \mathbf{f}_1[\mathbf{p}(t); \mathbf{h}(t)] - \mathbf{x}(t), \quad \hat{\mathbf{s}}_2(t) := \mathbf{f}_2(\mathbf{x}(t)). \tag{13}$$

The algorithm's iteration is completed by an update in the dual domain moderated by a predetermined step size ϵ along the direction $-\hat{\mathbf{s}}(t)$

$$\boldsymbol{\lambda}(t+1) = \left[\begin{array}{l} \boldsymbol{\lambda}_1(t) - \epsilon\Big(\mathbf{f}_1[\mathbf{p}(t); \mathbf{h}(t)] - \mathbf{x}(t)\Big) \\ \boldsymbol{\lambda}_2(t) - \epsilon\mathbf{f}_2(\mathbf{x}) \end{array}\right]^+ \tag{14}$$

where the operator $[\cdot]^+$ denotes projection in the positive orthant. The stochastic subgradient descent algorithm on the dual function consists of iterative application of (11)-(14).

The definition in (13) is similar to the expression for dual functions' subgradients. In fact, it is not difficult to prove that the expected value of $\hat{\mathbf{s}}(t)$ is a subgradient of the dual function. Since ergodic averages of primal variables obtained from deterministic dual subgradient algorithms converge to a near optimal operating point [7], it is not unreasonable to expect that this property will be retained by its stochastic counterpart. As it turns out this is not easy to prove because the algorithm descends in the dual domain while convergence is sought in the primal domain. It is nonetheless true as stated in the following theorem [9].

Theorem 1 *Consider the optimization in* (8) *and sequences* $\mathbf{x}(\mathbb{N})$ *and* $\mathbf{p}(\mathbb{N})$ *generated by* (11)-(14). *Let* $\hat{S}^2 \geq \mathbb{E}\left[\|\hat{\mathbf{s}}(t)\|^2 \,\middle|\, \boldsymbol{\lambda}(t)\right]$ *be a bound on the second moment of the norm of the stochastic subgradients and assume that there exists strictly feasible* $\mathbf{x}_0 \in \mathcal{X}$ *and* $\mathbf{p}_0(\mathbf{h})$ *such that* $\mathbb{E}[\mathbf{f}_1(\mathbf{p}_0(\mathbf{h}); \mathbf{h})] - \mathbf{x}_0 > 0$ *and* $\mathbf{f}_2(\mathbf{x}_0) > 0$. *Then:*

(i) **Almost sure feasibility.** *Sequences* $\mathbf{x}(\mathbb{N})$ *and* $\mathbf{p}(\mathbb{N})$ *are feasible with probability 1, i.e.,*

$$\lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{t} \mathbf{x}(u) \leq \lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{t} \mathbf{f}_1[\mathbf{p}(u); \mathbf{h}(u)] \qquad \text{a.s.,} \tag{15}$$

$$\mathbf{f}_2\left[\lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{t} \mathbf{x}(u)\right] \geq \mathbf{0} \qquad \text{a.s.} \tag{16}$$

(ii) **Almost sure near optimality.** *The ergodic average of* $\mathbf{x}(\mathbb{N})$ *almost surely converges to a value with optimality gap smaller than* $\epsilon\hat{S}^2/2$*, i.e,*

$$P - f_0\left[\lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{t} \mathbf{x}(u)\right] \leq \frac{\epsilon\hat{S}^2}{2} \qquad \text{a.s..} \tag{17}$$

The ergodic limit $\mathbf{x} := (1/t)\sum_{u=1}^{t} \mathbf{x}(u)$ satisfies the constraints in (8) and the objective function evaluated at $\mathbf{x}$ is within $\epsilon\hat{S}^2/2$ of optimal. Since $\mathcal{X}$ and $\mathcal{P}(\mathbf{h})$ are compact sets it follows that the bound $\hat{S}^2$ is finite. Therefore, reducing ϵ it is possible to make $\mathbf{f}_0(\mathbf{x})$ arbitrarily close to P and as a consequence $\mathbf{x}$ is an arbitrarily good approximation of an optimal $\mathbf{x}^*$. The optimal resource allocation $\mathbf{p}^*(\mathbf{h})$, however, is not computed by the algorithm. Rather, (15) implies that asymptotically the algorithm is drawing power allocation realizations $\mathbf{p}(t)$ from power allocation distributions $m_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})]$ that are close to optimal $m^*_{\mathbf{p}(\mathbf{h})}[\mathbf{p}(\mathbf{h})]$. This is not a drawback in practice because realizations $\mathbf{p}(t)$ are sufficient for implementation. In that sense, (11)-(14) yields an optimal power allocation policy, i.e., allocate $\mathbf{p}(t)$ units at time t, that supports optimal network variables $\mathbf{x}$ in an ergodic sense.

4. STOCHASTIC LEARNING ALGORITHM

The SLA is obtained by writing (11), (12) and (14) in explicit form. Introduce dual variables ν_i^k, ξ_{ij}, μ_i and λ_{ij} respectively associated with the flow conservation, rate, power and link capacity constraints in (7). After reordering terms, the Lagrangian in (9) can then be written as

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \mathbf{p}(\mathbf{h}), \boldsymbol{\lambda}) = & \sum_{i,k} U_i^k(a_i^k) - \nu_i^k a_i^k + \sum_{i,j,k} r_{ij}^k \left(\nu_i^k - \nu_j^k - \xi_{ij}\right) \\ & + \sum_{ij} c_{ij}(\xi_{ij} - \lambda_{ij}) + \sum_i \mu_i p_i - V_i(p_i) \\ & + \sum_{f \in \mathcal{F}} \mathbb{E}\left[\sum_{i,j} \lambda_{ij} C_{ij}^f\left[\mathbf{h}, \mathbf{p}(\mathbf{h})\right] - \mu_i p_{ij}^f(\mathbf{h})\right] \end{aligned} \tag{18}$$

The maximization of the Lagrangian in (18) can be decomposed into separate maximizations with respect to the power allocation $\mathbf{p}(h)$ and the primal variables a_i^k, r_{ij}^k, c_{ij} and p_i as indicated in (11)-(12). The expression in (18) further uncovers that primal variables a_i^k, r_{ij}^k, c_{ij} and p_i appear in only one summand and that vectors $\mathbf{p}^f(\mathbf{h})$ appear in different

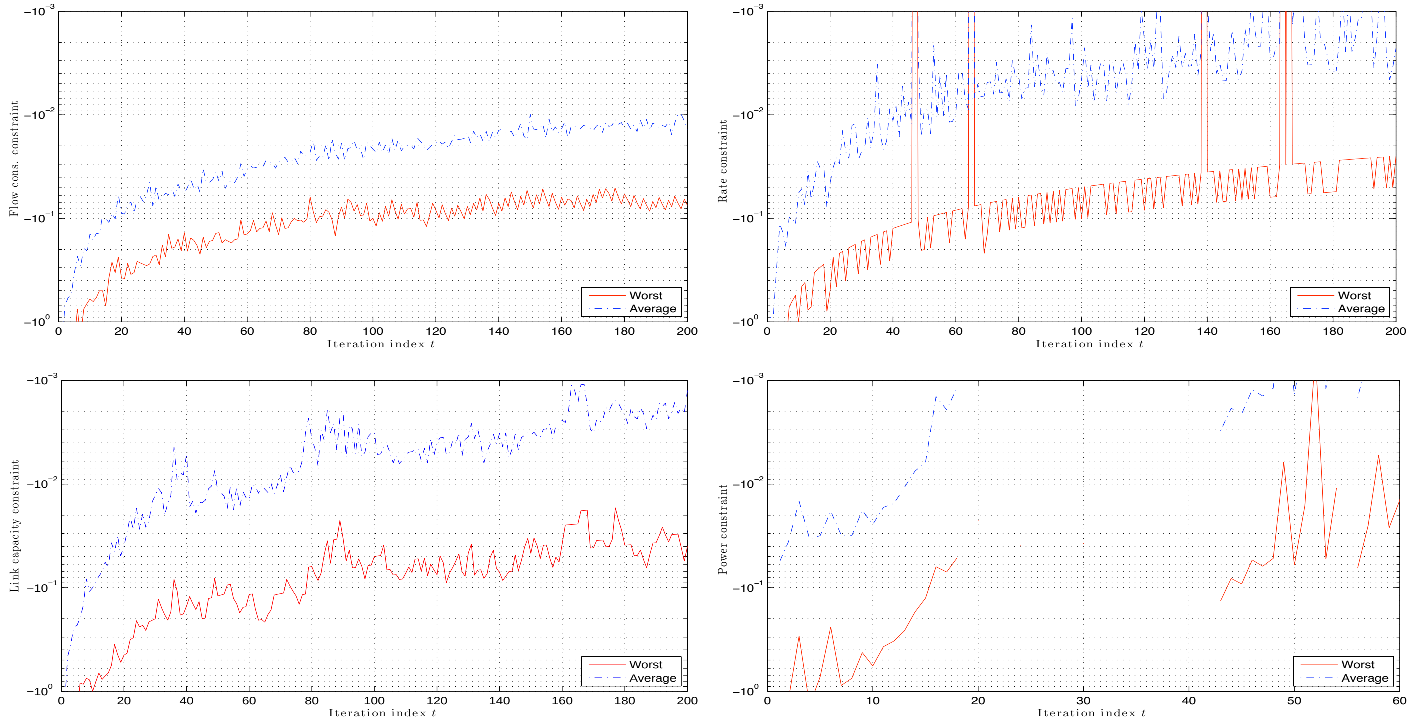

Fig. 2. Feasibility of Stochastic learning algorithm (SLA) iterates. It takes about 100 iterations for all constraints to be satisfied with a gap smaller than 10^{-1}. The constraint with slowest convergence is flow conservation. This is expected because determination of routing variables r_{ij}^k and admissible rates a_i^k implicitly necessitates propagation of information from the destination to the sources.

summands of $\mathcal{L}(\mathbf{x}, \mathbf{p}(\mathbf{h}), \boldsymbol{\lambda})$. It is then possible to further decompose the maximization of $\mathcal{L}(\mathbf{x}, \mathbf{p}(\mathbf{h}), \boldsymbol{\lambda})$ into separate maximizations with respect to a_i^k, r_{ij}^k, c_{ij}, p_i and $\mathbf{p}^f(\mathbf{h})$. These separate maximizations constitute the primal iteration of SLA

$$
\begin{aligned}
a_i^k(t) &= \operatorname*{argmax}_{a_i^k \in [a_{\min}, a_{\max}]} \quad U_i^k(a_i^k) - \nu_i^k(t) a_i^k \\
r_{ij}^k(t) &= \operatorname*{argmax}_{r_{ij}^k \in [r_{\min}, r_{\max}]} \quad \left(\nu_i^k(t) - \nu_j^k(t) - \xi_{ij}(t)\right) r_{ij}^k \\
c_{ij}(t) &= \operatorname*{argmax}_{c_{ij} \in [c_{\min}, c_{\max}]} \quad (\xi_{ij}(t) - \lambda_{ij}(t))\, c_{ij} \\
p_i(t) &= \operatorname*{argmax}_{p_i \in [p_{\min}, p_{\max}]} \quad \mu_i(t) p_i - V_i(p_i) \\
\mathbf{p}^f(t) &= \operatorname*{argmax}_{\mathbf{p}^f \in \mathcal{P}^f} \sum_{i,j} \lambda_{ij}(t) C_{ij}^f \left[\mathbf{h}^f(t), \mathbf{p}^f\right] - \mu_i(t) p_{ij}^f. \qquad (19)
\end{aligned}
$$

The dual iteration requires computation of the stochastic subgradient as per (13) and a descent in the dual domain as per (14). These two steps are combined to yield the expressions

$$
\begin{aligned}
\nu_i^k(t+1) &= \left[\nu_i^k(t) - \epsilon\left[\sum_{j \in n(i)} \left(r_{ij}^k(t) - r_{ji}^k(t)\right) - a_i^k(t)\right]\right]^+ \\
\xi_{ij}(t+1) &= \left[\xi_{ij}(t) - \epsilon\left[c_{ij}(t) - \sum_k r_{ij}^k(t)\right]\right]^+ \\
\mu_i(t+1) &= \left[\mu_i(t) - \epsilon\left[p_i(t) - \sum_{f \in \mathcal{F}} \sum_{j \in n(i)} p_{ij}^f(t)\right]\right]^+ \\
\lambda_{ij}(t+1) &= \left[\lambda_{ij}(t) - \epsilon\left[\sum_{f \in \mathcal{F}} C_{ij}^f\left[\mathbf{h}(t), \mathbf{p}(t)\right] - c_{ij}(t)\right]\right]^+ \qquad (20)
\end{aligned}
$$

As per Result (i) of Theorem 1 variables $a_i^k(t)$, $r_{ij}^k(t)$, $c_{ij}(t)$ and $p_i(t)$ combined with power allocation $\mathbf{p}(t)$ almost surely satisfy the limit inequalities in (1), (2), (4), and (5). According to Result (ii) we further have that, with probability 1,

$$
\lim_{t\to\infty} \sum_{i,k} U_i^k\left[\frac{1}{t}\sum_{u=1}^{t} a_i^k(u)\right] - \sum_i V_i\left[\frac{1}{t}\sum_{u=1}^{t} p_i(u)\right] \geq P - \frac{\epsilon \hat{S}^2}{2}. \qquad (21)
$$

The utility can be made arbitrarily close to the optimal P by reducing the step size ϵ.

Remark 1 The SLA algorithm with iterations (19) and (20) is similar but different from the stochastic network optimization algorithms of [3,4]. SLA and stochastic network optimization deal differently with non-convex constraints. In stochastic network optimization, the non-convex constraints are eliminated through the introduction of a capacity region and are left implicit in the definition of the dual function. In SLA, the non-convex constraints are incorporated in the definition of the dual function. Convergence properties of SLA and stochastic network optimization differ too. Results in stochastic network optimization establish that the expected value of ergodic limits satisfy problem constraints with a small gap. The feasibility result for SLA iterates is that constraints are almost surely and exactly satisfied by ergodic limits [cf. (15) and (16)]. Both guarantees, i.e., almost sure convergence and exact satisfaction of constraints, are important in practice. Near optimality in [3,4] pertains also to the ergodic mean, i.e., the expected value of the time average of iterates yields a utility that is close to optimal. In SLA, ergodic averages almost surely convergence to a near optimal point [cf. (17)]. This is also an important guarantee in applications. A more technical difference is the restriction to fading models with a finite number of states in [3,4] that is not required here.

5. NUMERICAL RESULTS

SLA of (19)-(20) is implemented to find optimal operating variables for the network in Fig. 1 using adaptive modulation and coding (AMC) over an interference limited physical layer. The metric determining link

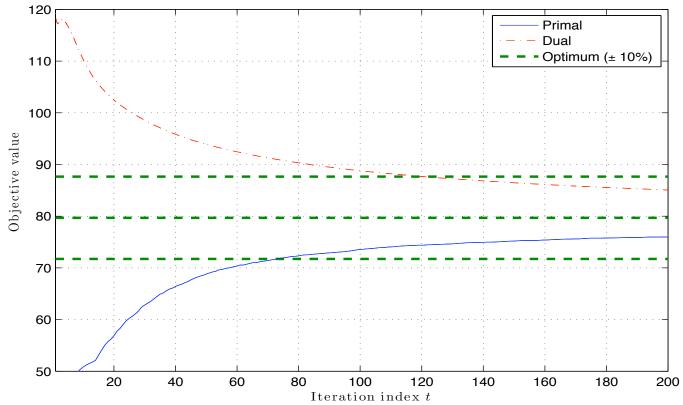

Fig. 3. Convergence of primal and dual objective to optimal value. Notice that primal and dual values approach each other and that to achieve an optimality gap smaller than 10% takes about 70 iterations.

qualities is the signal to interference plus noise ratio (SINR) given by

$$\text{SINR}_{ij}^f(t) = \frac{h_{ij}^f(t)p_{ij}^f(t)}{N_j^f + (1/S)\sum_{(m,n)\neq(i,j)} h_{mj}^f(t)p_{mn}^f(t)} \tag{22}$$

where N_j^f denotes the noise power, S the spreading gain and the summation index $(m,n) \neq (i,j)$ signifies sum aver all m and n except when both $m = i$ and $n = j$. With AMC, the map from $\text{SINR}_{ij}^f(t)$ in (22) to link capacities is of the form

$$C_{ij}^f(\mathbf{h}(t), \mathbf{p}(t)) = \sum_{l=1}^{L} \alpha_l \mathbb{I}\left(\beta_l \leq \text{SINR}_{ij}^f(t) \leq \beta_{l+1}\right) \tag{23}$$

where $\mathbb{I}(\cdot)$ denotes the indicator function, β_l transition powers and α_l the rate associated with mode l. According to (23), α_l units of information are transmitted when $\text{SINR}_{ij}^f(t)$ is between β_l and β_{l+1}. As it also happens for most wireless physical layers, $C_{ij}^f(\mathbf{h}(t), \mathbf{p}(t))$ is not a convex function of the power allocation $\mathbf{p}(t)$ – indeed $C_{ij}^f(\mathbf{h}(t), \mathbf{p}(t))$ is even discontinuous due to transitions between AMC modes.

Henceforth, nodes operate on 5 frequency bands with spreading gain $S = 16$. Three AMC modes yielding rates 1, 2 and 3 bits/s/Hz are used with transitions at SINR 1, 3 and 7. Fading channels are generated as i.i.d. Rayleigh with average powers $1/2$ for the links $4 \leftrightarrow 7$, $5 \leftrightarrow 9$, $7 \leftrightarrow 11$, $9 \leftrightarrow 10$, $11 \leftrightarrow 8$, $10 \leftrightarrow 6$, $8 \leftrightarrow 4$ and $6 \leftrightarrow 5$ and 1 for the remaining links. Noise power is $N_i^f = 0.1$ for all terminals and frequency bands. The maximum average power consumption per terminal is $p_{\max} = 2$ – chosen so that if a terminal with 4 neighbors spreads power uniformly across all neighbors and frequencies the signal to noise ratio is 0dB. Powers p_i are also constrained to be positive, i.e., $p_{\min} = 0$. A spectral mask $\mathcal{P}(\mathbf{h}) := \{p_{ij}^f(\mathbf{h}) : 0 \leq p_{ij}^f(\mathbf{h}) \leq p_{\text{mask}}\}$ is further defined with $p_{\max} = 2$ the same value used to limit average power consumption. Link capacities and routing variables are constrained by $c_{\min} = r_{\min} = 0$ bits/s/Hz and $c_{\max} = r_{\max} = 6$ bits/s/Hz. Four flows with destination at terminals 1, 7, 8 and 14 are considered with all other terminals required to deliver at least $a_{\min} = 0.5$ bits/s/Hz and at most $a_{\max} = 2$ bits/s/Hz to each of these flows. The optimality criteria is sum rate, i.e., $U_i^k(a_i^k) = a_i^k$ for all i, k. Powers p_i are absent from the objective, i.e., $V_i(p_i) = 0$ for all i. Steps size is $\epsilon = 10^{-2}$.

Convergence of the algorithm is corroborated by Figs. 2 and 3. As guaranteed by Theorem 1 optimality and feasibility are indeed achieved. Feasibility of time averages is tested for the constraints in (1), (2), (4), and (5). The average and worst case violation for each type of constraint are shown in Fig. 2. It takes about 100 iterations for all constraints to be satisfied with a gap smaller than 10^{-1}. Blanks in the plot of power constraint violations $P_i(t)$ correspond to times at which all constraints were satisfied, i.e., $P_i(t) \geq 0$, for all i. The constraint with slowest convergence is the flow conservation inequality in (1). This is because determination of routing variables r_{ij}^k and admissible rates a_i^k implicitly necessitates propagation of information from destinations to sources.

To test optimality we compute ergodic primal and dual objectives as a function of time. Since the objective is sum rate maximization of admission control variables, the ergodic primal objective is $P(t) := (1/t)\sum_{u=1}^{t}\sum_{i,k} a_i^k(u)$. Ergodic primal and dual objectives are shown in Fig. 3. It is seen that they approach each other and that to achieve an optimality gap smaller than 10% takes about 70 iterations. The optimal value in Fig. 3 is found by running SLA until the gap between primal and dual values is less than 1%.

6. CONLCUSIONS

Operation of wireless networks necessitates determination of admission control rates, routes, link capacities, average powers and power allocations across fading states. We proposed a stochastic learning algorithm (SLA) to determine optimal operating points. Defining optimal network operation in an ergodic sense we showed that SLA almost surely leads to feasible operating points while guaranteeing arbitrarily close to optimal performance. This holds true even though: (i) the optimization problem associated with optimal network design is not convex; (ii) power allocation functions across fading states leads to a variational problem with an infinite number of variables; and (iii) the probability distribution of fading is learnt from online observations.

Lack of convexity and large dimensionality imply a prohibitive computational cost. Although the proposed SLA does not completely eliminate high computational cost it does afford a significant reduction. In practice, SLA reduces complexity to the solution of an instantaneous power allocation problem at the physical layer. These problems are not convex but have manageable computational cost in small and medium-sized networks. The use of locally optimal and heuristic power allocations in large networks is a future research direction. In its current state SLA requires centralized computation and availability of global channel state information. Further research is necessary to lift these restrictions.

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