Dithering and Betweenness Centrality in Weighted Graphs

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***Abstract*—This paper applies dithering to design a node centrality measure for weighted graphs. The construction is an improvement on the stable betweenness centrality measure which, in turn, was introduced as a robust alternative to the well-known betweenness centrality. We interpret any given graph as the mean representation of a distribution of graphs and define the dithered centrality value as the expected centrality value across all graphs in the distribution. We show that the dithered stable betweenness centrality measure preserves robustness in the presence of noise while improving the behavior of stable betweenness. Numerical experiments demonstrate the advantages of dithering by comparing the performance of betweenness, stable betweenness and dithered stable betweenness centralities in terms of robustness to noise, dependence on the number and quality of alternative paths, and distribution of centrality values across the graph.**

***Index Terms*—Networks, graphs, centrality, betweenness, dithering.**

I. Introduction

The topology of a graph or network imposes an influence structure over its nodes. Central nodes have major impact in the flow of communication and the evolution of network dynamics, e.g. the distribution of power in exchange networks [1], whereas peripheral nodes have limited effect. Node centrality measures are tools designed to identify such influential agents. Although several centrality measures can be found in the literature, the most common being degree [2], [3], closeness [4], [5], stress [6], eigenvector [7], and betweenness [8] centrality, the latter has been extensively used in the study of both technological [9] and social [10] networks. However, betweenness centrality was shown to be unstable, i.e. sensitive to noisy data, and a stable alternative measure was proposed [11], [12]. In this paper we apply dithering to improve the stable betweenness centrality measure.

Dithering is a common technique used to reduce quantization error in digital signal processing by adding random noise to the signal [13], e.g. in digital image [14] and digital audio processing [15]. In our case, we consider the input graph to be a signal and, by adding zero-mean random noise to its weights, we obtain a random graph picked from a distribution of which the original signal graph is the mean. Instead of computing centrality directly on the mean graph, our dithering method consists in obtaining numerous random perturbations of the graph, computing centrality in each of them independently and then averaging the centrality results. Our first contribution is the formalization of dithering in the context of graphs (Section IV). We then define the dithered stable betweenness centrality and show that dithering preserves the stability property while solving undesirable behaviors of the betweenness centrality and its stable alternative (Section IV-A). Finally, through numerical experiments we illustrate the advantages of dithering in terms of robustness to noise (Section V-A), dependence on the number and quality of paths (Section V-B), and distribution of centrality values across the graph (Section V-C).

II. Preliminaries

We define a directed graph or network $G = (V, E, W)$ as a triplet formed by a finite set of n nodes or vertices V, a set of directed edges $E \subset V \times V$ where $(x, y) \in E$ represents an edge from $x \in V$ to $y \in V$, and a set of positive weights $W : E \to \mathbb{R}_{++}$ defined on each edge. The weights are associated to dissimilarities between nodes, i.e. the higher the weight the more dissimilar the nodes are. The graphs considered here do not contain self-loops, i.e., $(x, x) \notin E$ for all $x \in V$. For any given sets V and E, denote by $\mathcal{G}_{(V,E)}$ the space of all graphs with V as node set and E as edge set. This implies that two graphs $G, H \in \mathcal{G}_{(V,E)}$ can only differ in their weights.

Given a graph (V, E, W) and $x, x' \in V$, a path $P(x, x')$ which starts at x and finishes at x' is an *ordered* sequence of nodes, $P(x, x') = [x = x_0, x_1, \ldots, x_l = x']$, such that $e_i = (x_i, x_{i+1}) \in E$ for $i = 0, \ldots, l-1$. The length of a given path is the sum of the weights encountered when traversing its links in order. We define the shortest path function $s_G : V \times V \to \overline{\mathbb{R}}_+$ where the shortest path length $s_G(x, x')$ between nodes $x, x' \in V$ is

$$s_G(x, x') := \min_{P(x,x')} \sum_{i=0}^{l-1} W(x_i, x_{i+1}). \tag{1}$$

Given three arbitrary nodes $x, x', x'' \in V$, denote by $\sigma_{x'x''}$ the number of shortest paths from x' to x'', i.e. the number of paths $P(x', x'')$ of length $s_G(x', x'')$, and by $\sigma_{x'x''}(x)$ the number of these shortest paths that go through node x. The betweenness centrality $C_B(x)$ for any given node $x \in V$ is defined as [8]

$$C_B(x) := \sum_{\substack{x', x'' \in V \\ x' \neq x \neq x''}} \frac{\sigma_{x'x''}(x)}{\sigma_{x'x''}}. \tag{2}$$

In (2), we calculate the betweenness centrality value of a node $x \in X$ by sequentially looking at the shortest paths between any two nodes distinct from x and summing the proportion of shortest paths that contain node x. The higher the centrality value $C_B(x)$, the more central node x is.

III. Centrality and Stability

Node centrality is a measure of the importance of a node within a graph. This importance is based on the location of the node within the graph and not on the intrinsic nature of this node. More precisely, given a graph (V, E, W), a centrality measure $C : V \to \mathbb{R}_+$ assigns a nonnegative centrality value to every node such that the higher the value the more central the node is. Ideally, this detection should be invariant to small perturbations in the edge weights.

To formalize this notion of robustness against perturbations, we define the metric $d_{(V,E)} : \mathcal{G}_{(V,E)} \times \mathcal{G}_{(V,E)} \to \mathbb{R}_+$ on the space $\mathcal{G}_{(V,E)}$

$$d_{(V,E)}(G, H) := \sum_{e \in E} |W(e) - W'(e)|, \tag{3}$$

where $G = (V, E, W)$ and $H = (V, E, W')$. The metric $d_{(V,E)}$ enables the formal definition of stability presented next.

Definition 1 ([12]) *A centrality measure C is* stable *if, for every vertex set V, edge set E and any two graphs $G, H \in \mathcal{G}_{(V,E)}$,*

$$\left| C^G(x) - C^H(x) \right| \leq K_G \, d_{(V,E)}(G, H), \tag{4}$$

for every $x \in V$, where K_G is a constant for every graph G, $C^G(x)$ is the centrality value of node x in graph G and similarly for H.

The above definition states that a centrality measure is stable if the difference in centrality values for a given node in two different graphs is bounded by a constant K_G times the distance between these graphs. In particular, if graph H is a perturbed version of G, a stable centrality measure ensures that the change in centrality due to this perturbation is bounded; see Section V-A. Despite its extensive use, the betweenness centrality measure is not stable.

Work supported by NSF CCF-1217963.

978-1-4799-7088-9/14/$31.00

Proposition 1 ([11], [12]) *The betweenness centrality measure C_B in* (2) *is not stable in the sense of Definition 1.*

The instability of the betweenness centrality measure motivated the definition of the alternative stable betweenness centrality C_{SB}. Given an arbitrary graph $G = (V, E, W)$ and a node $x \in V$, define a new graph $G^x = (V^x, E^x, W^x)$ with $V^x = V \backslash \{x\}$, $E^x = E \backslash \{(x', x'') \,|\, x' = x \text{ or } x'' = x\}$, and $W^x = W|_{E^x}$. I.e., the graph G^x is constructed by deleting from G the node x and every edge directed to or from it. The stable betweenness centrality $C_{SB}(x)$ of any node $x \in V$ is given by [11], [12]

$$C_{SB}(x) := \sum_{\substack{x', x'' \in V \\ x' \neq x \neq x''}} s_{G^x}(x', x'') - s_G(x', x''). \tag{5}$$

Measure C_{SB} quantifies the centrality of a given node x by the change in the length of shortest paths once this node is removed. This means that the centrality of a node depends on the quality of the best alternative. In contrast to the traditional centrality measure, C_{SB} is stable.

Proposition 2 ([11]) *The stable betweenness centrality measure C_{SB} in* (5) *is stable as defined in Definition 1 with $K_G = 2\,n^2$.*

Stability is the main advantage of C_{SB} over C_B since it ensures robustness to noise of the former measure in contrast with the latter. However, the measure C_{SB} still inherits other undesirable properties from the traditional centrality measure C_B. In the following section we explain this in detail and propose another centrality measure that overcomes these limitations.

IV. Dithering

Given an arbitrary graph $G = (V, E, W)$, define the corresponding *dithered graph* $\boldsymbol{G} = (V, E, \boldsymbol{W})$ where the function $\boldsymbol{W}$ maps every edge $e \in E$ to an independent, real-valued, continuous random variable $\boldsymbol{W}(e) = \boldsymbol{e}$. In order for $\boldsymbol{W}$ to be valid, we must have that, for any two edges $a, e \in E$ and any real number η,

$$\mathbb{E}(\boldsymbol{e}) = W(e), \tag{6}$$

$$P(\boldsymbol{e} > 0) = 1, \tag{7}$$

$$W(e) \geq W(a) \quad \Rightarrow \quad P(\boldsymbol{e} \geq \eta) \geq P(\boldsymbol{a} \geq \eta). \tag{8}$$

The intuition is that any realization of $\boldsymbol{G}$ is a random perturbation of the original graph G. Requirement (6) ensures that the random variables representing the edge weights in $\boldsymbol{G}$ are unbiased with respect to the corresponding weights in the original graph G. Requirement (7) guarantees that with probability 1 every realization of $\boldsymbol{G}$ is a valid graph as defined in Section II. Finally, if we compare the probabilities that the weights of any two random edges in $\boldsymbol{G}$ are larger than a constant, requirement (8) assures that the random edge associated to the larger weight in the original network has a higher probability.

Example 1 (uniform δ-dithering) *A valid example of $\boldsymbol{W}$ satisfying requirements* (6)-(8) *is such that, for all $e \in E$,*

$$\boldsymbol{e} = W(e)(1 + \Delta)\,, \tag{9}$$

where Δ is uniformly distributed in $[-\delta, +\delta]$ for some $0 < \delta < 1$.

A. Dithered stable betweenness centrality

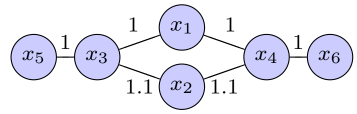
Given an arbitrary dithered graph $\boldsymbol{G} = (V, E, \boldsymbol{W})$, we may define the random variable $\boldsymbol{s_G}$ as the shortest path defined as in (1) but for random weights $\boldsymbol{W}(x_i, x_j)$ instead of deterministic ones $W(x_i, x_j)$. Using $\boldsymbol{s_G}$, we define the *dithered stable betweenness centrality* value of any node $x \in V$ as

$$C_{DSB}(x) := \mathbb{E}(\boldsymbol{C}_{SB}(x)) := \mathbb{E}\Bigg(\sum_{\substack{x', x'' \in X \\ x' \neq x \neq x''}} \boldsymbol{s}_{\boldsymbol{G}^x}(x', x'') - \boldsymbol{s_G}(x', x'') \Bigg). \tag{10}$$

Notice that C_{DSB} is defined as the expected value of the random variable $\boldsymbol{C}_{SB}$ which represents the stable betweenness centrality as in (5) of the dithered graph $\boldsymbol{G}$. Since the shortest path $\boldsymbol{s_G}$ is not a linear function of the weights, in general $\mathbb{E}(\boldsymbol{C}_{SB}(x)) \neq C_{SB}(x)$. Dithering preserves stability since C_{DSB} is stable as we show next.

Proposition 3 *The dithered stable betweenness centrality measure C_{DSB} in* (10) *is stable as defined in Definition 1 with $K_G = 2\,n^2$.*

Proof: See Appendix A in [16]. ■



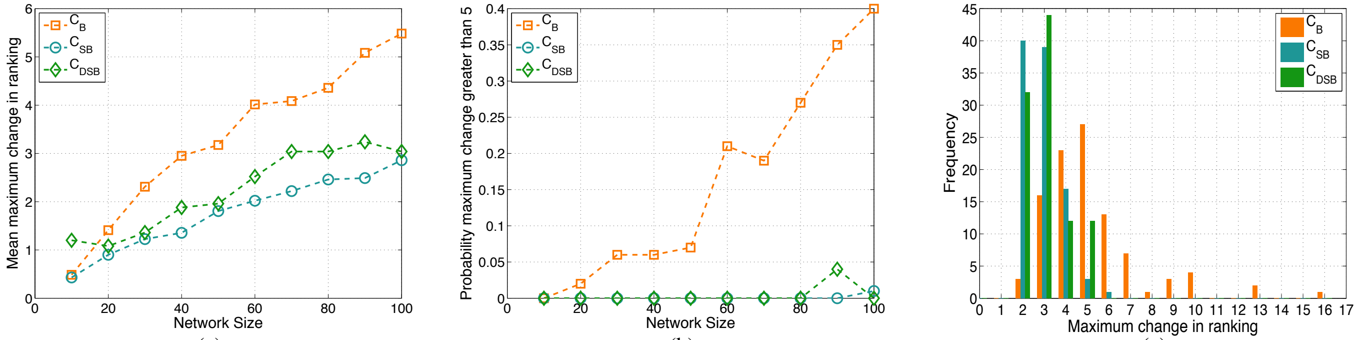
	C_B	C_{SB}	C_{DSB}
x_1	8.0	1.6	2.1
x_2	0.0	0.0	0.6
x_5	0.0	0.0	0.0

Fig. 1: Through dithering, C_{DSB} distinguishes the centralities of x_2 and x_5. This is not the case for C_B and C_{SB}.

The measure C_{DSB}, apart from being stable, solves other undesirable properties of C_B and C_{SB}; see Fig. 1. In the corresponding table, we compare the centrality values of three nodes computed using different centrality measures. For C_{DSB} we apply uniform 0.3-dithering as introduced in Example 1 and perform 1,000 Monte Carlo (MC) iterations to estimate the expected value in (10).

The three measures coincide in giving x_5 a null centrality value. This is reasonable since x_5 is not an intermediate node in any path between two nodes, let alone any shortest path. They also coincide in giving x_1 the highest value among these three, which again is reasonable since x_1 belongs to the shortest path between any pair of nodes in opposite sides of the network. Regarding x_2, the behavior of C_{DSB} is fundamentally different from the other two measures. We have that $C_B(x_2) = C_{SB}(x_2) = 0$ since x_2 does not belong to the shortest path between any pair of other nodes in the network. In this way, x_2 is indistinguishable from x_5. Nevertheless, the roles of x_2 and x_5 within the network's topology are different. The deletion of x_5 has no effect in the communication among the rest of the network justifying its null centrality value. In contrast, if x_2 is deleted, the network topology *does* change. Before the deletion we had two comparable paths to go from any node in the left to any node in the right and vice versa, whereas after the deletion there is only one choice, making the network more sensitive to the failure of node x_1. Measures C_B and C_{SB} are not aware of this impact, thus, assigning a null centrality value to node x_2. By introducing dithering, the proposed measure C_{DSB} overcomes this limitation and assigns a positive centrality value to x_2. This occurs because, although in expectation the paths through x_1 are shorter than those through x_2, for some realizations of the dithered graph, x_2 is the preferred intermediate node to traverse from right to left and vice versa. Thus, in expectation, the centrality value of x_2 is strictly positive. The behavior of C_{DSB} is further illustrated in Section V.

In general, a closed form for the expected value in (10) is not available. Hence, as in the previous example, we estimate the C_{DSB} centrality value of a node by computing its corresponding C_{SB} centrality value in k realizations of the corresponding dithered graph and averaging them. Since the complexity of C_{SB} is at most $O(n^2 m + n^3 \log n)$ [12], the complexity of C_{DSB} becomes $O(kn^2 m + kn^3 \log n)$.



(a) (b) (c)

Fig. 2: Robustness indicators when noise is introduced in random networks for the three centrality measures: betweenness (orange square), stable betweenness (cyan circle), and dithered stable betweenness (green diamond). (a) Mean of the maximum change recorded when perturbing a random network as a function of network size. (b) Probability that the maximum change in the ranking when perturbing a network exceeds 5 positions as a function of the network size. (c) Histogram of the maximum change recorded when perturbing random networks with 100 nodes.

V. Numerical experiments

We illustrate the advantages of dithering as a tool for computing centrality by analyzing the application of C_B, C_{SB} and C_{DSB} to different graphs. We show the practical implications of stability (Section V-A), we demonstrate that C_{DSB} has an intuitive dependence on the number and quality of alternative paths (Section V-B) and, finally, we reveal that C_{DSB} provides a smooth centrality distribution across comparable paths (Section V-C).

A. Robustness to noise

For a given node set V of size $n \geq 10$, we define a random network as one where an edge (x, x') belongs to E with probability $q = 10/n$. In case an edge exists between two nodes, the weight of this edge is randomly picked from a uniform distribution in $[0.5, 1.5]$. We analyze the robustness of the centrality rankings output by C_B, C_{SB}, and C_{DSB} when these networks are perturbed by random noise. Given a network, we build a perturbed version of it by multiplying the weight of each edge by a uniform random number in $[0.99, 1.01]$.

For the following experiment, we generate 100 random networks of n nodes, where n varies from 10 to 100 in multiples of 10. We then generate a perturbed version of each of these networks by applying the aforementioned random noise. For every network, we generate a centrality ranking of the nodes, i.e. we sort the nodes in decreasing order of centrality value, and compare it with the centrality ranking of the perturbed version of that network. For C_{DSB} we apply uniform 0.1-dithering and 100 MC iterations for every network.

The property of stability [cf. Definition 1] makes C_{SB} and C_{DSB} robust to noise, a characteristic that C_B does not posses. We begin by analyzing the maximum variation in ranking position experienced by a node when perturbing the network. In Fig. 2a we plot the mean of this indicator among the networks analyzed as a function of the network size. For example, for a network with 70 nodes, the perturbation generates a maximum change of 4.1 positions on average for the C_B ranking, 2.2 positions on average for the C_{SB} ranking, and 3.0 positions on average for the C_{DSB} ranking. Although all measures present an increase of the maximum change with the size of the network, the rate of increase is fastest for C_B, generating big performance differences between the measures for larger networks.

In Fig. 2b we plot the probability that the maximum change in the ranking generated by a perturbation is greater than 5 positions as a function of the network size. E.g., for networks of 100 nodes, there is a 0.4 probability that the betweenness centrality ranking undergoes a variation greater than 5 positions while this probability is less than 0.05 for the other two measures. To facilitate the understanding of Figs. 2a and 2b, in Fig. 2c we present the histogram of the maximum change suffered by the rankings when perturbing a network for the particular case of networks with 100 nodes. In this way, the mean of the orange histogram corresponds to the orange square for networks of 100 nodes in Fig. 2a, the mean of the cyan histogram corresponds to the cyan circle, and so on. To relate the histograms with Fig. 2b, notice that the orange histogram is the only one with a relevant portion of its weight in changes of 6 positions or more. This accounts for the big difference in the probabilities of having changes greater than 5 positions for networks of 100 nodes between the orange marker and the rest in Fig. 2b. Having a longer tail, the silhouette of the orange C_B histogram is essentially different from the other two. E.g., for one of the studied networks, the C_B ranking presents a change of 16 positions whereas the largest variation for the two other measures combined is of 6 positions. This is an empirical example of the instability of betweenness centrality.

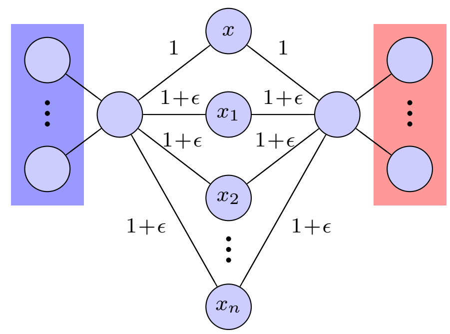
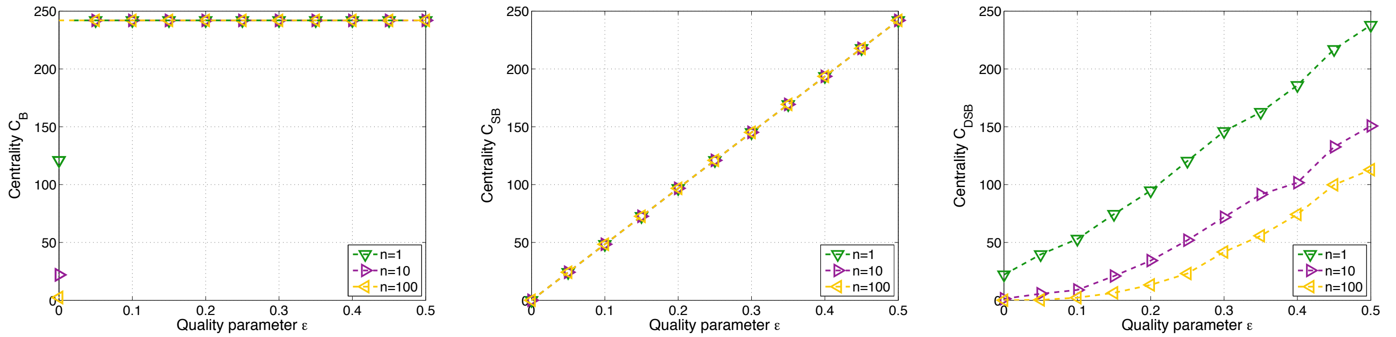


Fig. 3: The blue and the red components of the graph are joined by $n+1$ paths, one of them being optimal. We analyze C_B, C_{SB} and C_{DSB} as a function of the number of alternative paths n and their quality ϵ.

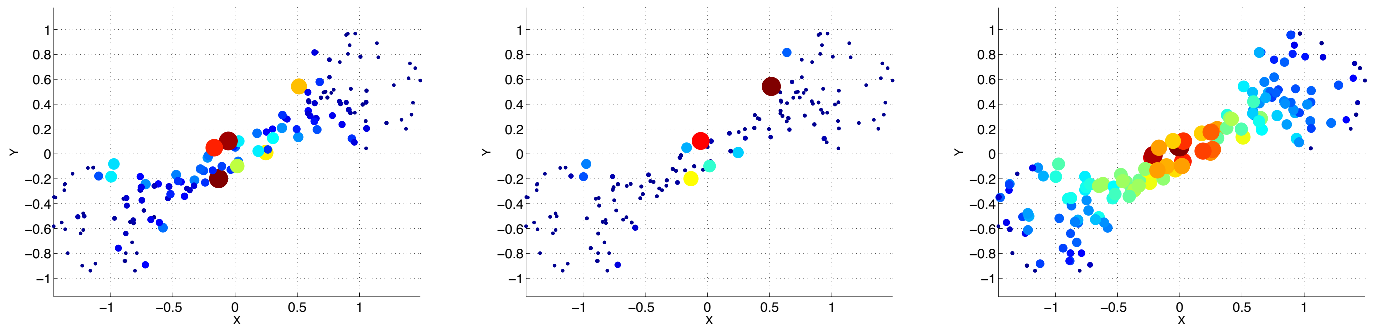
B. Dependence on alternative paths

The betweenness centrality value of a node is completely determined by the shortest paths between every pair of nodes in the network. The stable betweenness centrality, in addition, takes into account the best alternative path once the node being studied is deleted. However, C_{SB} is myopic to the number of comparable alternatives for shortest paths that exist. When adding dithering to obtain C_{DSB}, the centrality values depend on both the quality and number of alternative paths. To illustrate this we analyze the centrality value of node x in the graph in Fig. 3. In this graph we have two components of 10 nodes each – the blue and the red – which are connected by a series of *bridges*. We compute the centrality of x as given by the three measures C_B, C_{SB}, and C_{DSB} for different values of n, i.e. the amount of alternative paths, and varying ϵ, i.e. the quality of the alternative paths. For C_{DSB} we apply uniform 0.2-dithering and 100 MC iterations. In Fig. 4 we plot the centrality values as a function of ϵ for $n = 1$ in green, $n = 10$ in purple and $n = 100$ in yellow.



(a) (b) (c)

Fig. 4: Centrality values of x in the graph in Fig. 3 as a function of ϵ for $n \in \{1, 10, 100\}$. (a) Betweenness centrality C_B. Centrality values do not depend on ϵ or n as long as $\epsilon > 0$. (b) Stable betweenness centrality C_{SB}. Centrality values depend on ϵ but are independent of the number of alternative paths n. (c) Dithered stable betweenness centrality C_{DSB}. Centrality values increase with ϵ and decrease with n.



(a) (b) (c)

Fig. 5: Scatter plot of a *dumbbell-like* graph. The higher the centrality of a point, the larger its area and the redder its color. (a) C_B. Centrality is very susceptible to minor variations in the position. (b) C_{SB}. Most nodes have null centrality values since many comparable alternative paths exist. (c) C_{DSB}. The whole linear part of the graph is detected as central, with a smooth distribution of centrality values across the nodes.

First consider the behavior of C_B; see Fig. 4a. For strictly positive values of ϵ, the centrality value is constant regardless of the value of n or ϵ. This occurs because the bridge through x is strictly better than the rest and, thus, C_B ignores the number and quality of alternative bridges. For $\epsilon = 0$ though, every bridge is equally good. Hence, the greater n, the less portion of shortest paths go through x and the smaller its centrality value. From Fig. 4a, it is immediate that C_B is not a continuous centrality measure as shown in [12]. For the case of stable betweenness in Fig. 4b, centrality varies in a continuous way with ϵ since the quality of alternative paths is taken into account. However, just like traditional betweenness centrality, C_{SB} is impervious to the number of alternative paths n. By adding dithering, C_{DSB} behaves in a more intuitive way; see Fig. 4c. For a fixed number of paths n, the centrality of x grows with ϵ, i.e. it grows when the quality of alternative paths decreases. Moreover, for a fixed ϵ, the centrality of x decreases when n increases. This means that when multiple alternative paths exist, the centrality of a node lying in one of these paths is reduced.

C. Multiple path detection

Consider the point cloud in $\mathbb{R}^2$ depicted in Fig. 5. Notice that this *dumbbell-like* finite metric space consists of two circular clusters linked by scattered points in a linear fashion. We build a weighted graph G from this point cloud where for every pair of points we draw an edge between them weighted by their euclidean distance if this distance is less than 1. Otherwise, there is no edge between them. We then compute the centrality value of each node as given by the three measures C_B, C_{SB} and C_{DSB}. For C_{DSB} we apply uniform 0.2-dithering and 100 MC iterations. The area and the color of the points depend on their centrality. The more central a node is, the larger its area. Similarly, more central nodes are red in color as opposed to the blue color of less central points. For every graph, the centrality is normalized to be contained in $[0, 1]$.

Notice that in the centrality pattern given by C_B in Fig. 5a, some of the nodes in the linear portion of the graph are very central while other nodes close to these have very small centrality values. However, there is no fundamental difference between the central nodes and the rest, they were all generated randomly in a rectangular portion of the plane. What occurs is that belonging or not to a shortest path is extremely sensitive to small random variations. The measure C_{SB} does not solve this problem; see Fig. 5b. In fact, most of the nodes in the linear part of the graph have almost null centrality values. This occurs because if almost any node is deleted, a comparable path can be found through other nodes, bounding its centrality. In this way, C_{SB} is incapable of differentiating most of the nodes in the linear part of the graph from peripheral nodes in the circular clusters. By adding dithering, C_{DSB} correctly portrays the centrality profile of the graph; see Fig. 5c. Nodes belonging to most shortest paths still have the highest centralities but nodes belonging to comparable alternative paths have comparable centrality values. In this way, there is a smooth distribution of centrality across the points in the linear part of the graph since C_{DSB} detects multiple paths as opposed to just the shortest one. Moreover, points belonging to the linear portion of the graph can be easily distinguished from peripheral nodes in the circular clusters as opposed to the C_{SB} case. This is reasonable since nodes belonging to both categories – linear and peripheral – play fundamentally different roles in dynamical processes within the network.

VI. Conclusion

Dithering as a tool to design more reliable centrality measures was formally introduced and was used to develop an improvement on the stable betweenness centrality measure. This new measure preserves stability while following a more intuitive behavior in terms of quality and number of alternative paths. Furthermore, different advantages of dithering were illustrated through three numerical experiments.

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