Control with Random Access Wireless Sensors

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Abstract— **We consider multiple sensors randomly accessing a shared wireless medium to transmit measurements of their respective plants to a controller. To mitigate the packet collisions arising from simultaneously transmitting sensors, we appropriately design the sensor access rates. This is posed as an optimization problem, where the total transmit power of the sensors is minimized, and control performance for all control loops needs to be guaranteed. Control performance of each loop is abstracted as a desired expected decrease rate of a given Lyapunov function. By establishing an equivalent convex optimization problem, the optimal access rates are shown to be decoupled among sensors. Moreover, based on the Lagrange dual problem, we develop an easily implementable distributed procedure to find the optimal sensor access rates.**

I. Introduction

The abundance of wireless sensing devices in modern control environments, for example, smart buildings, creates a need for sharing the available wireless medium in an efficient manner. The design of efficient mechanisms for sensors to access the shared medium, in a way that provides control performance guarantees while also being easily implementable, for example decentralized, arises as an important challenge.

The prevalent approach to the problem sharing a wireless (or wired) communication medium in networked control systems is centralized scheduling. Scheduling can be static, specifying for example that sensors transmit in some predefined periodic sequence, and this periodic sequence is designed to meet control objectives, see, e.g., [1]–[3]. Deriving optimal scheduling sequences is recognized as a hard combinatorial problem [4]. Scheduling can also be dynamic, where a central authority decides which device accesses the medium at each time step. This dynamic decision can be based on plant state information, e.g., [5], [6], or the wireless channel conditions [7].

In contrast to centralized scheduling, we consider a random access mechanism to share the wireless medium between the sensors. Each sensor independently and randomly decides whether to transmit plant state measurements over the channel to a controller. This mechanism is decentralized and easy to implement as it does not require predesigned sequences of how sensors access the medium, or a central authority to take scheduling decisions. The drawback of this decentralized approach however is that packet collisions can occur from simultaneously transmitting sensors, resulting in lost packets and control performance degradation. Sensor access rates need to be designed to mitigate these effects.

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Control under random access communication mechanisms has drawn limited attention, to the best of our knowledge. Comparisons between different medium access mechanisms for networked control systems and the impact of packet collisions have been considered in [8], [9], including random access and related Aloha-like schemes (where after a packet collision the involved sensors wait for a random time interval and retransmit). Stability conditions under packet collisions were examined in [10]. In contrast to these works, our goal is to design the medium access mechanism so that desired control performance is guaranteed. Related work appears in [11], which instead considers the Aloha-like scheme and characterizes what retransmission policies lead to stability. Besides closed loop control, sensor transmission over collision channels for optimal remote estimation is considered recently in [12].

We consider multiple control loops over a shared wireless channel (Section II), and we are interested in designing the rate at which the sensor of each loop should be accessing the medium in order to ensure control performance for all loops. We propose a Lyapunov-like control performance requirement, motivated from our work on centralized scheduling [7]. Each control system is abstracted via a given Lyapunov function which is desired to decrease at predefined rates and stochastically due to the random packet losses and collisions on the shared medium. These control requirements are shown to be equivalent to a minimum packet success rate on each link.

We examine the design of sensor access rates that satisfy the Lyapunov control performance requirements and minimize the average transmit power of the sensors. We show that this is equivalent to a convex optimization problem, and a characterization of the optimal sensor access rates is established based on Lagrange duality (Section III). This characterization reveals an intuitive decoupled form; each sensor should access the channel at a rate proportional to the desired control performance of its corresponding control loop, and inverse proportional to its transmit power and the aggregate collision effect it causes on all other control loops. Similar decoupled structures are known in the context of random access wireless networks [13]–[15], where the relevant quantity of interest are data rates on links or general utility objectives. In our case we focus on packet success rates for desired control system performance.

In Section IV we derive a decentralized procedure converging to the optimal access rates, which has an interpretation of optimizing the dual problem. The procedure is easy to implement as it does not require the sensors to coordinate among themselves, or to know what the other sensors try to achieve. We conclude with a numerical example and some

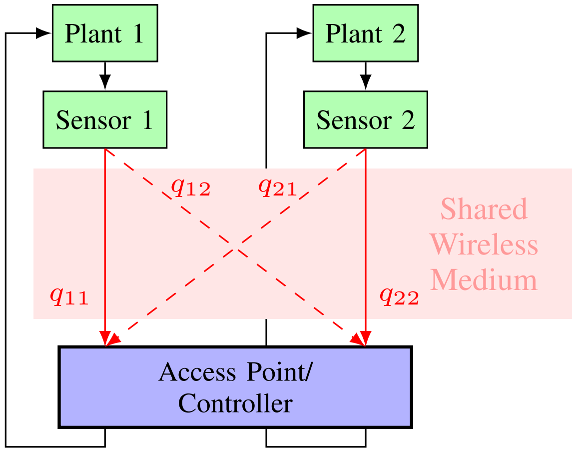

Fig. 1. Random access architecture over a shared wireless medium for $m = 2$ control loops. Each sensor randomly decides whether to transmit to a common access point computing the plant control inputs. When only sensor i is transmitting on the shared medium, the message is successfully decoded with probability q_{ii}. When another sensor j transmits at the same slot, there is a probability q_{ji} of causing a collision at sensor i's transmission, rendering i's packet lost. The goal is to design the rate at which every sensor accesses the medium in order to guarantee control performance for all control loops.

remarks (Sections V, VI).

II. Problem Formulation

We consider a wireless control architecture where m independent plants are controlled over a shared wireless medium. Each sensor i ($i = 1, 2, ..., m$) measures and transmits the output of plant i to an access point responsible for computing the plant control inputs. Packet collisions might arise on the shared medium between simultaneously transmitting sensors. The case for $m = 2$ control loops is shown in Fig. 1. We are interested in designing a mechanism for each sensor to independently decide whether to access the medium (random access) in a way that guarantees desirable control performance for all control systems.

Our goal is to design communication aspects of the problem, hence we assume the dynamics for all m control systems are fixed, meaning that controllers have been already designed. Let us indicate with $\gamma_{i,k} \in \{0, 1\}$ the success of the transmission at time k for link/system i, which as we make clear next is a random event. We suppose the system evolution is described by a switched linear time invariant model,

$$x_{i,k+1} = \begin{cases} A_{c,i}\, x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 1 \\ A_{o,i}\, x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 0 \end{cases} . \tag{1}$$

Here $x_{i,k} \in \mathbb{R}^{n_i}$ denotes the state of control system i at each time k, which can in general include both plant and controller states. At a successful transmission the system dynamics are described by the matrix $A_{c,i} \in \mathbb{R}^{n_i \times n_i}$, where 'c' stands for closed-loop, and otherwise by $A_{o,i} \in \mathbb{R}^{n_i \times n_i}$, where 'o' stands for open-loop. We assume that $A_{c,i}$ is asymptotically stable, implying that if system i successfully transmits at each slot the state evolution of $x_{i,k}$ is stable. The open loop matrix $A_{o,i}$ may be unstable. The additive terms $w_{i,k}$ model an independent (both across time k for each system i, and across systems) identically distributed (i.i.d.) noise process with mean zero and covariance $W_i \succeq 0$. An example of the above networked control system model (1) is presented next.

Example 1. Suppose each closed loop i consists of a linear plant and a linear output of the form

$$\begin{aligned} x_{i,k+1} &= A_i x_{i,k} + B_i u_{i,k} + w_{i,k}, \\ y_{i,k} &= C_i x_{i,k} + v_{i,k}, \end{aligned} \tag{2}$$

where $w_{i,k}$ and $v_{i,k}$ are i.i.d. Gaussian disturbance and measurement noise respectively. Each wireless sensor i transmits the output measurement $y_{i,k}$ to the controller. A dynamic control law adapted to the packet drops keeps a local controller state $z_{i,k}$, which may represent a local estimate of the plant state [16], and applies plant input $u_{i,k}$ as

$$\begin{aligned} z_{i,k+1} &= F_i\, z_{i,k} + \gamma_{i,k}\, (F_{c,i}\, z_{i,k} + G_i\, y_{i,k}), \\ u_{i,k} &= K_i\, z_{i,k} + \gamma_{i,k}\, (K_{c,i}\, z_{i,k} + L_i\, y_{i,k}), \end{aligned} \tag{3}$$

i.e., it corrects appropriately the local state and input whenever a measurement is received. The overall closed loop system is obtained by joining plant and controller states into

$$\begin{aligned} \begin{bmatrix} x_{i,k+1} \\ z_{i,k+1} \end{bmatrix} &= \begin{bmatrix} A_i + \gamma_{i,k} B_i L_i C_i & B_i K_i + \gamma_{i,k} B_i K_{c,i} \\ \gamma_{i,k} G_i C_i & F_i + \gamma_{i,k} F_{c,i} \end{bmatrix} \\ &\cdot \begin{bmatrix} x_{i,k} \\ z_{i,k} \end{bmatrix} + \begin{bmatrix} I & \gamma_{i,k} B_i L_i C_i \\ 0 & \gamma_{i,k} G_i \end{bmatrix} \begin{bmatrix} w_{i,k} \\ v_{i,k} \end{bmatrix} \end{aligned} \tag{4}$$

which is of the form (1). □

We assume communication takes place in time slots, and at every time k each sensor i randomly and independently decides to access the channel with some constant probability $\alpha_i \in [0, 1]$. The vector of sensor access rates $\alpha \in [0, 1]^m$ is the variable that needs to be designed in our problem. Packet drops in the shared wireless medium occur due to two reasons. First, if only sensor i transmits at a time slot, the message is not always successfully decoded at the access point/controller because of noise added to the transmitted signal on the wireless channel [17]. We assume that successful decoding occurs with some known constant positive probability $q_{ii} \in (0, 1]$. Given that sensor i randomly decides to transmit at a rate α_i, we conclude that in isolation the ith loop in (1) successfully closes with probability $\alpha_i\, q_{ii}$ at each time step.

The second reason for packet drops in our setup is interference due to other sensors transmitting at the same time slot as sensor i does. In particular when another sensor j transmits at the same slot, a collision on sensor i's packet occurs with some probability q_{ji}, leading to packet i being lost. The values q_{ji} are known constants and for simplicity are assumed positive $q_{ji} \in (0, 1]$ – the case where some sensors do not interfere with each other is similarly handled as discussed later in Remark 3. This collision model subsumes: i) the conservative case where simultaneous transmissions certainly lead to collisions ($q_{ji} = 1$) usually considered in control literature, e.g., [10], [11], ii) the case where simultaneously transmitted packets are not always lost ($q_{ji} <$

1), e.g., due to the capture phenomenon [18], and iii) the case where different sensors j, ℓ interfere differently on link i, e.g., $0 < q_{ji} < q_{\ell i} \leq 1$ if sensor ℓ is closer than sensor j.

Given that sensor j randomly decides to transmit with probability α_j, the probability that transmission i is affected by sensor j equals the product $\alpha_j\, q_{ji}$. To sum up, the combined effect from all sensors on packet success at link i is expressed as

$$\mathbb{P}(\gamma_{i,k}=1) = \alpha_i\, q_{ii} \prod_{j\neq i} \Big[1-\alpha_j q_{ji}\Big]. \tag{5}$$

This expression states that the probability of system i in (1) closing the loop at time k equals the probability that transmission i is successfully decoded at the receiver, multiplied by the probability that no other sensor $j \neq i$ is causing collisions on ith transmission. The product in this expression is a consequence of the fact that all sensors independently decide to access the channel [1]. In (5) the communication parameters q_{ji}, $i, j \in \{1, \ldots, m\}$ are given, and the variables to be designed are the sensor access rates α.

The random packet success on link i modeled by (5) causes each control system i in (1) to switch in a random fashion between the two modes of operation (open and closed loop). As a result the access rate vector α to be designed affects the performance of all control systems. The following proposition characterizes, via a Lyapunov-like abstraction, a connection between control performance and the packet success rate.

Proposition 1 (Control performance abstraction)**.** *Consider a switched linear system i described by (1) with $\gamma_{i,k}$ being an i.i.d. sequence of Bernoulli random variables, and a quadratic function $V_i(x_i) = x_i^T P_i x_i$, $x_i \in \mathbb{R}^{n_i}$ with a positive definite matrix $P_i \in S_{++}^{n_i}$. Then the function decreases with an expected rate $\rho_i < 1$ at each step, i.e., we have*

$$\mathbb{E}\left[V_i(x_{i,k+1})\,\big|\,x_{i,k}\right] \leq \rho_i\, V_i(x_{i,k}) + Tr(P_i W_i) \tag{6}$$

for all $x_{i,k} \in \mathbb{R}^{n_i}$, if and only if

$$\mathbb{P}(\gamma_{i,k}=1) \;\geq\; c_i, \tag{7}$$

where $c_i \geq 0$ is computed by the semidefinite program

$$c_i = \min\{\theta \geq 0 :\; \theta A_{c,i}^T P_i A_{c,i} + (1-\theta) A_{o,i}^T P_i A_{o,i} \preceq \rho_i P_i\} \tag{8}$$

Proof. The expectation over the next system state $x_{i,k+1}$ on the left hand side of (6) accounts via (1) for the randomness introduced by the process noise $w_{i,k}$ as well as the random success $\gamma_{i,k}$. In particular we have that

$$\begin{aligned}\mathbb{E}\left[V_i(x_{i,k+1})\,\big|\,x_{i,k}\right] &= \mathbb{P}(\gamma_{i,k}=1)\; x_{i,k}^T A_{c,i}^T P_i A_{c,i} x_{i,k} \\ &+ \mathbb{P}(\gamma_{i,k}=0)\; x_{i,k}^T A_{o,i}^T P_i A_{o,i} x_{i,k} + Tr(P_i W_i),\end{aligned} \tag{9}$$

[1]Precisely, if the vector $\delta \in \{0,1\}^m$ indicates which of the m sensors transmit at time k, we have $\mathbb{P}(\gamma_{i,k} = 1\,|\,\delta) = \delta_i q_{ii} \prod_{j\neq i}[1-q_{ji}]^{\delta_j}$. Taking expectation over the independent Bernoulli variables $\delta_i \sim \text{Bern}(\alpha_i)$ yields (5).

Here we used the fact that the random variable $\gamma_{i,k}$ is independent of the system state $x_{i,k}$. Plugging (9) at the left hand side of (6) we get for $x_{i,k} \neq 0$

$$\mathbb{P}(\gamma_{i,k}=1) \;\geq\; \frac{x_{i,k}^T (A_{o,i}^T P_i A_{o,i} - \rho_i P_i) x_{i,k}}{x_{i,k}^T (A_{o,i}^T P_i A_{o,i} - A_{c,i}^T P_i A_{c,i}) x_{i,k}}. \tag{10}$$

Since conditions (6) needs to hold at any value of $x_{i,k} \in \mathbb{R}^{n_i}$, we can rewrite (10) as $\mathbb{P}(\gamma_{i,k}=1) \geq c_i$ where

$$c_i = \sup_{y\in\mathbb{R}^{n_i}, y\neq 0} \frac{y^T (A_{o,i}^T P_i A_{o,i} - \rho_i P_i) y}{y^T (A_{o,i}^T P_i A_{o,i} - A_{c,i}^T P_i A_{c,i}) y}. \tag{11}$$

This is equivalent to the semidefinite program (8). □

The interpretation of the quadratic function $V_i(x_i)$ in this proposition is that it acts as a Lyapunov function for the control system, guaranteeing not only stability but also performance – see also Remark 1 for more details. When the loop closes the Lyapunov function of the system state decreases, while in open loop it increases, and (6) describes an overall decrease in expectation over the packet success. The required packet success rate computed in (8) satisfies $c_i \leq 1$ as long as $A_{c,i}^T P_i A_{c,i} \preceq \rho_i P_i$, i.e., $V_i(x_i)$ is chosen as a Lyapunov function for the stable mode $A_{c,i}$ in (1).

In this paper we assume that quadratic Lyapunov functions $V_i(x_i)$ and desired expected decrease rates ρ_i are given for each control system. They present a control interface for communication design over a shared wireless medium. We aim to design the sensor access rates α so that the Lyapunov functions *for all systems i* decrease *in expectation* at the desired rates $\rho_i < 1$ at any time k. By the above proposition, these control performance requirements can be transformed to necessary and sufficient packet success rates (7) for each link i, computes by (8). Hence we need to ensure that (7) holds for all links i.

Besides control performance, it is desired that the sensors' channel access mechanism makes an efficient use of their power resources. Suppose that when system i decides to access the channel it transmits with a constant power $p_i > 0$. We pose then the design of the sensor access rates α that minimize the total expected power expenditure $\sum_{i=1}^m \alpha_i\, p_i$ subject to the desired control performance (cf. (5), (6)) for all plants as

$$\underset{\alpha\in\mathcal{A}}{\text{minimize}} \quad \sum_{i=1}^m \alpha_i p_i \tag{12}$$

$$\text{subject to} \quad c_i \leq \alpha_i\, q_{ii} \prod_{j\neq i}\Big[1-\alpha_j q_{ji}\Big],\; i \in \{1,\ldots,m\}. \tag{13}$$

Here we restrict attention to access rates α in a closed set

$$\mathcal{A} = \prod_{i=1}^m \mathcal{A}_i, \quad \mathcal{A}_i = [\alpha_{i,\min}, \alpha_{i,\max}], \tag{14}$$

subset of the unit cube $[0,1]^m$. This choice is purely for technical reasons, without restricting the feasible set. Intuitively each sensor i can neither choose α_i too close to 0 otherwise it cannot meet its packet success requirement in (13), nor too close to 1 otherwise it causes significant packet collisions on other sensors. We assume that for all $\alpha \in \mathcal{A}$ the right hand sides of the constraints (13) are strictly positive.[2]

In the following section we proceed to characterize the optimal access rates α^*, by transforming the original non-convex problem (12)-(13) into an equivalent convex one. This way we reveal and exploit a simple decoupled structure. Each sensor i independently accesses the channel at a rate that trades off the goal of closed loop i with the effect of collisions on all other closed loops $j \neq i$ collectively. Later in Section IV we develop a decentralized procedure to find these optimal access rates.

Remark 1. In this paper we are concerned with communication design for control performance, in contrast to the problem of determining what communication designs guarantee stability, commonly examined in the literature, e.g. [1], [2], [11], [16]. The Lyapunov-like abstraction (6) of Proposition 1 provides such a characterization of control performance, which also implies stability. If (6) holds for each time step $k = 0, \dots, N$, then by taking the expectation at both sides and by iterating backwards in time we find that

$$\begin{aligned} \mathbb{E}V_i(x_{i,N}) &\leq \rho_i \, \mathbb{E}V_i(x_{i,N-1}) + Tr(P_i W_i) \\ &\leq \dots \leq \rho_i^N \, \mathbb{E}V_i(x_{i,0}) + \sum_{k=0}^{N-1} \rho_i^k \, Tr(P_i W_i). \end{aligned} \tag{15}$$

Hence, system states have second moments that decay exponentially with rate ρ_i with respect to initial states, and in the limit remain bounded by $Tr(P_i W_i)/(1-\rho_i)$, since the sum in (15) converges due to $\rho_i < 1$.

As a technical sidenote, an advantage of the Lyapunov performance approach is that it defines a convex region (a lower bound) for the packet success rate in (7), which is easy to employ in our random access design in (13), and thus presents a suitable interface between control and communication design. On the contrary, it is known that a jump linear system of the form (1) is (mean square) stable if and only if the spectral radius of $\mathbb{P}(\gamma_i = 1) A_{c,i} \otimes A_{c,i} + \mathbb{P}(\gamma_i = 0) A_{o,i} \otimes A_{o,i}$ is less than 1 [19]. However the spectral radius of a non-symmetric matrix is not convex in general, hence it is unclear how to best examine stability in our random access framework. □

III. Control-aware Random Access Design

In this section we consider the optimal random access design problem and characterize the form of the optimal solution. This optimal design problem in (12)-(13) is not convex because the functions appearing in the right hand side of the constraints (13) are not concave. However, we can express the problem in an equivalent convex form.

Taking the logarithm at each side of the constraint (13), by monotonicity, preserves the feasible set of variables. Then the logarithm of the product on the right hand side of (13) becomes a sum of logarithms, so that we can rewrite the optimal random access design problem equivalently as

$$\underset{\alpha \in \mathcal{A}}{\text{minimize}} \quad \sum_{i=1}^{m} \alpha_i p_i \tag{16}$$

$$\begin{aligned} \text{subject to} \quad & \log(c_i) \leq \log(\alpha_i \, q_{ii}) + \sum_{j \neq i} \log\Big(1 - \alpha_j q_{ji}\Big), \\ & \text{for all } i \in \{1, \dots, m\}. \end{aligned} \tag{17}$$

This equivalent problem is convex in the access rate variables α, because the logarithm functions appearing in the constraints are concave.

Of course, once we have rewritten the random access design problem (12)-(13) in its equivalent convex form (16)-(17), we can solve efficiently for the optimal rate α^* using off-the-shelf convex optimization algorithms [20]. Then each sensor i in Fig. 1 can store the respective access rate α_i^* and operate according to it. However, apart from tractability, the convex reformulation permits a characterization of the form of the optimal access rates, as we show next.

Proposition 2 (Optimal sensor access rates)**.** *Consider the design of optimal sensor access rates in (12)-(13), and suppose that a strictly feasible solution exists. Then there exists a matrix of non-negative elements $\nu^* \in \mathbb{R}_+^{m \times m}$ such that the optimal sensor access rate α_i^* for each sensor $i \in \{1, \dots, m\}$ can be expressed as*

$$\alpha_i^* = \left[\frac{\nu_{ii}^*}{p_i + \sum_{j \neq i} \nu_{ji}^*} \right]_{\mathcal{A}_i}, \tag{18}$$

where $[.]_{\mathcal{A}_i}$ denotes the projection on the set $\mathcal{A}_i$ in (14).

This proposition is perhaps surprising because it states that each sensor can select its access rate optimally in a simple decoupled way, as long as the matrix ν^* is available. That is because α_i^* in (18) only depends on parameters pertinent to system i, i.e., its transmit power p_i, and is independent of what other sensors are trying to achieve. It is also independent of any other parameters of system i, such as the required minimum success c_i.

All the information about the optimal rate in (18) is encoded in the matrix v^*, which as we will explain is the optimal Lagrange multiplier of an appropriately defined problem. Intuitively ν_{ii}^* can be thought of as the requirement for control performance of closed loop i, and similarly ν_{ji}^* as the collision effect that sensor i has on the closed loop j. The optimal access rate for sensor i in (18) tries to tradeoff the requirement on loop i and the collective negative effect $(\sum_{j \neq i} \nu_{ji}^*)$ on all other control loops $j \neq i$. A high transmit power p_i also implies that sensor i should access the channel at a low rate α_i^* to limit expenditures.

Proof of Proposition 2. We argued that the original problem (12)-(13) is equivalent to the convex optimization in (16)-(17). We further introduce an auxiliary problem that is equivalent to (16)-(17). For notational convenience we change notation from variables α_i to variables α_{ii} belonging in the diagonal of an $m \times m$ matrix. Moreover, at the right hand

[2] As an example of how to choose each interval $\mathcal{A}_i$ in (14), it can be seen from (13) that each sensor i needs to ensure $\alpha_i \geq c_i / q_{ii}$, which is positive, as well as $\alpha_i \leq \min_j (1 - c_j)/q_{ij}$ and $\alpha_i \leq 1$.

side of (17) we replace the variables α_j, $j \neq i$ with auxiliary variables α_{ji} to write (17) as

$$\log(c_i) \leq \log(\alpha_{ii}\, q_{ii}) \ + \sum_{j\neq i} \log\Big(1 - \alpha_{ji} q_{ji}\Big). \tag{19}$$

Intuitively we would like the new variables α_{ji} to behave like α_j in the original problem, hence we also introduce an additional coupling constraint $\alpha_{ji} \geq \alpha_{jj}$ to the problem. Overall we formulate the auxiliary convex optimization problem

$$\underset{\alpha\in\mathcal{A}'}{\text{minimize}} \quad \sum_{i=1}^{m} \alpha_{ii} p_i \tag{20}$$

$$\text{subject to} \quad \log(c_i) \leq \log(\alpha_{ii}\, q_{ii}) \ + \sum_{j\neq i} \log\Big(1 - \alpha_{ji} q_{ji}\Big), \quad \text{for all } i \in \{1, \dots, m\}, \tag{21}$$

$$\alpha_{jj} \leq \alpha_{ji}, \quad \text{for all } i \neq j \in \{1, \dots, m\} \tag{22}$$

where the set $\mathcal{A}' \subset [0, 1]^{m\times m}$ is appropriately constructed so that $\alpha_{ji} \in \mathcal{A}_j$ for all $i, j \in \{1, \dots, m\}$.

Problem (20)-(22) is equivalent to (16)-(17). Every feasible solution of (17) can be converted to a feasible solution for (21)-(22) via $\alpha_{ji} = \alpha_j$ for all $i, j \in \{1, \dots, m\}$ with the same objective. Reversely, let $\alpha \in \mathcal{A}'$ be a feasible solution for (21)-(22). We can assume without loss of generality that all constraints (22) are satisfied with equality, because if say $\alpha_{jj} < \alpha_{ji}$ for some j, i then we can reduce the value of α_{ji} to α_{jj} without changing the objective value and without violating the constraint (21) since $\log(1 - \alpha_{ji} q_{ji})$ is increasing. The diagonal elements α_{ii} of this solution immediately become a feasible solution to problem (17) with equal objective.

Let us then define the Lagrange dual problem of (20). Consider non-negative dual variables $\nu_{ii} \geq 0$ associated with each one of the constraints (21), and $\nu_{ij} \geq 0$ associated with the constraints (22) – note the order of indices. Then the Lagrangian function is defined as

$$\begin{aligned} L(\alpha, \nu) &= \sum_{i=1}^{m} \alpha_{ii} p_i \\ &+ \sum_{i=1}^{m} \nu_{ii} \Big[\log(c_i) - \log(\alpha_{ii}\, q_{ii}) \ - \sum_{j\neq i} \log(1 - \alpha_{ji} q_{ji}) \Big] \\ &+ \sum_{i=1}^{m} \sum_{j\neq i} \nu_{ij} \Big[\alpha_{jj} - \alpha_{ji} \Big]. \end{aligned} \tag{23}$$

Moreover by a rearrangement of the terms in the Lagrangian in (23), in particular by a careful interchange of the indices in the sums, we get

$$\begin{aligned} L(\alpha, \nu) = &\sum_{i=1}^{m} \Big[(p_i + \sum_{j\neq i} \nu_{ji})\, \alpha_{ii} - \nu_{ii} \log(\alpha_{ii}\, q_{ii}) \Big] \\ &+ \sum_{i=1}^{m} \sum_{j\neq i} \Big[- \nu_{ii}\, \log(1 - \alpha_{ji} q_{ji}) - \nu_{ij}\, \alpha_{ji} \Big] \\ &+ \sum_{i=1}^{m} \nu_{ii} \log(c_i). \end{aligned} \tag{24}$$

This rearrangement decouples the primal variables α_{ij} for all $i, j \in \{1, \dots, m\}$ and gives rise to the decoupled form of the optimal access rates in (18) as we argue next.

The Lagrange dual function of (20)-(22) is defined as

$$g(\nu) = \inf_{\alpha\in\mathcal{A}'} \ L(\alpha, \nu), \tag{25}$$

and the Lagrange dual problem is defined as

$$\underset{\nu\in\mathbb{R}_+^{m\times m}}{\text{maximize}} \quad g(\nu). \tag{26}$$

The strict feasibility of (12)-(13), by assumption of the proposition, implies that a strictly feasible solution exists for the equivalent problem (20)-(22). Since the problem (20)-(22) is convex and has a strictly feasible solution (Slater's condition), strong duality holds [20, Sec. 5.2], i.e., the optimal values of the primal problem (20) and its dual (26) are equal. Moreover the optimal access rate vector α^* is a minimizer of the Lagrangian function at the optimal dual point ν^*, i.e.,

$$\alpha^* \in \underset{\alpha\in\mathcal{A}'}{\text{argmin}} \ L(\alpha, \nu^*). \tag{27}$$

We then observe by (24) that the Lagrangian function is a strictly convex function in the variables α. Hence, the minimizers of the Lagrangian in (27) are unique and satisfy the first order condition

$$\frac{\partial L}{\partial \alpha_{ji}}(\alpha, \nu^*) = 0, \tag{28}$$

subject to the box constraints $\alpha_{ji} \in \mathcal{A}_j$ for all $i, j \in \{1, \dots, m\}$. Since the Lagrangian in (24) decouples among each primal variable α_{ji}, we see that

$$\frac{\partial L}{\partial \alpha_{ji}}(\alpha, \nu) = \begin{cases} (p_i + \sum_{j\neq i} \nu_{ji}) - \nu_{ii}/\alpha_{ii} & \text{if } i = j \\ \nu_{ii} q_{ji}/(1 - \alpha_{ji} q_{ji}) - \nu_{ij} & \text{if } i \neq j. \end{cases} \tag{29}$$

The optimal value of the diagonal elements α^*_{ii}, which are equivalently the optimal values α^*_i of the original problem (16), are found by finding the value α_{ii} making the expression in the first branch in (29) equal to zero, projected to the set $\mathcal{A}_i$. This directly verifies (18) and completes the proof of the proposition.

We note here for future reference that the unique off-diagonal minimizers α_{ji}, $j \neq i$ of the Lagrangian are

$$\underset{\alpha_{ji}\in\mathcal{A}_j}{\text{argmin}} \ L(\alpha, \nu) = \left[\frac{1}{q_{ji}} - \frac{\nu_{ii}}{\nu_{ij}} \right]_{\mathcal{A}_j}, \tag{30}$$

according to the first order condition (28) and the second branch in (29). $\square$

The decoupled structure of the optimal sensor access rates according to Proposition 2 relies on knowing the (optimal dual) values ν^*. In the following section we develop a distributed iterative procedure to obtain the desired ν^*, which is easily implementable in the architecture of Fig. 1. In particular the access point/controller maintains variables ν and communicates them to the sensors via the reverse channel, and the sensors adapt their access rates according to the decoupled form of Proposition 2.

Remark 2. The fact that the sensor access rates can be designed in a decoupled way is known in random access protocols [13]–[15]. Mathematically the problem studied in these works is similar to the one in (12)-(13), and the same logarithm transformation is employed to express it as a convex optimization as we do in (16)-(17). The context differs however, since in general wireless data networks the quantity of interest is the achieved data rates or fairness and other general utility functions, in contrast to the packet success rates used for control systems here. Moreover, the references typically assume that collisions happen with certainty in simultaneous transmissions. Here we employ a more general model ($q_{ji} \in (0,1]$) where packets can still be recovered after simultaneous transmissions, and this differentiates the mathematical formulation in (16). It is also worth noting that our approach to the problem via the dual domain, and the dual optimization to be presented in the next section, are similarly followed in [14], [15]. □

IV. Implementation of control-aware random access

We develop a distributed iterative algorithm to determine the values ν^* which, according to Proposition 2, are sufficient in order to select optimal sensor access rates α^*. Under the interpretation of ν^* as the optimal dual variables of an appropriately defined primal problem, the iterates $\nu(t)$ of the algorithm follow a dual subgradient optimization method.

The steps of the iterative procedure are shown in Algorithm 1. At each iteration t, the access point/controller of Fig. 1 keeps a matrix of dual variables $\nu(t)$ and sends to the sensors via the reverse channel the information required for them to select their access rates independently. In particular, each sensor i gets to know the values $\nu_{ii}(t)$ and $\sum_{j\neq i}\nu_{ji}(t)$, and selects its access rate $\alpha_i(t)$ in (31) as if these were the optimal dual values (cf. (18)).

At the end of each iteration t the access point updates the dual values to $\nu(t+1)$ to prepare for the next iteration. The dual variable update in (35) is a step towards the matrix direction $s(t) \in \mathbb{R}^{m\times m}$, computed as follows. The access point gets to know the access rates $\alpha_i(t)$ selected by each sensor at this iteration, and computes the auxiliary variables $\alpha_{ji}(t)$ by (32). The elements of the direction matrix $s(t)$ are computed via (33)-(34). Then the procedure repeats.

This algorithm is guaranteed to converge to the optimal access rates, as we state next.

Proposition 3 (Sensor access rates optimization)**.** *Consider the design of optimal sensor access rates in (12)-(13), and suppose that a strictly feasible solution exists. The iterations of Algorithm 1 with stepsize in (35) satisfying*

$$\sum_{t\geq 1}\varepsilon(t)^2 < \infty,\ \sum_{t\geq 1}\varepsilon(t) = \infty, \tag{36}$$

converge to the optimal access rates, i.e., $\alpha(t)\to\alpha^*$.

Proof. Consider the reformulation of problem (12)-(13) in (20)-(22), the Lagrangian function $L(\alpha,\nu)$ defined in (23), the dual function $g(\nu)$ defined in (25), and the dual problem in (26).

Algorithm 1 Distributed random access implementation

Initialize $\nu(0) \in \mathbb{R}_+^{m\times m}$ at the access point/controller, $t \leftarrow 0$

loop At period t

The access point/controller sends to each sensor i the values $\nu_{ii}(t)$, $\sum_{j\neq i}\nu_{ji}(t)$.

Each sensor i computes

$$\alpha_i(t) \leftarrow \left[\frac{\nu_{ii}(t)}{p_i + \sum_{j\neq i}\nu_{ji}(t)}\right]_{\mathcal{A}_i} \tag{31}$$

and for the rest of the period t it accesses the channel with rate $\alpha_i(t)$.

The access point/controller measures the access rates $\alpha_i(t)$ selected by all sensors $i=1,\ldots,m$ during the period and computes the auxiliary variables

$$\alpha_{ji}(t) \leftarrow \left[\frac{1}{q_{ji}} - \frac{\nu_{ii}(t)}{\nu_{ij}(t)}\right]_{\mathcal{A}_j} \quad \text{for all } j\neq i, \tag{32}$$

and the matrix $s(t)\in\mathbb{R}^{m\times m}$ with diagonal elements

$$s_{ii}(t) \leftarrow \log(c_i) - \log(\alpha_i(t)\, q_{ii}) - \sum_{j\neq i}\log\Big(1-\alpha_{ji}(t)q_{ji}\Big) \tag{33}$$

for all $i\in\{1,\ldots,m\}$, and offdiagonal elements

$$s_{ij}(t) \leftarrow \alpha_j(t) - \alpha_{ji}(t) \tag{34}$$

for all $i,j\in\{1,\ldots,m\}$, $i\neq j$.

The access point/controller computes the new matrix

$$\nu(t+1) \leftarrow \Big[\nu(t) + \varepsilon(t)\, s(t)\Big]_+ \tag{35}$$

where $[\ ]_+$ denotes the elementwise projection to the non-negatives $\mathbb{R}_+^{m\times m}$.

end loop

We argue that the matrix $s(t)$ computed by the algorithm in (33)-(34) is a subgradient direction of the dual function $g(.)$ at the point $\nu(t)$, i.e., it satisfies

$$g(\nu') - g(\nu(t)) \leq \mathrm{Tr}((\nu' - \nu(t))\, s(t)) \tag{37}$$

for all $\nu' \in \mathbb{R}_+^{m\times m}$. This can be shown as follows.

The values $\alpha_i(t)$ selected during the algorithm in (31) are minimizers of the Lagrangian at $\nu(t)$ because they satisfy the first order condition $\partial L/\partial\alpha(\alpha,\nu(t)) = 0$ where the gradient is derived in (29). Similarly $\alpha_{ji}(t)$ in (32) are Lagrangian minimizers at $\nu(t)$ (cf.(30)). Hence the dual function at $\nu(t)$, which is the minimum of the Lagrangian, equals $g(\nu(t)) = L(\alpha(t),\nu(t))$.

Moreover the values $s(t)$ computed in (31)-(32) using the Lagrangian minimizers $\alpha(t)$ are directly interpreted as the slacks of $\alpha(t)$ in the primal constraints (21)-(22). As a result, the dual function $g(\nu(t)) = L(\alpha(t),\nu(t))$ can be equivalently written as

$$g(\nu(t)) = \sum_{i=1}^{m}\alpha_i(t)p_i + \mathrm{Tr}(\nu(t)s(t)), \tag{38}$$

where we substituted the constraint slack $s(t)$ in the right hand side of the Lagrangian $L(\alpha(t), \nu(t))$ by (23).

Then at any point ν' we have by definition of the dual function in (25) that $g(\nu') \leq L(\alpha(t), \nu')$, and using again the constraint slack interpretation of $s(t)$ we have

$$g(\nu') \leq \sum_{i=1}^{m} \alpha_i(t) p_i + \mathrm{Tr}(\nu' s(t)). \tag{39}$$

Subtracting (38) from (39) by sides yields (37).

To sum up, at each iteration of the algorithm, the dual variable $\nu(t)$ according to (35) moves towards a subgradient direction of the dual function $g(\nu(t))$. Additionally the subgradient $s(t)$ is bounded due to the restriction $\alpha(t) \in \mathcal{A}'$. More precisely, by the implicit assumption that the access rates sets in (14) are such that the right hand sides of (13) are strictly positive, the logarithms in (33) are finite. Under the bounded subgradient condition, convergence of $\nu(t)$ to the optimal dual variable ν^* for stepsizes in (36) relies on standard subgradient method arguments – see, e.g., [21, Prop. 8.2.6] for a proof.

We close by showing that $\nu(t) \to \nu^*$ implies $\alpha(t) \to \alpha^*$. This follows by continuity, because by Proposition 2 the optimal values α^* are provided as a continuous function of the optimal dual variables ν^* in (18). $\square$

Apart from converging to the optimal operating point, Algorithm 1 is easily implementable in the wireless control architecture of Fig. 1. The sensors decide upon their access rates independently without coordination among themselves. Moreover the sensors do not need to know the whole problem information, for example, what the other sensors are trying to achieve or what control loops they are operating on. In fact each sensor does not even need to know how many other sensors are sharing the same wireless medium. That is because sensor i only needs to know how much collision effect it causes on all other sensors collectively (captured by the value of the sum $\sum_{j\neq i} \nu_{ji}(t)$).

The access point/controller, on the other hand, needs to know the packet success rates required for control performance of each control loop, as well as the channel collision pattern described by the values q_{ji}. It is worth noting that according to the algorithm the access point also needs to know all the sensor access rates $\alpha(t)$ at each iteration, which can be: 1) computed since the access point also knows all dual variables, 2) estimated online using the empirical packet receptions, 3) sent from each sensor to the access point within the transmitted packets.

The caveat of this distributed implementation is that it requires information exchange between sensors and the access point, hence it introduces some communication overhead. This overhead however burdens mainly the access point which is typically a base station with more capabilities compared to the simpler wireless sensors.

Remark 3. The problem formulation can be modified to include the case where some sensors are not interfering with each other. If sensor j never causes collisions to transmission i we have $q_{ji} = 0$. Define the subset of sensors that affect

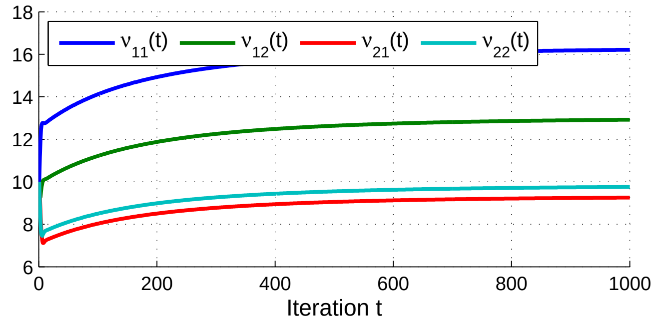

Fig. 2. Evolution of dual variables during the optimization algorithm. The elements of the matrix $\nu(t)$ converge to the optimal values ν^* required to obtain the optimal sensor access rates.

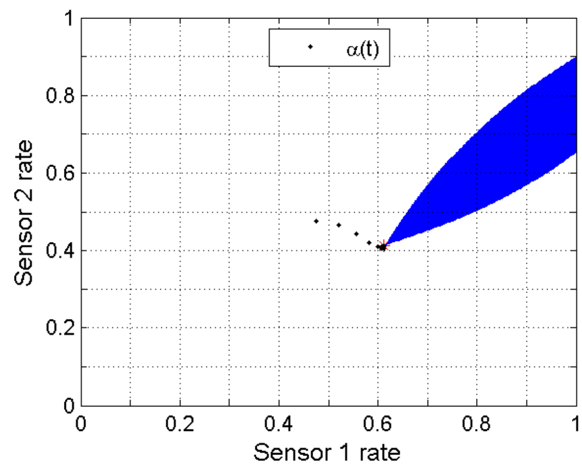

Fig. 3. Sensor access rates for the numerical example in Section V. The feasible set of sensor access rates that meet the control performance requirements of the two control systems is shown in shaded. After few iterations the access rates $\alpha(t)$ selected by the optimization algorithm converge close to the feasible point with the lowest utilization.

link i as $I_i = \{j \neq i : q_{ji} > 0\}$, and conversely the subset of links that are affected by sensor i as $O_i = \{j \neq i : q_{ij} > 0\}$. Then the packet success probability in (5) is modified so that the product is over the interfering sensors $\prod_{j\in I_i}$. Similarly the optimal sensor access rates in (18) are modified to include the sum $\sum_{j\in O_i} \nu^*_{ji}$. Algorithm 1 is also modified so that no 'coupling' variables α_{ji}, ν_{ij} are needed when $j \notin I_i$. $\square$

V. Numerical Simulations

We present a numerical example of the random access design. As in Fig. 1 we consider $m = 2$ identical scalar control systems of the form (1), with open (unstable) and closed (stable) loop dynamics $A_{o,i} = 1.1$, $A_{c,i} = 0.4$ respectively. The two respective wireless sensors transmit to the access point/controller over a shared channel with success and collision parameters

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} 0.95 & 0.6 \\ 0.6 & 0.95 \end{bmatrix} \tag{40}$$

i.e., in isolation 5% of the messages are dropped, and collisions happen with probability 60% in simultaneous transmissions. The transmit powers are taken equal $p_i = 1$.

The systems and the channel are symmetric, but we model an asymmetric control performance requirement. A Lyapunov function $V_i(x) = x^2$ for each plant state is required to decrease with expected rates $\rho_1 = 0.75$ and $\rho_2 = 0.95$ respectively (cf. (6)). System 1 is more demanding, also shown by the required packet success rates $c_1 \approx 0.44$, $c_2 \approx 0.25$ of the two sensors, computed via (8).

We solve the random access design problem (12)-(13) by Algorithm 1, which as explained in Section IV solves the problem in the dual domain. The dual variables $\nu(t)$ of the algorithm converge as shown in Fig. 2. We also plot the evolution of the sensor access rates $\alpha(t)$ during the algorithm in Fig. 3, along with the set of all access rates that are feasible with respect to the control performance requirements (13). We observe that the sensor rate iterates $\alpha(t)$ start from infeasible values, and moves towards the extreme point of the feasible set with the lowest access rates, so that the power expenditure in (12) is minimized. In fact after only a few iterations of the algorithm the sensor access rates are very close to the optimal point, which is

$$\alpha_1^* \approx 0.61, \ \alpha_2^* \approx 0.41. \tag{41}$$

As expected, sensor 1 is accessing the shared channel at a higher rate than sensor 2 in order to achieve the more demanding control performance requirement of system 1. Moreover, both sensors access the channel at a rate higher than the necessary packet success rates, i.e., $\alpha_i^* > c_i$. This happens because the sensors need to counteract the effect of packet collisions, as well as packet drops due to decoding errors. In comparison to an ideal channel without collisions (but with packet drops) where each sensor would access the channel at rates c_i/q_{ii}, the increase in channel access is 47% for sensor 1, and 75% for sensor 2.

VI. Concluding remarks

We design a random access mechanism for sensors transmitting measurements of multiple plants over a shared wireless channel to a controller. The goal of the design is to mitigate the effect of packet collisions from simultaneous transmissions and to guarantee control performance for all control systems. Via a Lyapunov function abstraction, control performance is transformed to required packet success rates of each closed loop. We show that the optimal random access design has a form decoupled between the sensors, and we develop a distributed procedure to obtain the optimal design.

An advantage of the distributed random access design procedure which we aim to explore in future research is the ability to track and adapt to changes in the problem parameters, for example, changes to the channel collision pattern, control performance requirements, or the admission of new control loops in the architecture. Future work also includes opportunistic adaptation to channel fading as in our centralized scheduling in [7], as well as adaptation to plant states as in, e.g., the scheduling in [5], the single link case in [17], or the remote estimation in [12]. Furthermore we aim to generalize the presented random access mechanism in other cases where control/estimation objectives can be abstracted in desired packet success rates, as in, e.g., the intermittent Kalman filter of [22].

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