INTERPOLATION OF GRAPH SIGNALS USING SHIFT-INVARIANT GRAPH FILTERS

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ABSTRACT

New schemes to recover signals defined in the nodes of a graph are proposed. Our focus is on reconstructing bandlimited graph signals, which are signals that admit a sparse representation in a frequency domain related to the structure of the graph. The schemes are designed within the framework of linear shift-invariant graph filters and consider that the seeding signals are injected only at a subset of interpolating nodes. After several sequential applications of the graph-shift operator –which computes linear combinations of the information available at neighboring nodes– the seeding signals are diffused across the graph and the original bandlimited signal is eventually recovered. Conditions under which the recovery is feasible are given, and the corresponding schemes to recover the signal are proposed. Connections with the classical interpolation in the time domain are also discussed.

Index Terms— Graph signal processing, Interpolation, Signal reconstruction, Graph shift operator, Graph filter

1. INTRODUCTION

Interpolation is a cornerstone problem in classical signal processing. With the emergence of new fields of knowledge such as network science and big data, there is a pressing need to extend the results existing for classical time-varying signals to signals defined on graphs [1,2]. This not only entails modifying the algorithms currently available for time-varying signals, but also gaining intuition on what concepts are preserved (and lost) when a signal is defined, not in the classical time grid, but in a more general graph domain.

In our approach, interpolated signals are obtained as the output of a graph filter applied to a seeding signal. Graph filters are the generalization of the classical time invariant system when the signals are defined in a general graph structure as opposed to the classical time domain. Seeding signals are graph signals defined on a subset of the nodes in the graph. Hence, in order to define an interpolation scheme, two objects must be specified: the interpolating graph filter and the seeding signal. Different from both what occurs in the classical time domain and the usual approach followed for interpolating graph signals [3–6], we show that if graph filters are used as interpolators, the values of the seeding signals at the seeding nodes do not correspond with those of the signal to interpolate. Moreover, we study the conditions under which interpolation is possible, provide a scheme to locally achieve recovery of bandlimited signals, and illustrate the application of this scheme by reconstructing a graph signal.

Preliminary concepts are introduced in Section 2. The interpolation schemes and their recovery conditions are presented in Section 3. Section 4 discusses the connections with the interpolation of time-varying signals. Numerical results and concluding remarks in Sections 5 and 6 close the paper. [1]

2. BANDLIMITED GRAPH SIGNALS AND FILTERS

General notation and the concept of the graph-shift operator [2] are introduced in Section 2.1, bandlimited graph-signals in Section 2.2, and graph filters in Section 2.3.

2.1. General modeling considerations

Let $\mathcal{G}$ denote a directed graph with a set of nodes or vertices $\mathcal{N}$ (with cardinality N) and a set of links $\mathcal{E}$, such that if node i is connected to j, then $(i,j) \in \mathcal{E}$. Since $\mathcal{G}$ is directed, the set $\mathcal{N}_i := \{j \ \ |(j,i) \in \mathcal{E}\}$ stands for the (incoming) neighborhood of i. The focus of the paper is not on analyzing $\mathcal{G}$, but a graph signal defined on the set of nodes $\mathcal{N}$. Formally, such a signal can be represented as a vector $\mathbf{x} = [x_1, ..., x_N]^T \in \mathbb{R}^N$ (where the i-th component represents the value of the signal at node i) or, alternatively, as a function $f : \mathcal{N} \to \mathbb{R}$.

The graph $\mathcal{G}$ can be endowed with the so-called *graph-shift operator* $\mathbf{S}$ [2]. The shift $\mathbf{S}$ is a $N \times N$ matrix whose

* The authors' work was supported by NSF CCF-1217963.

† The author's work was supported by the Spanish Ministry of Economy grant No TEC2013- 41604-R.

[1] **Notation:** Generically, the entries of a matrix $\mathbf{X}$ and a (column) vector $\mathbf{x}$ will be denoted as X_{ij} and x_i; however, when contributing to avoid confusion, the alternative notation $[\mathbf{X}]_{ij}$ and $[\mathbf{x}]_i$ will be used. The notation ^{T} and ^{H} stands for transpose and transpose conjugate, respectively; $\text{diag}(\mathbf{x})$ is a diagonal matrix satisfying $[\text{diag}(\mathbf{x})]_{ii} = [\mathbf{x}]_i$; $\mathbf{e}_i$ is the ith $N \times 1$ canonical vector (all entries of $\mathbf{e}_i$ are zero except the ith one, which is one); $\mathbf{E}_K := [\mathbf{e}_1, ..., \mathbf{e}_K]$ is a tall matrix collecting the K first canonical vectors; and $\mathbf{0}$ is the all-zero matrix (when not clear from the context, a subscript indicating the dimensions will be used). The modulus (remainder) obtained after dividing x by N will be denoted as $\text{mod}_N(x)$.

entry S_{ji} can be non-zero only if $i = j$ or if $(i, j) \in \mathcal{E}$. Widely-used choices for $\mathbf{S}$ are the adjacency matrix of the graph [2] and the Laplacian [1]. The intuition behind $\mathbf{S}$ is to represent a linear transformation that can be computed locally at the nodes of the graph. More rigorously, if $\mathbf{y}$ is defined as $\mathbf{y} = \mathbf{S}\mathbf{x}$, then node i can compute y_i provided that it has access to the value of x_j for $j \in \mathcal{N}_i$. We assume that $\mathbf{S}$ can be decomposed as $\mathbf{S} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, where $\mathbf{\Lambda}$ is diagonal.

Remark: *Graph support of time signals.* To establish connections with time-varying signals, it is convenient to define here the directed-chain graph $\mathcal{G}_{dc}$, with node set $\mathcal{N} = \{1, 2, \ldots, N\}$ and edge set $\mathcal{E}_{dc} = \{(i, \mathrm{mod}_N(i) + 1)\}_{i=1}^{N}$. This graph is the natural support of (circulant) time varying signals [2]. Its adjacency and Laplacian matrices are denoted, respectively, as $\mathbf{A}_{dc}$ and $\mathbf{L}_{dc} := \mathbf{I} - \mathbf{A}_{dc}$. Setting $\mathbf{S}$ either to $\mathbf{A}_{dc}$ or to $\mathbf{L}_{dc}$ implies that $\mathbf{V}$ is equal to the Fourier basis $\mathbf{F}$, where $F_{ij} := \frac{1}{\sqrt{N}} e^{+\mathrm{j}\frac{2\pi}{N}(i-1)(j-1)}$ with $\mathrm{j} := \sqrt{-1}$. Setting $\mathbf{S} = \mathbf{A}_{dc}$ has the additional advantage of satisfying $\Lambda_{ii} = e^{-\mathrm{j}\frac{2\pi}{N}(i-1)}$, i.e., the eigenvalues of the shift operator correspond to the classical discrete frequencies.

2.2. Bandlimited graph signals

The common practice when addressing the problem of sampling signals in graphs is to assume that the graph-shift operator $\mathbf{S}$ plays a key role in explaining the signals of interest $\mathbf{x}$. More specifically, that $\mathbf{x}$ can be expressed as a linear combination of a *subset* of the columns of $\mathbf{V} = [\mathbf{v}_1, ..., \mathbf{v}_N]$, or, equivalently, that the vector $\widehat{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x}$ is sparse [7]. In this context, vectors $\mathbf{v}_i$ are interpreted as the graph frequency basis, $\widehat{x}_i$ as the corresponding signal frequency coefficients, and $\mathbf{x}$ as a K-bandlimited graph signal. The superscript $\widehat{\ }$ will be used to emphasize that the signal pertains to the frequency domain. We will assume that the set of active frequencies are known and, without loss of generality, that those are the first K ones. Then, with $\widehat{\mathbf{x}}_K := [\widehat{x}_1, \ldots, \widehat{x}_K]^T$ being a $K \times 1$ vector collecting the coefficients associated with the active frequencies, it holds that $\mathbf{x}$ is a K-bandlimited signal if

$$\widehat{\mathbf{x}} = [\widehat{x}_1, \ldots, \widehat{x}_K, 0, \ldots, 0]^T \tag{1}$$

$$\mathbf{x} = \mathbf{V}\widehat{\mathbf{x}} = \mathbf{V}_K\widehat{\mathbf{x}}_K, \tag{2}$$

where $\mathbf{V}_K := \mathbf{V}\mathbf{E}_K = [\mathbf{v}_1, ..., \mathbf{v}_K]$.

2.3. Graph filters

Let $\mathbf{H} : \mathbb{R}^N \to \mathbb{R}^N$ be a graph signal operator. We are interested in operators of the form $\mathbf{H} := \sum_{l=0}^{L-1} h_l \mathbf{S}^l$, i.e., in linear transformations that can be expressed as a polynomial of the graph-shift operator. This type of transformations are called shift-invariant graph filters [2]. Two of the main reasons to focus on such a class of transformations are: i) there exists a theory that connects those filters with the classical time-invariant filters, facilitating the analysis and design of $\mathbf{H}$; and ii) since the application of the graph operator to a graph signal can be computed locally (cf. related discussion in Section 2.1), transformations of the form $\sum_{l=0}^{L-1} h_l \mathbf{S}^l$ can be implemented locally too (for example, with $L - 1$ exchanges of information among neighbors).

Note that the graph filter $\mathbf{H}$ can be written as $\mathbf{H} := \mathbf{V}\big(\sum_{l=0}^{L-1} h_l \mathbf{\Lambda}^l\big)\mathbf{V}^{-1}$. The diagonal matrix $\widehat{\mathbf{H}} := \sum_{l=0}^{L-1} h_l \mathbf{\Lambda}^l$ can be viewed as the frequency response of $\mathbf{H}$ and it can be alternatively written as $\widehat{\mathbf{H}} = \mathrm{diag}(\widehat{\mathbf{h}})$, where vector $\widehat{\mathbf{h}}$ is a vector that contains the N frequency coefficients of the filter. By defining the $N \times L$ Vandermonde matrix

$$\mathbf{\Psi} := \begin{pmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{L-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_N & \ldots & \lambda_N^{L-1} \end{pmatrix} \tag{3}$$

and the vector containing the time coefficients of the filter as $\mathbf{h} := [h_0, \ldots, h_{L-1}]^T$, then $\widehat{\mathbf{h}} = \mathbf{\Psi}\mathbf{h}$ and therefore

$$\mathbf{H} = \textstyle\sum_{l=0}^{L-1} h_l \mathbf{S}^l = \mathbf{V}\mathrm{diag}\big(\mathbf{\Psi}\mathbf{h}\big)\mathbf{V}^{-1} = \mathbf{V}\mathrm{diag}(\widehat{\mathbf{h}})\mathbf{V}^{-1}. \tag{4}$$

Remark: *Frequency response in the time domain.* For the case of $\mathcal{G}_{dc}$, the shift $\mathbf{S}_{dc}$ has a strong structure that can be leveraged when designing signal processing algorithms for time-varying signals. For example, since $\mathbf{A}_{dc}$ and $\mathbf{L}_{dc}$ are circulant, $\mathbf{V}^{-1}$ and $\mathbf{V}$ correspond to $\mathbf{F}^H$ and $\mathbf{F}$. Similarly, when $\mathbf{S}_{dc} = \mathbf{A}_{dc}$, the Vandermonde matrix $\mathbf{\Psi}$ is $\sqrt{N}\mathbf{F}^H$. Matrix $\mathbf{F}$ and some of its submatrices are unitary and both row and column full-rank Vandermonde. As it will be shown in the ensuing sections, some of these useful properties are lost when considering signals defined in more general graph supports, rendering the interpolation problem more challenging.

3. INTERPOLATING SIGNALS USING A SUBSET OF SEEDING NODES

Consider a set of P nodes serving as seeds for interpolating the K-bandlimited graph signal $\mathbf{x}$. Since $\mathbf{x}$ (or equivalently $\widehat{\mathbf{x}}$) has K degrees of freedom, for the time being we will assume that $P \geq K$ (we will be more concrete later on). To facilitate exposition we will also assume, without loss of generality, that the seeding nodes are the first P ones $i = 1, \ldots, P$. Let $\bar{x}_i$ denote the value that node i injects into the network and, based on it, define the $P \times 1$ and $N \times 1$ seeding vectors as

$$\bar{\mathbf{x}}_P = [\bar{x}_1, \ldots, \bar{x}_P]^T \tag{5}$$

$$\bar{\mathbf{x}} = [\bar{x}_1, \ldots, \bar{x}_P, 0, \ldots, 0]^T. \tag{6}$$

Then, given a bandlimited signal $\mathbf{x}$, our goal is to design $\mathbf{H}$ and $\bar{\mathbf{x}}$ such that

$$\mathbf{x} = \mathbf{H}\bar{\mathbf{x}}, \tag{7}$$

where $\mathbf{H}$ has the particular structure of a shift-invariant graph filter [cf. Section 2.3]. Note that the values in $\bar{\mathbf{x}}$ are not forced to be the same than those in the signal to reconstruct $\mathbf{x}$. The

reason for doing this will become clear in Section 4. Since $\mathbf{x}$ is bandlimited, it is reasonable to write the previous equation in the frequency domain. To do this, both sides of (7) are left multiplied by $\mathbf{V}^{-1}$, which yields

$$\widehat{\mathbf{x}}=\mathbf{V}^{-1}\mathbf{H}\bar{\mathbf{x}}=\mathbf{V}^{-1}\mathbf{V}\text{diag}(\mathbf{\Psi h})\mathbf{V}^{-1}\bar{\mathbf{x}}=\text{diag}(\mathbf{\Psi h})\widehat{\bar{\mathbf{x}}}, \quad (8)$$

where for the second equality we have used (4). We will now use the fact that $\bar{\mathbf{x}}$ is sparse [cf. (6)]. To do so, let us define the $N \times P$ matrix $\mathbf{U}_P := \mathbf{V}^{-1}\mathbf{E}_P$. Then, the frequency representation of the seeding signal $\bar{\mathbf{x}}$ is

$$\widehat{\bar{\mathbf{x}}} = \mathbf{V}^{-1}\bar{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{E}_P\bar{\mathbf{x}}_P = \mathbf{U}_P\bar{\mathbf{x}}_P. \quad (9)$$

Our initial goal of designing $\mathbf{H}$ and $\bar{\mathbf{x}}$ such that $\mathbf{x} = \mathbf{H}\bar{\mathbf{x}}$ can be reformulated as designing $\mathbf{h}$ and $\bar{\mathbf{x}}_P$ such that

$$\widehat{\mathbf{x}} = \text{diag}\big(\mathbf{\Psi h}\big)\big(\mathbf{U}_P\bar{\mathbf{x}}_P\big), \quad (10)$$

which is a bilinear system of N equations and $L+P$ variables. Define the row-selection matrices $\mathbf{C} = [\mathbf{e}_{K+1}, \ldots, \mathbf{e}_N]^T$ and $\bar{\mathbf{C}} = (\mathbf{E}_K)^T$. Then, leveraging the fact of $\widehat{\mathbf{x}}$ being sparse [cf. (1)], the N equations in (10) can be split into two groups

$$\mathbf{0}_{N-K}=\mathbf{C}\text{diag}(\mathbf{\Psi h})\big(\mathbf{U}_P\bar{\mathbf{x}}_P\big)=\text{diag}(\mathbf{C\Psi h})\big(\mathbf{C}\mathbf{U}_P\bar{\mathbf{x}}_P\big) \quad (11)$$

$$\widehat{\mathbf{x}}_K=\bar{\mathbf{C}}\text{diag}(\mathbf{\Psi h})\big(\mathbf{U}_P\bar{\mathbf{x}}_P\big)=\text{diag}(\bar{\mathbf{C}}\mathbf{\Psi h})\big(\bar{\mathbf{C}}\mathbf{U}_P\bar{\mathbf{x}}_P\big). \quad (12)$$

Note that the conditions in (11) are the same for any K-bandlimited signal. On the other hand, the conditions in (12) depend on the specific signal to be interpolated. A natural approach is to use the filter coefficients $\mathbf{h}$ –which are related to the global behavior of the network– to guarantee that (11) holds, while using the seeding signal $\bar{\mathbf{x}}_P$ to satisfy (12) and, hence, to guarantee that the output of the interpolation is $\mathbf{x}$. This is explored in Sections 3.1 and 3.2.

3.1. Low-pass interpolation

The conditions under which the approach just described is guaranteed to find a feasible solution are given in the form of two propositions. The proofs are not complicated, but due to space limitations we include them in an online appendix[2] and not in the main body of the paper. Ensuing discussions describe the actual procedure to interpolate the signal.

Proposition 1 *Let D denote the number of distinct eigenvalues in $\{\lambda_k\}_{k=K+1}^{N}$. Then, it holds that if $L > D$, for any $\mathbf{\Psi}$ there exist infinitely many nonzero $L \times 1$ vectors $\mathbf{h}^*$ such that, after setting $\mathbf{h} = \mathbf{h}^*$, (11) is satisfied for all $\mathbf{U}_P$ and $\bar{\mathbf{x}}_P$.*

Since $\mathbf{\Psi}$ is a Vandermonde matrix, it can be shown that for $L > D$, $\mathbf{C\Psi}$ is rank deficient and the dimension of its kernel space is $L - D$. Hence, setting $\mathbf{h}$ to any nonzero element of the kernel space will satisfy (11). By default, we will assume that $L = D + 1$ and set the coefficients $\mathbf{h}^*$ that solve (11) equal to the unit vector that spans the one-dimensional kernel of $\mathbf{C\Psi}$. Note also that for the case where all the eigenvalues are distinct, we need $L > N - K$. An alternative way to design the filter coefficients that annihilate a specific set of frequencies is to use the "successive nulling of eigenvalues" approach in [8], which relies on a slightly different definition of a graph filter.

Once the coefficients of the filter are designed, the next step is to find the optimum seeding signal. Substituting $\mathbf{h} = \mathbf{h}^*$ into (12) yields

$$\widehat{\mathbf{x}}_K = \text{diag}\big(\bar{\mathbf{C}}\mathbf{\Psi h}^*\big)\big(\bar{\mathbf{C}}\mathbf{U}_P\bar{\mathbf{x}}_P\big). \quad (13)$$

Then, the following result holds.

Proposition 2 *The system of K equations in (13) is guaranteed to have a solution w.r.t. $\bar{\mathbf{x}}_P$ if the two following conditions hold: i) $\lambda_{k_1} \neq \lambda_{k_2}$ for any $(\lambda_{k_1}, \lambda_{k_2})$ pair such that $k_1 \leq K$ and $k_2 > K$; and ii) $\text{rank}(\bar{\mathbf{C}}\mathbf{U}_P) \geq K$.*

Different from the time domain, where all the eigenvalues of the shift operator (frequencies) are distinct, in the more general graph domain there can be graph topologies that give rise to shift operators with repeated eigenvalues. Condition *i)* states that: a) if the graph has such a property; and b) the signal of interest is zero in some of the repeated frequencies and non-zero in others, then the interpolation procedure described in this section will fail. The reason is that the graph filters $\mathbf{H}$ always give the same frequency response if the corresponding eigenvalues are the same. Therefore, it is not possible for $\mathbf{H}$ to annihilate one of the frequencies without annihilating the other. We will see that a way to fix this problem is to use the schemes in Section 3.2. Condition *ii)* requires the rank of the $K \times P$ matrix $\bar{\mathbf{C}}\mathbf{U}_P$ being at least K. At the very least, this requires P, the number of seeding nodes, to be equal to K, the number of frequencies present in $\mathbf{x}$. However, there may be cases where setting $P = K$ will fail. To see why this is true, notice that $[\mathbf{U}_P]_{k,p}$ can be viewed as how strongly node p expresses frequency k. Suppose for example that there exists a k such that that $[\mathbf{U}_P]_{k,p} = 0$ for all nodes $p = 1, \ldots, P$, then it is not possible to reconstruct a signal whose kth frequency coefficient $[\widehat{\mathbf{x}}]_k$ is non-zero using that set of nodes. This problem is also present when sampling graph signals by observing the value of the signal in a subset of nodes [9].

3.2. Augmented low-pass interpolation

The schemes in Section 3.1 implement a graph filter of order $L-1$. For the case of a shift operator without repeated eigenvalues, this implies that $L-1$ has to be at least $N-K$, which can be a high number for large networks. Since the order of $\mathbf{H}$ corresponds to the number of applications of $\mathbf{S}$, when the filter is implemented in a distributed setup, $L-1$ will account for the number of times every node needs to exchange information with its neighbors. Although not all nodes in the network will have to implement $L-1$ exchanges (nodes that are

[2] http://www.seas.upenn.edu/∼ssegarra

far from the seeding nodes will not be required to exchange information until the "diffused" seeding signal reaches them), the signaling overhead can be a problem. In this context it is natural to ask for solutions that reduce the order of the filter by increasing the number of seeding nodes P.

Mathematically, the problem to solve is very similar to the one in Section 3.1. Suppose that L is given and use this value to define the augmented $(N-L+1)\times 1$ frequency vector as

$$\widehat{\mathbf{x}}_{K,L} := [\widehat{x}_1, ..., \widehat{x}_K, 0, ..., 0]^T. \tag{14}$$

The problem is now to find $\mathbf{h}$ and $\bar{\mathbf{x}}_P$ such that [cf. (11)-(12)]

$$\mathbf{0}_{L-1} = \text{diag}\big(\mathbf{C}\mathbf{\Psi}\mathbf{h}\big)\big(\mathbf{C}\mathbf{U}_P\bar{\mathbf{x}}_P\big) \tag{15}$$

$$\widehat{\mathbf{x}}_{K,L} = \text{diag}\big(\bar{\mathbf{C}}\mathbf{\Psi}\mathbf{h}\big)\big(\bar{\mathbf{C}}\mathbf{U}_P\bar{\mathbf{x}}_P\big), \tag{16}$$

where $\mathbf{C}$ and $\bar{\mathbf{C}}$ have $L-1$ and $N-L+1$ rows, respectively. The filter coefficients must be obtained now to annihilate the $L-1$ frequencies in (15) and the P seeding nodes must inject a signal such that, when multiplied by the frequency response of the filter, it replicates the frequency pattern in $\widehat{\mathbf{x}}_{K,L}$.

In (15) the filter is used to annihilate $L-1$ frequencies; however, if the shift operator $\mathbf{S}$ has repeated eigenvalues, more frequencies can be annihilated with the same filter degree. This reduces the required number P of seeding nodes. To be more concrete, recall that D stands for the number of distinct eigenvalues in $\{\lambda_k\}_{K+1}^N$, let m_d denote the multiplicity of each of the distinct eigenvalues (with $\sum_{d=1}^D m_d = N-K$) and assume that the indices d are chosen such that $m_d \geq m_{d+1}$. Suppose now that the filter is designed to annihilate the frequencies corresponding to the $L-1$ first distinct eigenvalues; i.e., those corresponding to $d=1, ..., L$–1. Then, $L' := \sum_{d=1}^{L-1} m_d \geq L$–1 frequencies will be annihilated. The signal in the seeding nodes is designed now to recover $\widehat{\mathbf{x}}_{K,L'}$ [cf. (16)], which is a smaller vector than $\widehat{\mathbf{x}}_{K,L}$, reducing the required number P of seeding nodes.

4. RELATING GRAPH AND TIME INTERPOLATION

In the classical time domain, interpolation of a sampled signal is performed using a sinc function, which corresponds to an ideal low-pass filter $\widehat{\mathbf{H}}_{LP}$ whose bandwidth is related to that of the bandlimited signal $\mathbf{x}$. If the sampling is performed at the minimum possible rate, the bandwidth of $\widehat{\mathbf{H}}_{LP}$ has to be exactly the same than that of $\mathbf{x}$. If the signal is oversampled, the bandwidth of $\widehat{\mathbf{H}}_{LP}$ can be larger. In other words, if more samples than the minimum required number are available, then $\widehat{\mathbf{H}}_{LP}$ does not need to cancel all the frequencies that are not present in $\mathbf{x}$. The schemes just presented reveal that this is also the case when the signals are defined in more general graph domains.

The main differences between the reconstruction schemes presented and classical time interpolation come from the fact that the basis $\mathbf{V}$ of a general graph shift $\mathbf{S}$ is not as structured as the Fourier basis $\mathbf{F}$. Among these differences, we remark the fact that for general graphs the seeding values $\bar{\mathbf{x}}_P$ do not coincide with the values of the desired signal $\mathbf{x}$. This contrasts with the case of interpolation from uniform sampling for classical time varying signals where $\bar{\mathbf{x}}_P$ is a subset of the signal $\mathbf{x}$. In fact, it can be rigorously shown that requiring such a condition for general graphs would lead to an infeasible interpolation. To be concrete, suppose that $P=K$, define the $K\times 1$ vector $\bar{\mathbf{x}}_K := \mathbf{E}_K^T\mathbf{x}$ and, based on it the $N\times 1$ vector $\bar{\mathbf{x}} := [\bar{\mathbf{x}}_K^T, \mathbf{0}]^T$. Recall that we want to design a filter $\mathbf{H}$ that using the fact that $\mathbf{x}$ is sparse in the frequency domain achieves that $\mathbf{x} = \mathbf{H}\bar{\mathbf{x}}$. Firstly, we leverage the fact that $\bar{\mathbf{x}}$ and $\widehat{\mathbf{x}}$ are sparse to write

$$\bar{\mathbf{x}}_K = \mathbf{E}_K^T\mathbf{x} = \mathbf{E}_K^T\mathbf{V}\widehat{\mathbf{x}} = \mathbf{E}_K^T\mathbf{V}\mathbf{E}_K\widehat{\mathbf{x}}_K = \mathbf{V}_{K,K}\widehat{\mathbf{x}}_K, \tag{17}$$

where $\mathbf{V}_{W,Q} := \mathbf{E}_W^T\mathbf{V}\mathbf{E}_Q$ is the submatrix of the basis $\mathbf{V}$ formed by the first W rows and Q columns. Secondly, we write the goal of $\mathbf{x} = \mathbf{H}\bar{\mathbf{x}}$ into the frequency domain as $\widehat{\mathbf{x}} = \text{diag}(\widehat{\mathbf{h}})\mathbf{V}^{-1}\bar{\mathbf{x}}$ and use again the sparsity of $\bar{\mathbf{x}}$ and $\widehat{\mathbf{x}}$ to write

$$\begin{aligned}\widehat{\mathbf{x}}_K &= \mathbf{E}_K^T\text{diag}(\widehat{\mathbf{h}})\mathbf{V}^{-1}\mathbf{E}_K\bar{\mathbf{x}}_K = \text{diag}(\mathbf{E}_K^T\widehat{\mathbf{h}})\mathbf{E}_K^T\mathbf{V}^{-1}\mathbf{E}_K\bar{\mathbf{x}}_K \\ &= \text{diag}(\widehat{\mathbf{h}}_K)\mathbf{U}_{K,K}\bar{\mathbf{x}}_K,\end{aligned} \tag{18}$$

where $\widehat{\mathbf{h}}_K := \mathbf{E}_K^T\widehat{\mathbf{h}}$ contains the first K components of $\widehat{\mathbf{h}}$ and $\mathbf{U}_{W,Q} := \mathbf{E}_W^T\mathbf{V}^{-1}\mathbf{E}_Q$. Substituting (17) into (18) yields

$$\widehat{\mathbf{x}}_K = \text{diag}(\widehat{\mathbf{h}}_K)\mathbf{U}_{K,K}\mathbf{V}_{K,K}\widehat{\mathbf{x}}_K. \tag{19}$$

Since (19) must hold for all $\widehat{\mathbf{x}}_K$, it can only be satisfied if $\text{diag}(\widehat{\mathbf{h}}_K)\mathbf{U}_{K,K}\mathbf{V}_{K,K}$=$\mathbf{I}$. This requires matrix $(\mathbf{U}_{K,K}\mathbf{V}_{K,K})$ to be diagonal. While this is true when $K=N$, this is not true for a general K. However, in the time domain where $\mathbf{V}=\mathbf{F}$, for some cases the multiplication of sub matrices is guaranteed to be diagonal. For example, when the K seeding nodes are equally spaced, the submatrices $\mathbf{U}_{K,K}$ and $\mathbf{V}_{K,K}$ are also Fourier and then it follows that $\mathbf{U}_{K,K}\mathbf{V}_{K,K} = K/N\,\mathbf{I}$. This not only implies that (19) is satisfied, but also reveals that all the entries in $\widehat{\mathbf{h}}_K$ must be set to N/K. In words, that the optimal low-pass interpolator in the time domain has the same response for all the active frequencies.

5. NUMERICAL RESULTS

Consider the 20-node graph $\mathcal{G}$ depicted in Fig. 1 and define the associated graph-shift operator $\mathbf{S}$ such that $S_{ii}=1$ for all i and $S_{ji}=-1$ for all (i,j) in the edge set $\mathcal{E}$ of $\mathcal{G}$. Moreover, consider the signal $\mathbf{x}$ on $\mathcal{G}$ depicted in Fig. 1(left) by the color of the nodes. Although seemingly random in the graph domain, the structure of the signal $\mathbf{x}$ is highly determined by the graph. To illustrate this, in Fig. 2 we present the frequency components $\widehat{\mathbf{x}}$ of the signal in the graph frequency basis given by the columns of $\mathbf{V}$ (which are the eigenvectors of the operator $\mathbf{S}$) [cf. (2)]. From Fig. 2, it is immediate that $\mathbf{x}$ has a bandwidth of 5, thus, following the procedure described in Section 3.1, we interpolate this signal using 5 seeding nodes.

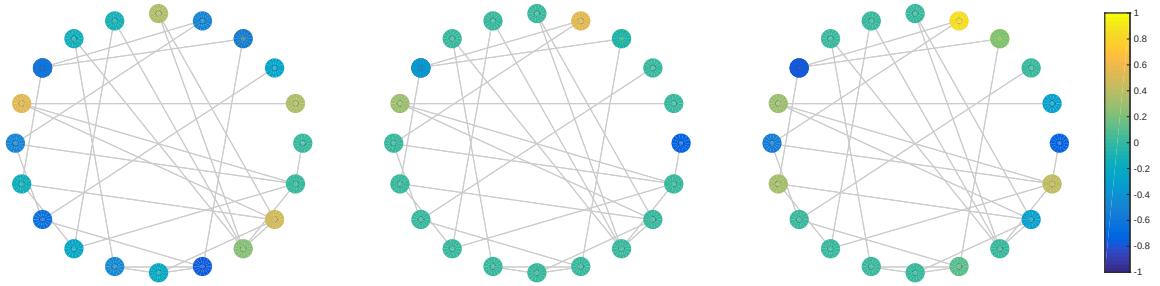

Fig. 1: (left) The graph $\mathcal{G}$ and the signal $\mathbf{x}$ defined on it given by the node colors. (mid) The seeding signal $\bar{\mathbf{x}}$ used to reconstruct $\mathbf{x}$. The number of seeding nodes equals the bandwidth of the signal [cf. Fig. 2]. (right) One shift of the seeding signal $\mathbf{S}\bar{\mathbf{x}}$. The weighted sum of successive shifts leads to perfect reconstruction of $\mathbf{x}$, where the weights are given by the optimal filter coefficients $\mathbf{h}^*$.

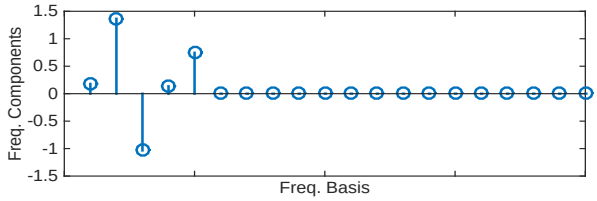

Fig. 2: Frequency components $\widehat{\mathbf{x}}$ of graph signal $\mathbf{x}$ [cf. Fig. 1(left)].

In Fig. 1(mid) we present the seeding signal $\bar{\mathbf{x}}$. Notice that this signal is non-zero only for the five seeding nodes. Further notice that the value of the seeding signal $\bar{\mathbf{x}}$ in the seeding nodes differs from the value of the original signal $\mathbf{x}$ in the seeding nodes [cf. Fig. 1(left)], as discussed in Section 4. The seeding signal propagates through the graph by local interactions given by the shift operator $\mathbf{S}$. E.g., in Fig. 1(right) we can observe the value of $\mathbf{S}\bar{\mathbf{x}}$ after one application of the shift, where only the seeding nodes and their neighbors present a non-zero signal value. The original signal $\mathbf{x}$ is then recovered by a weighted sum of successive applications of $\mathbf{S}$ to the seeding signal $\bar{\mathbf{x}}$ where the weights are given by the optimal filter coefficients [cf. Proposition 1]. Indeed, the fact that all eigenvalues of $\mathbf{S}$ are distinct ensures the fulfillment of Proposition 1 for a filter of degree 15 as well as the fulfillment of condition *i)* in Proposition 2. By simply checking that $\text{rank}(\bar{\mathbf{C}}\mathbf{U}_5) = 5$ [cf. condition *ii)* in Proposition 2], we are guaranteed perfect reconstruction, as observed in practice.

6. CONCLUSIONS

This work designed schemes for the interpolation of bandlimited graph signals using data injected by a subset of nodes. The schemes can be viewed as implementations of a graph filter –a polynomial of the graph-shift operator– that accounts for the sparse structure of the graph and can be implemented distributedly. We showed that a K-bandlimited signal can be recovered using K nodes if the interpolator corresponds to a low-pass filter in the (graph) frequency domain. In contrast to classical time-varying signals, it was also shown that if the interpolating nodes use as seeds the values of the original signal in those nodes, the interpolation is not feasible.

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