

# STABILITY AND CONTINUITY OF CENTRALITY MEASURES IN WEIGHTED GRAPHS

Santiago Segarra

Alejandro Ribeiro

Department of Electrical and Systems Engineering, University of Pennsylvania

## ABSTRACT

This paper introduces a formal definition of continuity and generalizes an existing notion of stability for node centrality measures in weighted graphs. It is shown that the frequently used measures of degree, closeness and eigenvector centrality are stable and continuous whereas betweenness centrality is neither. Numerical experiments in synthetic and real-world networks show that both stability and continuity are desirable in practice since they imply different levels of robustness in the presence of noisy data. In particular, a stable alternative of betweenness centrality is shown to exhibit resilience against noise while preserving its notion of centrality.

**Index Terms**— Networks, graphs, centrality, continuity, stability.

## 1. INTRODUCTION

In any graph or network, the topology determines an influence structure among the nodes or agents. Peripheral nodes have limited impact on the dynamics of the network whereas central nodes have a major effect on the behavior of the whole graph, e.g. migration in biological networks [1]. Moreover, recognizing the most influential nodes in a network helps in designing optimal ways to influence it, e.g. attack vulnerability of networks [2]. Node centrality measures are tools designed to identify such important agents. However, node importance is a rather vague concept and can be interpreted in various ways, giving rise to multiple coexisting centrality measures, the most common being degree [3,4], closeness [5,6], eigenvector [7], and betweenness [8] centrality.

The ability of a centrality measure to be robust to noise in the network data is of practical importance. Stability has been utilized to compare centrality measures in the past [9–11]. In these papers, an empirical approach was followed by comparing stability indicators measured in both random and real-world networks for different centrality measures. However, no formal theory was developed explaining the different behaviors among measures. [12] represents the first attempt towards formalizing the stability of centrality measures in networks, where betweenness centrality was shown to be unstable and a stable alternative was proposed.

Our first contribution is the generalization of this notion of stability from complete networks to arbitrary directed graphs. We then show that degree, closeness, and eigenvector centrality are stable measures (Section 3). Moreover, we introduce the concept of continuity as a milder requirement for robustness and analyze the continuity of frequently used centrality measures (Section 4). Finally, through numerical experiments in synthetic and real-world networks, we illustrate how stability and continuity are correlated with practical robustness indicators and show that the alternative definition of betweenness centrality behaves better than the standard one while preserving a similar notion of centrality (Section 5).

The proofs of Propositions 1 to 8 can be found in [13].

## 2. PRELIMINARIES

We define a directed graph or network  $G = (V, E, W)$  as a triplet formed by a finite set of  $n$  nodes or vertices  $V$ , a set of directed edges  $E \subset V \times V$  where  $(x, y) \in E$  represents an edge from  $x \in V$  to  $y \in V$ , and a set of positive weights  $W : E \rightarrow \mathbb{R}_{++}$  defined on each edge. The weights can be associated to similarities between nodes, i.e. the higher

the weight the more similar the nodes are, or dissimilarities, depending on the application. The graphs considered here do not contain self-loops, i.e.,  $(x, x) \notin E$  for all  $x \in V$ . For any given sets  $V$  and  $E$ , denote by  $\mathcal{G}_{(V,E)}$  the space of all graphs with  $V$  as node set and  $E$  as edge set. An alternative representation of a graph is through its adjacency matrix  $A \in \mathbb{R}^{n \times n}$ . If there exists an edge from node  $i$  to node  $j$ , then  $A_{ij}$  takes the value of the corresponding weight. Otherwise,  $A_{ij}$  is null.

Given a graph  $(V, E, W)$  and  $x, x' \in V$ , a path  $P(x, x')$  which starts at  $x$  and finishes at  $x'$  is an *ordered* sequence of nodes,  $P(x, x') = [x = x_0, x_1, \dots, x_l = x']$ , such that  $e_i = (x_i, x_{i+1}) \in E$  for  $i = 0, \dots, l-1$ . Specifically when  $W$  is associated to dissimilarities, we define the length of a given path as the sum of the weights encountered when traversing its links in order. We define the shortest path function  $s_G : V \times V \rightarrow \mathbb{R}_+$  where the shortest path length  $s_G(x, x')$  between nodes  $x, x' \in V$  is

$$s_G(x, x') := \min_{P(x, x')} \sum_{i=0}^{l-1} W(x_i, x_{i+1}). \quad (1)$$

## 3. NODE CENTRALITY AND STABILITY

Node centrality is a measure of the importance of a node given its location within a graph. More precisely, given a graph  $(V, E, W)$ , a centrality measure  $C : V \rightarrow \mathbb{R}_+$  assigns a nonnegative centrality value to every node such that the higher the value the more central the node is. Ideally, this detection should be invariant to small perturbations in the edge weights.

To formalize this notion of robustness against perturbations, we define the metric  $d_{(V,E)} : \mathcal{G}_{(V,E)} \times \mathcal{G}_{(V,E)} \rightarrow \mathbb{R}_+$  on the space of graphs  $\mathcal{G}_{(V,E)}$  containing  $V$  as node set and  $E$  as edge set, as follows

$$d_{(V,E)}(G, H) := \sum_{e \in E} |W(e) - W'(e)| = \sum_{i,j} |A_{ij} - A'_{ij}|, \quad (2)$$

where  $G = (V, E, W)$  and  $H = (V, E, W')$ , and have  $A$  and  $A'$  as adjacency matrices, respectively.  $d_{(V,E)}$  is a well-defined metric since it is the  $\ell_1$  distance between two vectors obtained by stacking the values in  $W$  and  $W'$ . This metric enables the following definition of stability.

**Definition 1** A centrality measure  $C$  is stable if, for every vertex set  $V$ , edge set  $E$  and any two graphs  $G, H \in \mathcal{G}_{(V,E)}$ ,

$$|C^G(x) - C^H(x)| \leq K_G d_{(V,E)}(G, H), \quad (3)$$

for every  $x \in V$ , where  $K_G$  is a constant for every graph  $G$ ,  $C^G(x)$  is the centrality value of node  $x$  in graph  $G$  and similarly for  $H$ .

The above definition states that a centrality measure is stable if the difference in centrality values for a given node in two different graphs is bounded by a constant  $K_G$  times the distance between these graphs. Definition 1 extends the one in [12] from complete networks to arbitrary directed graphs. The constant  $K_G$  depends on the whole graph  $G$ , as opposed to just its size [12], and must be valid for every graph  $H$ . In particular, if  $H$  is a perturbed version of  $G$ , any stable centrality measure ensures that the change in centrality due to this perturbation is bounded. This generates a robust measure in the presence of noise as we illustrate through examples in Section 5. In the following sections we analyze the stability of several centrality measures.

### 3.1. Degree centrality

Degree centrality is a local measure of the importance of a node within a graph. In directed graphs  $(V, E, W)$ , degree centrality of a node  $x$  is unfolded into two different measures, out-degree  $C_{OD}$  and in-degree  $C_{ID}$  centrality, computed as

$$C_{OD}(x) := \sum_{x' | (x, x') \in E} W(x, x'), \quad C_{ID}(x) := \sum_{x' | (x', x) \in E} W(x', x). \quad (4)$$

The out-degree centrality of  $x$  is given by the sum of the weights of the edges that originate in  $x$  whereas the in-degree centrality is given by the sum of the weights of the edges that finish in  $x$ . For undirected graphs, both notions coincide and we call them degree centrality  $C_D$ . Although the degree centrality measure has a number of limitations related to its locality [14], it is stable as we state next.

**Proposition 1** *The degree  $C_D$ , out-degree  $C_{OD}$  and in-degree  $C_{ID}$  centralities in (4) are stable as defined in Definition 1 with  $K_G = 1$ .*

The degree centrality measure is applied to similarity graphs. In this way, a high degree centrality value of a given node means that this node has a large number of neighbors and is closely connected to them. A consequence of Proposition 1 is the limited effect that a perturbation in the weights of a graph has on the centrality values; see Section 5.

### 3.2. Closeness centrality

Closeness is a relevant centrality measure when we are interested in how fast information can spread from one node to every other node in a network. A common definition of closeness centrality is the one in [5] where the centrality  $C_C(x)$  of a node  $x$  in a graph  $G = (V, E, W)$  is defined as the inverse of the sum of the shortest path lengths from this node to every other node in the graph. However, as done in [15], we will work with the *decentrality* version  $\bar{C}_C$ , where the lower the value the more central the node,

$$C_C(x) := \left( \sum_{x' \in V} s_G(x, x') \right)^{-1}, \quad \bar{C}_C(x) := \sum_{x' \in V} s_G(x, x'). \quad (5)$$

Since we are ultimately interested in the centrality ranking being impervious to perturbations, it is immediate that the ranking stability of  $C_C$  and of  $\bar{C}_C$  are equivalent since they are related by a strictly decreasing function. In the following proposition, we show stability of closeness decentrality.

**Proposition 2** *The closeness decentrality measure  $\bar{C}_C$  in (5) is stable as defined in Definition 1 with  $K_G = n$ .*

For (5) to make sense, the weights in  $W$  must represent dissimilarities between the nodes. Some alternative definitions of closeness centrality exist [16, 17] including that in [6] where the measure in (5) is normalized by  $n - 1$ . However, since normalization constants can be absorbed into  $K_G$ , stability does not depend on the appearance of normalization terms.

### 3.3. Betweenness centrality

Centrality can be interpreted as the possibility of a node to control the communication or the optimal flow within a graph. Betweenness centrality takes this position by giving higher centrality values to nodes that fall within the shortest path of many pairs of nodes. Formally, if denote by  $\sigma_{x'x''}$  the number of shortest paths from  $x'$  to  $x''$  and by  $\sigma_{x'x''}(x)$  the number of these shortest paths that go through node  $x$ , then the betweenness centrality  $C_B(x)$  for given node  $x$  is defined as [8]

$$C_B(x) := \sum_{\substack{x', x'' \in V \\ x' \neq x \neq x''}} \frac{\sigma_{x'x''}(x)}{\sigma_{x'x''}}. \quad (6)$$

In (6), the betweenness centrality value of node  $x$  is computed by sequentially looking at the shortest paths between any two nodes distinct from  $x$  and summing the proportion of shortest paths that contain node  $x$ . Despite its use in the study of technological [18] and social [19] networks, betweenness centrality is not stable.

**Proposition 3 ([12])** *The betweenness centrality measure  $C_B$  in (6) is not stable in the sense of Definition 1.*

As was the case for  $C_C$ , betweenness centrality should be applied to dissimilarity graphs. The instability of the betweenness centrality measure entails an undesirable behavior when applied to synthetic and real-world networks as shown in Section 5.

### 3.4. Eigenvector centrality

The eigenvector centrality  $C_E$  of a node does not depend on the number of neighbors but rather on how important its neighbors are. The importance of its neighbors in turn depends on how important *their* neighbors are, and so on. This recurrence relation translates into an eigenvector equation in terms of the adjacency matrix  $A$  of the graph being studied [7]

$$\lambda C_E = A C_E, \quad (7)$$

where  $C_E = (C_E(x_1), \dots, C_E(x_n))^T$ . The solution of (7) is not uniquely determined, since every pair  $(\lambda, C_E)$  of eigenvalues and eigenvectors solves the equation. However, for undirected and connected graphs the Perron-Frobenius Theorem [20] ensures that the eigenvector corresponding to the maximal eigenvalue contains all positive components. Thus,  $C_E$  in (7) is defined as the normalized dominant eigenvector of  $A$ . Eigenvector centrality is a stable measure as the following proposition shows.

**Proposition 4** *The eigenvector centrality measure  $C_E$  in (7) is stable as defined in Definition 1 with  $K_G = 4/(\lambda_n - \lambda_{n-1})$  where  $\lambda_n \geq \dots \geq \lambda_1$  are the eigenvalues of the adjacency matrix of  $G$ .*

The eigenvector centrality measure is applied to similarity graphs. In contrast to the cases for degree and closeness centrality,  $K_G$  for eigenvector centrality depends on the weights of the graph rather than just its size. This difference does not impact the practical implementation of eigenvector centrality as we see in Section 5.

### 3.5. Stable betweenness centrality

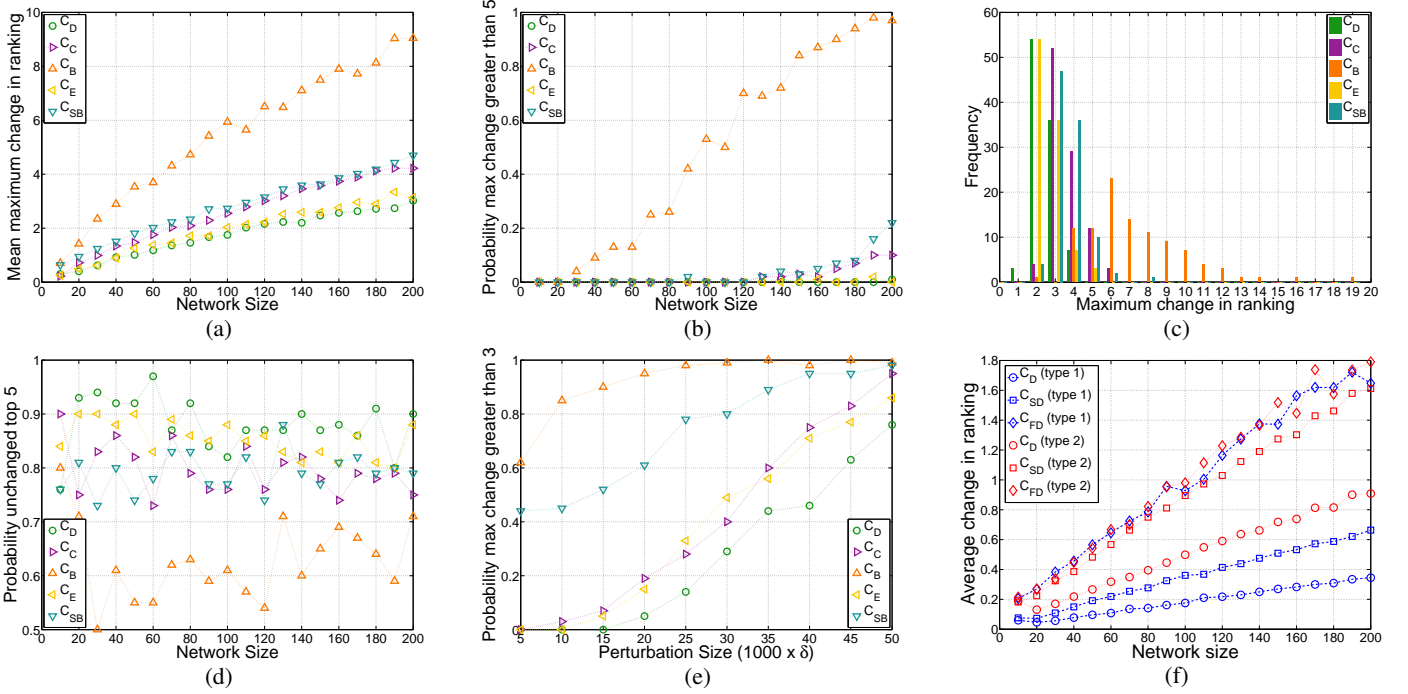
In [12], the stable betweenness centrality measure  $C_{SB}$  was introduced as a stable alternative to the traditional betweenness centrality. Given an arbitrary graph  $G = (V, E, W)$  and a node  $x \in V$ , define a new graph  $G^x = (V^x, E^x, W^x)$  with  $V^x = V \setminus \{x\}$ ,  $E^x = E \setminus \{(x, x') \mid x' = x \text{ or } x'' = x\}$ , and  $W^x = W|_{E^x}$ . I.e., the graph  $G^x$  is constructed by deleting from  $G$  the node  $x$  and every edge directed to or from it. The stable betweenness centrality  $C_{SB}(x)$  of any node  $x \in V$  is given by

$$C_{SB}(x) := \sum_{\substack{x', x'' \in V \\ x' \neq x \neq x''}} s_{G^x}(x', x'') - s_G(x', x''). \quad (8)$$

Measure  $C_{SB}$  quantifies the centrality of a given node  $x$  by the change in the length of shortest paths once this node is removed. This means that the centrality of a node depends on the quality of the best alternative. In contrast to the traditional centrality measure,  $C_{SB}$  is stable.

**Proposition 5 ([12])** *The stable betweenness centrality measure  $C_{SB}$  in (8) is stable as defined in Definition 1 with  $K_G = 2n^2$ .*

As was the case for  $C_B$ , definition (8) should be applied to graphs where the weights represent dissimilarities between nodes. In practice,  $C_{SB}$  presents a more robust behavior than  $C_B$ ; see Section 5.



**Fig. 1:** Comparison of stability indicators for all centrality measures: degree (green circle), closeness (purple right triangle), betweenness (orange upwards triangle), eigenvector (yellow left triangle), and stable betweenness (cyan downwards triangle). (a) Mean of the maximum change in random networks. (b) Probability that the maximum change exceeds 5 positions in random networks. (c) Histogram of the maximum change recorded when perturbing random networks with 150 nodes. (d) Probability that the top 5 ranking remains unchanged when perturbing a random network. (e) Probability that the maximum change exceeds 3 positions in  $G_I$ . (f) Average change in ranking for degree (circles), degree squared (squares) and floor degree (diamonds) centrality measures under two types of noise: small (blue) and large perturbations (red).

#### 4. CONTINUITY OF CENTRALITY MEASURES

We define a continuous centrality measure as one in which the centrality values of every node in a given graph are a continuous function of the weights in the edges of this graph.

**Definition 2** Let  $G = (V, E, W)$  be an arbitrary graph with adjacency matrix  $A$ . For every matrix  $B$  such that  $B_{ij} = 0$  if  $A_{ij} = 0$  and  $B + A \geq 0$  element-wise, define the graph  $H = (V, E, W')$  whose adjacency matrix is  $A + B$ . Then, a centrality measure  $C$  is continuous if for every node  $x \in V$ ,

$$C^H(x) \rightarrow C^G(x) \quad \text{as} \quad \|B\|_2 \rightarrow 0, \quad (9)$$

where  $C^G(x)$  is the centrality of  $x$  in graph  $G$  and similarly for  $H$ .

In the above definition, matrix  $B$  can be interpreted as a perturbation defined on the edges of graph  $G$ . A continuous centrality measure ensures that as this perturbation vanishes, the centrality values tend to those in graph  $G$ . Continuity is a weaker notion than stability since the latter implies the former as we show next.

**Proposition 6** If a centrality measure  $C$  is stable as in Definition 1 then it is continuous as in Definition 2.

As stated in Section 3, a centrality measure is a function of a graph that assigns a nonnegative real number to each node. This definition enables the existence of a wide variety of measures. In particular, centrality measures which are continuous but not stable.

**Proposition 7** If a centrality measure  $C$  is continuous as in Definition 2 then it need not be stable as in Definition 1.

Proposition 6 guarantees that degree, closeness, eigenvector and stable betweenness centrality are continuous centrality measures. Proposition 7 leaves open the question of whether betweenness centrality is continuous or not. The result below shows that it is not.

**Proposition 8** The betweenness centrality measure  $C_B$  in (6) is not continuous as defined in Definition 2.

Being not only unstable but discontinuous further hinders practical applicability of  $C_B$ , making  $C_{SB}$  an appealing alternative as we illustrate in the next section.

#### 5. NUMERICAL EXPERIMENTS

Stability and continuity regulate the behavior of centrality measures in the presence of noise. We empirically validate three facts: betweenness centrality is fundamentally different from the other measures (Section 5.1), continuity and stability encode different robustness properties (Section 5.2), and the stable betweenness alternative  $C_{SB}$  retains the same centrality notion as the original  $C_B$  (Section 5.3).

For a given node set  $V$  of size  $n \geq 10$ , we define a random network as one where an undirected edge  $(x, x')$  belongs to  $E$  with probability  $q = 10/n$ . The weight of this edge is randomly picked from a uniform distribution in  $[0.5, 1.5]$ . We consider these weights to be indication of dissimilarities. Notice that the centrality rankings obtained by applying a centrality measure based on dissimilarities – e.g., closeness – and one based on similarities – e.g., degree – on the same graph are not comparable. Thus, for every random graph we generate a similarity based graph with the same nodes and edges but where the weights are computed as 2 minus the edges in the original graph, hence also contained in  $[0.5, 1.5]$ .

As real-world data, we use a network that records interactions between sectors of the U.S. economy [21]. More precisely, the economic network  $G_I = (V_I, E_I, W_I)$ , contains as nodes the 61 industrial sectors of the economy as defined by the North American Industry Classification System (NAICS). There exists an edge  $(x, x') \in E_I$  if part of the output of sector  $x$  is used as input to sector  $x'$ , and the weight  $W_I(x, x')$  is given by how much output of  $x$  – in dollars – is productive input of  $x'$ . We consider  $W_I(x, x')$  as a measure of similarity and use the inverse  $1/W_I(x, x')$  as weights for the centrality measures that require dissimilarity graphs.

**Table 1:** Average (upper triangular part) and maximum (lower triangular part) variation of centrality ranking across different measures.

	$C_D$	$C_C$	$C_B$	$C_E$	$C_{SB}$	$C_{DS}$
$C_D$	0	11.3	11.6	7.3	13.1	5.3
$C_C$	43.8	0	10.3	9.9	12.7	8.5
$C_B$	44.7	41.6	0	14.6	<b>4.2</b>	8.3
$C_E$	30.0	38.9	55.5	0	16.6	8.5
$C_{SB}$	51.1	51.3	<b>18.9</b>	61.5	0	10.0
$C_{DS}$	22.3	34.4	34.3	33.7	42.4	0

### 5.1. Robustness indicators

We analyze the robustness of the centrality rankings when the random networks are perturbed. Given a network, we build a perturbed version of it by modifying every edge weight with probability  $p$ . The perturbed edge weights are multiplied by a uniform random number in  $[1 - \delta, 1 + \delta]$ . In this section we use type 1 noise where  $p_1 = 1$  and  $\delta_1 = 0.01$ . For the following experiment, we generate 100 random networks of  $n$  nodes, where  $n$  varies from 10 to 200 in multiples of 10. We then generate a perturbed version of each of these networks. For every network, we generate a centrality ranking of the nodes, i.e. we sort the nodes in decreasing order of centrality value, and compare it with the centrality ranking of the perturbed version of that network. We perform this comparison for the rankings output by the five centrality measures in Section 3.

We begin by analyzing the maximum variation in ranking position experienced by a node when perturbing the network as a function of its size; see Fig. 1a. E.g., for a network with 100 nodes, the perturbation generates a maximum change of 1.8 positions on average for the  $C_D$  ranking and 5.9 positions on average for the  $C_B$  ranking. All measures experience an approximately linear increase of the maximum change with the size of the network, but the rate of increase is fastest for  $C_B$ , generating big performance differences between the measures for larger networks. We are also interested in the distributions of these variations for the different centrality measures. Thus, we plot the probability that the maximum change in the ranking generated by a perturbation is greater than 5 positions in Fig. 1b. E.g., for over 90% of the networks of 180 nodes, the betweenness centrality ranking undergoes a variation greater than 5 positions when perturbed while this percent is smaller than 10% for the other measures. To facilitate the understanding of Figs. 1a and 1b, in Fig. 1c we present the histogram of the maximum change found in the rankings when perturbing a network for the particular case of networks with 150 nodes for all measures. The mean of these histograms correspond to the markers of the same color for networks with 150 nodes in Fig. 1a. To relate the histogram with Fig. 1b, notice that only the orange histogram has a considerable portion of its weight for changes of 6 positions or more, translating into a big difference in probabilities between the orange marker and the rest in Fig. 1b. Having a longer tail, the silhouette of the orange  $C_B$  histogram is essentially different from the rest. E.g., for one of the studied networks, the  $C_B$  ranking presents a change of 19 positions whereas the largest variation for all other measures combined is of 8 positions.

Another indication of robustness is the position where the first change in the ranking occurs. In Fig. 1d, we plot the probability that the top 5 nodes in the ranking retain their positions after perturbing the network. Observe that there is no clear trend with the size of the network but probabilities oscillate around different values for different centrality measures. In this way, we can state that for around 75% to 95% of the networks there is no change in the top 5 centrality ranking for all measures except for betweenness centrality where this percentage falls to 60% on average.

The same is true for a symmetrized version of the economic network  $G_I$ . In this case, as opposed to the random networks, the network size is fixed. Thus, we analyze performance as a function of the magnitude of the perturbation. A perturbation magnitude of  $\delta$  implies that every weight in the network is multiplied by a random number in  $[1 - \delta, 1 + \delta]$ . For every perturbation level, we generate 100 perturbed networks. In Fig. 1e, we plot the probability of having a change of more than 3 positions in the ranking for varying perturbation levels. As expected due to its instability, this probability is consistently highest for  $C_{SB}$ , and the difference with

**Table 2:** Comparison of the centrality rankings for the economic network  $G_I$ .  $C_B$  and  $C_{SB}$  output the most similar rankings.

Rk.	$C_B$	$C_{SB}$	$C_C$	$C_{OD}$	$C_{ID}$
1	<b>Real estate</b>	<b>Prof. Serv.</b>	<b>Prof. Serv.</b>	<b>Prof. Serv.</b>	Food
2	<b>Construction</b>	<b>Real Estate</b>	Oil & gas	<b>Real Estate</b>	<b>Real Estate</b>
3	<b>Prof. Serv.</b>	<b>Construction</b>	Petroleum	Oil & gas	Petroleum

other measures is maximized for perturbation of  $\delta = 0.02$  and smaller.

### 5.2. Effects of continuity and stability

Given that betweenness centrality is neither continuous nor stable and the rest of the measures analyzed in Section 3 are both, it is unclear the lack of which property is responsible for the low robustness of betweenness centrality. In order to answer this question, we introduce the following two variants of degree centrality – degree squared  $C_{DS}$  and floor degree  $C_{FD}$  – defined as follows for every node  $x \in V$  in a graph  $(V, E, W)$

$$C_{DS}(x) := \sum_{x' | (x, x') \in E} (W(x, x'))^2, \quad C_{FD}(x) := \sum_{x' | (x, x') \in E} \text{floor}(W(x, x')). \quad (10)$$

Degree centrality  $C_D$  is both continuous and stable and it can be shown [13] that  $C_{DS}$  is continuous but not stable, and  $C_{FD}$  is neither continuous nor stable. In Fig. 1f we plot the average change in rankings, i.e. the expected rank variation of any given node in the network, output by the three measures when perturbing networks of different sizes. We do this for two types of noise, type 1 as defined in Section 5.1 (plotted in blue) and type 2 noise with parameters  $p_2 = 0.1$  and  $\delta_2 = 0.1$  (plotted in red). As expected, degree centrality has the highest robustness followed by degree squared and floor degree being the less robust of the three measures under both types of noise. However, notice that for noise of small magnitude (type 1) the degree squared behaves more similar to degree centrality, showing a robust behavior in the presence of noise. For larger magnitudes of noise (type 2), degree squared centrality has a similar behavior to the unstable floor degree centrality. This points towards the fact that continuity provides robustness under small perturbations while the stronger concept of stability provides robustness for more general perturbations.

### 5.3. Ranking similarity across measures

Finally, we compare the centrality rankings across different measures. In order to do this, we pick 100 random networks of size 100 nodes and compute the average and maximum change for a pair of rankings output by different measures; see Table 1. E.g., the mean average ranking variation of nodes ranked by the degree  $C_D$  and the eigenvector  $C_E$  centralities is 7.3 positions. Moreover, the mean maximum variation between two given rankings output by the betweenness  $C_B$  and the closeness  $C_C$  centrality is 41.6 positions. Notice that the smallest variations – both in average and maximum – are achieved when comparing the rankings of the betweenness  $C_B$  and the stable betweenness  $C_{SB}$  centrality measures. This is empirical proof that both measures encode a similar centrality concept. Further observe that the variations between these two rankings are even smaller than the ones between degree  $C_D$  and squared degree  $C_{DS}$  centrality, two measures with closely related definitions [cf. (4) and (10)].

To complete the analysis, we use the economic network  $G_I$  to further illustrate the resemblance between  $C_B$  and  $C_{SB}$ . In Table 2 we inform the three most central sectors of the economy as given by different centrality measures. The ranking output by stable betweenness centrality  $C_{SB}$  is the most similar to  $C_B$  since both measures share the top 3 economic sectors.

## 6. CONCLUSION

Stability and continuity, as formal characterizations of the robustness of centrality measures, were introduced. The most frequently used centrality measures were shown to be stable and continuous with the exception of betweenness centrality. We illustrated the robustness implications of stability and continuity in noisy random and real-world networks. Finally, we showed that the alternative stable version of betweenness centrality carries a similar centrality notion to the original one.

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