Social Learning, Information Heterogeneity, and Coordination

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ARO MURI 1st Year Review
with Pooya Molavi (MIT Economics), Alireza Tahbaz Salehi(Columbia GSB)
Outline

1. Task S2 Social learning, Heterogeneous Information and Network Structure
   ▶ motivation: diffusion/aggregation of information from heterogeneous sources
   ▶ model (from last year) and a behavioral foundation (new)
   ▶ rate of social learning: dependence on centrality and discriminative power of observations
   ▶ Informed agents vs. allocation of information
   ▶ Influential agents

2. Task S1 Learning for coordination/ emergence of conventions
   ▶ How do individuals coordinate when there is a pay-off relevant unknown state?
   ▶ How does one coordinate on the most advantageous yet unknown set of rules of action,
   ▶ Emergence of conventions (Shin & Williamson '96)
   ▶ reaching consensus on optimal action: when and how?
   ▶ Key new result: failure of optimal information aggregation is nongeneric
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   ▶ Key new result: failure of optimal information aggregation is nongeneric
Task S2: Information aggregation and social learning

Challenges

- Information is dispersed
- No central mechanisms for aggregation
- Interactions are local
- Related: Diffusion, gossip in
  - face to face communications
  - online social media
- examples:
  - diffusion of micro finance programs (Banerjee, Chandrasekhar, Duflo, Jackson (’13))
  - Coordinating events during popular uprisings (Ali (2011), Hassanpour (’12))
  - decision making in organizations (Calvó-Armengol, Beltran (’09))
  - Making consumption decisions (Kotler (’86))
  - Learning new agricultural techniques (Hagerstrand (’69), Rogers (’83))
Model (agents and observations)

- \( \{1, \ldots, n\} \): finite set of agents
- Agents want to learn an underlying state \( \theta \in \Theta \).
- \( t \in \mathbb{N} \): discrete time. ‘State’ drawn at \( t = 0 \) according to agents’ common prior.
- \( \omega_{it} \in S \): private observations of agent \( i \) at time \( t \)
- Conditional on \( \theta \) being realized, \( \omega_{it} \sim \ell_i^\theta \in \Delta S \).
- \( \ell_i = \{\ell_i^\theta\}_{\theta \in \Theta} \): agent \( i \)’s signal structure: what is the likelihood of \( \omega_{it} \in S \), if \( \theta \) is the truth?

Assumption (identifiability)

For all \( \theta, \hat{\theta} \in \Theta \), there exists \( i \) such that \( \ell_i^\theta \neq \ell_i^{\hat{\theta}} \). Globally, there is enough to discover the truth

Question posed last year:

Role of network and information structure?
Classical setting, no networks, What to expect?

Doob (1949), Blackwell and Dubins (1962)

Merging of opinions with increasing information: The belief of a Bayesian agent $i$ with absolutely continuous prior observing a stream of signals will merge to the truth; i.e., she will learn the likelihood function $\ell_i$. 

We can't disagree forever: Two agents with a common prior exchanging beliefs repeatedly will reach agreement; moreover, their consensus belief will generically be as if they commonly knew each others' private information.
Classical setting, no networks, What to expect?

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Merging of opinions with increasing information: The belief of a Bayesian agent $i$ with absolutely continuous prior observing a stream of signals will merge to the truth; i.e., she will learn the likelihood function $\ell_i$.

Geanakoplos and Polemarchakis (1982)
We can’t disagree forever: Two agents with a common prior exchanging beliefs repeatedly will reach agreement; moreover, their consensus belief will generically be as if they commonly knew each others’ private information.

What happens in the networked case?
The Bayesian Benchmark: Multi-agent setting

- Let $\mathcal{X} = \Theta \times \Omega \times \Gamma$ be the measurable space that captures all uncertainty.

- Assume agents have a common prior over the $\mathcal{X}$.
The Bayesian Benchmark: Multi-agent setting

- Let $\mathcal{X} = \Theta \times \Omega \times \Gamma$ be the measurable space that captures all uncertainty.

- Assume agents have a common prior over the $\mathcal{X}$.

**Theorem**

**Assume**

(a) agents’ common prior has full support over $\mathcal{X}$;
(b) the realized network is strongly connected;
(c) the realized state is identifiable.

Then all agents learn the true state asymptotically almost surely; i.e., $\mu_{it} \longrightarrow 1_{\theta^*}$ for all $i \in \mathcal{N}$.

Agents need to reason about too many things. Is there a simpler behavioral model?
What is a “reasonable” Non-Bayesian alternative? Extension/modification of DeGroot learning model (Golub and Jackson 2010) with these features:

- continuous flow of new information
- heterogenous stream of private observations
- asymptotic agreement with the Bayesian benchmark

Contributions this year

- Implications of the rate analysis
- Axiomatic construction of non-Bayesian models
Model (learning rule)

- At $t \in \mathbb{N}$ agents also observe beliefs of their neighbors.
- $\mu_{it} \in \Delta \Theta$: belief of agent $i$ at $t$
Model (learning rule)

- At $t \in \mathbb{N}$ agents also observe beliefs of their neighbors.
- $\mu_{it} \in \Delta \Theta$: belief of agent $i$ at $t$
- The update rule:

$$
\mu_{it+1} = a_{ii} \text{BU}(\mu_{it}; \omega_{it+1}) + \sum_{j \neq i} a_{ij} \mu_{jt}.
$$

- $i$: Bayesian posterior belief conditioned on private signal
- $ii$: beliefs of the neighbors
- Weights sum to one representing network connections.
- Is there a behavioral foundation for this model?

### Research question

How do the network structure and agents’ information endowments determine the extent of information aggregation?
Model (social network)

- $a_{ij} > 0 \iff$ Agent $j$ is a neighbor of agent $i$.
- $A = [a_{ij}]$ row-stochastic social interaction matrix.
- Weights can be time-varying and belief-dependent.
Model (social network)

- $a_{ij} > 0 \iff \text{Agent } j \text{ is a neighbor of agent } j$.
- $A = [a_{ij}]$ row-stochastic social interaction matrix.
- Weights can be time-varying and belief-dependent.

Assumption (strong connectivity)

There is a directed path from any agent to any other one (can be generalized to switching graphs).

- Guarantees that information can flow from any agent to any other.
Proposition

If identifiability and strong connectivity assumptions are satisfied,

$$\mu_{it}(\cdot) \longrightarrow \mathbf{1}_\theta(\cdot)$$

the rate is (up to first order)

$$r \approx \min_{\theta} \min_{\theta \neq \hat{\theta}} \sum_{i=1}^{n} v_i h_i(\theta, \hat{\theta}) + h.o.t$$

- The learning process asymptotically coincides with Bayesian learning
- Unlike Bayesian models, the model is tractable
- Rate is a convex combination of relative entropies $h_i(\theta, \hat{\theta})$ with weights as eigenvector centrality $v_i$.
- Consistent with empirical and theoretical observations in Jackson (2013, 2014)
Towards an axiomatic view

What should a reasonable model look like?

- If private signals are uninformative \( \Rightarrow \) beliefs updated as in DeGroot ’74

- If a signal is evidence in favor of a state, the posterior belief on that state should increase (increasing function of likelihood ratio)

- Update should be separable in terms of private signal and an aggregate of belief of neighbors

- Conjecture: All such updates that converge, have the same asymptotic rate (up to first order)

- One such example: Average log beliefs of neighbors with log private posterior J, Shahinpour 2012, Tahbaz-Salehi, Rahnama-Rad 2010

- Bayesian with limited memory/ recall: take the first step of Bayesian update and apply for future steps
Relative Entropy and Eigenvector Centrality

### Definition (relative entropy)

Given \( \hat{\theta} \neq \theta \),

\[
h_i(\theta, \hat{\theta}) = \sum_{s \in S} \ell^\theta_i(s) \log \left( \frac{\ell^\theta_i(s)}{\ell^\hat{\theta}_i(s)} \right)
\]

- \( h_i(\theta, \hat{\theta}) \): information in favor of \( \theta \) against \( \hat{\theta} \) when \( \theta \) is realized
- \( h_i(\theta, \hat{\theta}) = 0 \) \( \Rightarrow \) agent \( i \) cannot distinguish \( \theta \) and \( \hat{\theta} \)
- Larger \( h_i(\theta, \hat{\theta}) \) \( \Rightarrow \) easier to rule out \( \hat{\theta} \) when \( \theta \) is realized

### Eigenvector Centrality

#### Definition (eigenvector centrality)

Given \( A \), the eigenvector centrality of agent \( i \) is

\[
v_i = \sum_{j=1}^{n} v_j a_{ji}
\]
Under which allocation of signals is learning the fastest?

Proposition

Suppose

- agents’ signals are comparable with respect to \( \succeq_{UI} \);
- \( l_i \succeq_{UI} l_j \) if and only if \( v_i \geq v_j \).

Then, no reallocation of signals increases the rate of learning.

- Positive assortative matching of centralities and signal qualities maximizes the rate of learning.
- Intuition: Irrespective of the realized state, the most informative signals receive the most attention.
- Same insight as Matt’s talk: best Info given to the most central has the most effect.
New results: what if information endowments are incomparable?

- relative informativeness of agent $i$’s signals for $(\theta, \hat{\theta})$: 
  \[\gamma_i(\theta, \hat{\theta}) = \sup \{ \beta : h_i(\theta, \hat{\theta}) \geq \beta h_j(\theta, \hat{\theta}) \text{ for all } j \neq i \}\]

- specialty of agent $i$: 
  \[E_i = \{(\theta, \hat{\theta}) : \theta \neq \hat{\theta} \text{ and } \gamma_i(\theta, \hat{\theta}) \geq 1\}\]

<table>
<thead>
<tr>
<th>Definition (expertise)</th>
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<tbody>
<tr>
<td>relative expertise: $\gamma_i = \min { \gamma_i(\theta, \hat{\theta}) : (\theta, \hat{\theta}) \in E_i }$</td>
</tr>
<tr>
<td>absolute expertise: $\varepsilon_i = \min { h_i(\theta, \hat{\theta}) : (\theta, \hat{\theta}) \in E_i }$</td>
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Suppose that

- $E_i \neq \emptyset$ for all $i$;
- $\varepsilon_i \geq \varepsilon_j$ if and only if $v_i \leq v_j$.

Then, reallocations of signals do not increase the rate by more than $\alpha(\max_i \varepsilon_i)/(\min_i \gamma_i)$.

1st condition: Agents are all experts.

2nd condition: The least central agents have the highest absolute expertise.
Effect of Network topology?

Definition (regularity)

\[ A \preceq_{\text{reg}} A' \]

if

\[ \sum_{i=1}^{k} v[i] \leq \sum_{i=1}^{k} v'[i] \]

\[ \Downarrow \]

\[ \Downarrow \]

\[ v \Downarrow \quad \text{FOSD}^a \quad v' \Downarrow \]

\[ \downarrow \]

\[ \text{afirst-order stochastically dominates} \]

\[ \Downarrow \]

\[ \preceq_{\text{reg}} \]

\[ \downarrow \]
Consider the rate under the best allocation: $r^*$. Is $r^*$ higher for regular or irregular networks?
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**Proposition**

Suppose agents’ signals are comparable with respect to $\succeq_{UI}$. Then,

$$A \succeq_{reg} A' \implies r^* \leq r'^*$$
Network Regularity and Learning

- Consider the rate under the best allocation: \( r^* \).

Is \( r^* \) higher for regular or irregular networks?

**Proposition**

Suppose agents’ signals are comparable with respect to \( \succeq_{ul} \). Then,

\[
A \succeq_{reg} A' \implies r^* \leq r'^*
\]

- The gap can grow unboundedly in large networks.
Network Regularity and Learning: An Example

- $\Theta = \{\theta_0, \theta_1, \ldots, \theta_n\}$
- $S = \{\text{Head}, \text{Tail}\}$
- $\pi > \frac{1}{2}$

$\ell_\theta^i(s) :$

- $\theta_0 (\begin{pmatrix} 1 - \pi & \pi \\ \pi & 1 - \pi \end{pmatrix})$
- $\theta_1 (\begin{pmatrix} 1 - \pi & \pi \\ \pi & 1 - \pi \end{pmatrix})$
- $\vdots$
- $\theta_i (\begin{pmatrix} \pi & 1 - \pi \\ 1 - \pi & \pi \end{pmatrix})$
- $\theta_n (\begin{pmatrix} 1 - \pi & \pi \\ \pi & 1 - \pi \end{pmatrix})$

Proposition

| $A \; \succeq_{\text{reg}} \; A'$ | $\Rightarrow$ | $r^* \; \geq \; r'^*$ |

- Ordering of networks is reversed with expert agents!
- The gap does not grow unboundedly.
  $\Rightarrow$ Rates of learning in all large networks are similar.
Part 2: Learning to Coordinate
Motivating Example: Popular Uprising

- Pay-off relevant state = government willingness to use force
- Unknown, but individuals receive noisy signals about it.
- The optimal level of unrest for an individual depends on:
  - Her belief about the willingness of government to use force.
  - Her belief about the aggregate level of protest. Will optimal coordination occur?
  - Each agent only observes the action of her friends
- Lewis('69), Ullman-Margalit ('77), Gilbert ('90), Miller ('90), Shin & Williamson ('96), Acemoglu & Jackson (2013): conventions and norms viewed as equilibrium strategies in coordination games.

\[-(1 - \lambda)(a_i - \theta)^2 - \lambda (a_i - \bar{a})^2,\]

where $\lambda \in (0, 1)$ and $\bar{a} = \frac{1}{n} \sum_{j=1}^{n} a_j$. 
Model

- \{1, \ldots, n\}: finite set of agents
- Agents want to coordinate on \( \theta \in \mathbb{R} \).
- \( t \in \mathbb{N} \): discrete time
- \( a_{it} \): time \( t \) action of agent \( i \)
- \( -(1 - \lambda)(a_{it} - \theta)^2 - \lambda(a_{it} - a_{\text{average}})^2 \): payoff of agent \( i \) at \( t \) (could be a general coordination payoff)
- \( s_{it} \): time \( t \) private signal of agent \( i \)
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**Endogenous Signals**

\( s_{it} \) could be a function of the agents’ previous actions.
Social Network

- At $t$ agents observe the time $t - 1$ actions of their time $t - 1$ neighbors.

- Random and time-varying social network

Assumption (independence)

*Networks are independent of the signals.*

Assumption (connectivity)

*There is a directed path from any agent to any other one such that: the edges of the path appear almost surely infinitely often.*
Model (social network)

- “Connected” social network

- Agents observe the previous actions of their neighbors

- \( h_{it} \): agent \( i \)'s private history at time \( t \),
  \[ h_{i,t} = \{ h_{i,t-1}, \{ a_{j,t-1} \}_{j \in N_i}, s_{i,t-1} \} \]
  where \( h_{i,0} = \emptyset \)

- \( \mathcal{H}_{it} \): \( \sigma \)-field generated by \( h_{it} \)

- \( j \) is a neighbor of \( i \) \( \Rightarrow \) \( a_{jt-1} \) is \( \mathcal{H}_{it} \)-measurable

- \( \sigma_i : h_{it} \mapsto a_{it} \): strategy of agent \( i \)

- \( \sigma = (\sigma_1, \ldots, \sigma_n) \): strategy profile
Model (payoffs)

\[ u_i(a, \theta) = -(1 - \lambda)(a_i - \theta)^2 - \lambda(a_i - \bar{a}) \]

\[ \bar{a} = \frac{1}{n} \sum_{j=1}^{n} a_j \]

\[ U_{it;\sigma}(\sigma) = \mathbb{E}_{\sigma}[u_i(\sigma_1(h_{1t}), \ldots, \sigma_n(h_{nt}), \theta)|H_{it}] \]
Equilibrium

Definition (Myopic Perfect Bayesian equilibrium)

(a) Agents’ beliefs are consistent given their priors and strategies.
(b) Agents’ strategies are expected stage payoff maximizing given their beliefs.

\[ E_{\sigma^*}[u_i(\sigma_{i,t}^*, \sigma_{-i,t}^*, \theta) \mid h_{i,t}] \geq E_{\sigma^*}[u_i(\sigma_{i,t}, \sigma_{-i,t}^*, \theta) \mid h_{i,t}] \text{ for all } i. \]

- Agents are myopic (Folk theorems if non-myopic).
- FOC: Optimal action linear in private best-estimates and in actions (cf. Blume’s talk last year).
Equilibrium

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- Agents are myopic (Folk theorems if non-myopic).
- FOC: Optimal action linear in private best-estimates and in actions (cf. Blume’s talk last year).

Proposition

(a) An equilibrium exists.
(b) The equilibrium actions are unique up to sets of measure zero.
Consensus in Actions

- \(a_{it}^*\): equilibrium time \(t\) action of agent \(i\)

**Proposition**

For all \(i\),

\[a_{it}^* \rightarrow a_{i\infty}^* .\]
Consensus in Actions

- $a^*_t$: equilibrium time $t$ action of agent $i$

**Proposition**

For all $i$,

$$a^*_t \longrightarrow a^*_i$$

**Proposition**

For all $i, j$,

$$a^*_{i\infty} = a^*_{j\infty}$$
Consensus in Estimates

- $\mathcal{H}_it$: $i$’s information at time $t$
- $E^*$: equilibrium-induced probability distribution

**Proposition**

For all $i$,

$$a^*_{it} \rightarrow E^*[\theta|\mathcal{H}_i\infty].$$

- The coordination motive asymptotically disappears.
Consensus in Estimates

- $\mathcal{H}_{it}$: $i$’s information at time $t$
- $\mathbb{E}^*$: equilibrium-induced probability distribution

**Proposition**

For all $i$,

$$a^*_{it} \longrightarrow \mathbb{E}^* [\theta | \mathcal{H}_{i\infty}].$$

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**Proposition**

For all $i, j$,

$$\mathbb{E}^* [\theta | \mathcal{H}_{i\infty}] = \mathbb{E}^* [\theta | \mathcal{H}_{j\infty}]$$

- Agents reach consensus in their bests estimates of $\theta$. 
Optimal Information Aggregation

- $\mathcal{H}_\infty$: total information available to the agents at the end of the game

Best estimate of $\theta$ given all the information

$$E^*[\theta|\mathcal{H}_\infty],$$
Optimal Information Aggregation

- $\mathcal{H}_\infty$: total information available to the agents at the end of the game

Best estimate of $\theta$ given all the information

$$E^*[\theta | \mathcal{H}_\infty],$$

- $E^*[\theta | \mathcal{H}_\infty]$ optimal action given all the information

- $E^*[\theta | \mathcal{H}_\infty] \neq E^*[\theta | \mathcal{H}_i\infty] \Rightarrow$ failure of full information aggregation
An Example of Failure of Information Aggregation

- two connected agents
- supp $\mu = \{-1, 1\}$
- $\mu P(-1) = \mu P(1) = \frac{1}{2}$
- $S_1 = S_2 = \{H, T\}$
- signaling function

\[
(s_1 t, s_2 t) \sim \begin{cases} 
\frac{1}{2} \delta(H,H) + \frac{1}{2} \delta(T,T) & \text{if } \theta = 1, \\
\frac{1}{2} \delta(H,T) + \frac{1}{2} \delta(T,H) & \text{if } \theta = -1,
\end{cases}
\]
An Example of Failure of Information Aggregation

- two connected agents
- $\text{supp } \mu = \{-1, 1\}$
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\frac{1}{2} \delta(H, T) + \frac{1}{2} \delta(T, H) & \text{if } \theta = -1,
\end{cases}$$

- $0 = \mathbb{E}^*[\theta|\mathcal{H}_{i\infty}] \neq \mathbb{E}^*[\theta|\mathcal{H}_{\infty}] = \theta$.
- The example is nongeneric.
Exogenous Signals and Optimal Information Aggregation

Assumption (exogenous private signals)

Private signals are \textit{independent} of the history of the game.

Assumption (connectivity)

Network is \textit{fixed and strongly connected}.

Assumption (finite signal space)

Private signals belong to \textit{finite} spaces.
Exogenous Signals and Optimal Information Aggregation

Assumption (exogenous private signals)
*Private signals are independent of the history of the game.*

Assumption (connectivity)
*Network is fixed and strongly connected.*

Assumption (finite signal space)
*Private signals belong to finite spaces.*

Theorem
*For a generic set of priors and likelihood functions, information is optimally aggregated.*

\(^a\)a residual set of probability distributions given the topology of uniform convergence
Contribution to Bayesian Social Learning

» Special case: $\lambda = 0$

» Agents only care about learning the true state

» Agents communicate their best estimate of the state to their neighbors

» Agents reach consensus in their estimates

» The consensus estimate is generically optimal

» Punchline: no need to communicate full beliefs: just optimal estimates! counterpart to the results where agents communicate full posteriors:
  » Borkar and Varaiya (1978)
  » Geanakoplos, Polemarchakis (1982)
  » Jadbabaie, Molavi, Sandroni, Tahbaz-Salehi (2012)
  » Mueller-Frank (2013)
Conclusions and Future Work

- two models of social learning:
  - learning with heterogeneous information

- analytical characterization of rate of learning in terms of network and information structures.

- rate of learning under different allocations of information

- comparative statics of rate with respect to network structure

- How much memory recall is needed for update to converge? diameter of graph? what if actions are observed?

- future work: conventions and norms as coordination games with unknown state: non myopic case?

- optimal aggregation of information except sets of measure zero