Systemic Risk and Stability in Networks

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Introduction

- We live in an interconnected world, creating a myriad of efficiency, technological and communication gains, e.g., WWW.

- But interconnections also bring **systemic risk**: Failures or distress in some units spread to the rest of the system, through a series of cascading failures or distress.

**Examples:**

- Breakdown in the supply chain: e.g., caused by the 2011 earthquakes and tsunami in Japan and felt throughout the world.
- Collapse of investment broker Lehman Brothers, plunging world financial markets into turmoil.
- Small power surge in Ontarios electricity grid leading to major blackouts throughout the US.
Motivation

- Much recent interest in the relationship between systemic risk and network effects, mainly a consequence of the Financial Crisis.

“In the current crisis, we have seen that financial firms that become too interconnected to fail pose serious problems for financial stability and for regulators. Due to the complexity and interconnectivity of today’s financial markets, the failure of a major counterparty has the potential to severely disrupt many other financial institutions, their customers, and other markets.”

Charles Plosser
President and Chief Executive Officer,
Federal Reserve Bank of Philadelphia,
March 6, 2009

“If any one of the domestic companies should fail, we believe there is a strong chance that the entire industry would face severe disruption. Ours is in some significant ways an industry that is uniquely interdependent – particularly with respect to our supply base, with more than 90 percent commonality among our suppliers. Should one of the other domestic companies declare bankruptcy, the effect on Ford’s production operations would be felt within days – if not hours. Suppliers could not get financing and would stop shipments to customers. Without parts for the just-in-time inventory system, Ford plants would not be able to produce vehicles.”

Alan Mulally
President and Chief Executive Officer,
Ford Motor Co.,
November 18, 2008
A common conjecture in economics and finance is that “rings” are more fragile or unstable, and “complete” networks are more stable. [Allen and Gale (2000), Freixas, Parigi and Rochet (2000) and Kiyotaki and Moore (1997)].

But also the opposite perspective: more densely connected financial networks are more prone to systemic risk. [Battiston et al. (2009) and Blume et al. (2011)].

Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi (2011): with linear (e.g., input-output) interactions, rings as stable as complete networks.

Which perspective?
This Talk

- Two recent papers that investigate the relationship between systemic risk and network structure.
  - **Linear dominoes model**: Systemic Risk in Input-Output Economies [Acemoglu, Carvalho, Ozdaglar, Tahbaz-Salehi 11]
    - Common conjecture is wrong: Rings as stable as complete networks.
  - **Nonlinear dominoes model**: Systemic Risk in Financial Networks [Acemoglu, Ozdaglar, Tahbaz-Salehi 12]
    - For small shocks, rings most unstable and complete networks most stable.
    - For large shocks, there is a phase transition: complete networks become most unstable.
Related Literature

- Input-output networks:

- Financial networks:
Input-Output Economies

- An economy consisting of $n$ sectors, $\mathcal{I}_n = \{1, 2, \ldots, n\}$.
- The output of each sector is used by a subset of all sectors as input (intermediate goods) for production.
Production Side

- We assume that all sectors have Cobb–Douglas production technologies: output of sector \( i \), denoted by \( x_i \) is given by

\[
x_i = z_i^\alpha l_i^\alpha \prod_{j=1}^{n} x_{ij}^{(1-\alpha)w_{ij}},
\]

where

- \( l_i \): labor employed by sector \( i \),
- \( \alpha \in (0, 1] \): labor share,
- \( x_{ij} \): the amount of commodity \( j \) used in the production of good \( i \),
- \( w_{ij} \geq 0 \): share of good \( j \) in input use of sector \( i \). The supply relations is captured by the input-output matrix \( W_n = [w_{ij}]_{i,j \in \mathcal{I}_n} \).
  - \( w_{ij} = 0 \) if sector \( i \) does not use good \( j \) as input for production;
  - \( w_{ij} \) also corresponds to the entries of input-output tables.
- \( z_i \): productivity shock to sector \( i \). We assume that \( z_i \) are independent across sectors, and denote the distribution of \( \epsilon_i = \log(z_i) \) by \( F_i \).

- We use \( \mathcal{E}_n = (I_n, W_n, \{F_i\}_{i \in \mathcal{I}_n}) \) to refer to an economy of size \( n \).
Assumptions

Assumption

The input shares of any firm $i \in \mathcal{I}_n$ in the economy add up to one, i.e.,
\[ \sum_{j=1}^n w_{ij} = 1. \]

- This assumption guarantees that the production functions exhibit constant returns to scale to their labor inputs and intermediate goods provided by suppliers.
- It guarantees that the input-output matrix $W_n$ is a (row) stochastic matrix, i.e., all rows add up to one.

Assumption

Given a sequence of economies $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$ and for any sector $i \in \mathcal{I}_n$, $F_i$ is such that

1. $\mathbb{E} \epsilon_i = 0$, and
2. $\text{var}(\epsilon_i) = \sigma_i^2 \in (\underline{\sigma}^2, \bar{\sigma}^2)$, where $0 < \sigma < \bar{\sigma}$ are independent of $n$. 

Network Representation

- Intersectoral network: weighted, directed graph $G_n = (I_n, E_n, W_n)$.

- Degree of sector $j$: (value) share of $j$’s output in the total production of economy $d_j = \sum_{i=1}^{n} w_{ij}$.
Network Representation

- **Intersectoral network**: weighted, directed graph \( G_n = (\mathcal{I}_n, E_n, W_n) \).

- **Degree of sector \( j \)**: (value) share of \( j \)'s output in the total production of economy \( d_j = \sum_{i=1}^{n} w_{ij} \).

- Representative firm in sector \( i \) solves the problem:

\[
\max_{l_i, x_i, \{x_{ij}\}_{j \in \mathcal{I}_n}} \quad p_i x_i - hl_i - \sum_{j=1}^{n} p_j x_{ij} \\
\text{subject to} \quad x_i = e^{\alpha e_i l_i^\alpha} \prod_{j=1}^{n} x_{ij}^{(1-\alpha)w_{ij}}.
\]

- \( h \) is the market wage, \( p_i \) is the market price of good \( i \).
Consumer side

- Representative household endowed with one unit of labor.
- Preferences over all goods in the economy are given by
  \[ u(c_1, c_2, \ldots, c_n) = A_n \prod_{i=1}^{n} (c_i)^{1/n}. \]
- Representative consumer’s problem:
  \[
  \max_{\{c_i\}_{i \in I_n}} u(c_1, \ldots, c_n) \\
  \text{subject to} \quad p_1 c_1 + \cdots + p_n c_n = h
  \]
Competitive Equilibrium

Definition

In the competitive equilibrium of economy, the prices \((p_1, p_2, \ldots, p_n)\) and wage \(h\) are such that

(a) the representative consumer maximizes her utility,
(b) the representative firms in each sector maximize profits,
(c) labor and commodity markets clear.

\[
c_i^* + \sum_{j=1}^{n} x_{ji}^* = x_i^* \quad \forall i \in I_n
\]

\[
\sum_{i=1}^{n} l_i^* = 1.
\]
Competitive Equilibrium

- The aggregate value added in the economy is given by the equilibrium wage $h$.

- We define the aggregate output $y_n$ as

$$y_n \equiv \log(h).$$

Proposition

At the equilibrium, the aggregate output is a convex combination of the sectoral shocks:

$$y_n = \sum_{i=1}^{n} v_{n,i} \epsilon_i = v'_n \epsilon,$$

where $\epsilon = [\epsilon_1, \ldots, \epsilon_n]$ and $v_n$ is the influence vector given by

$$v_n \equiv \frac{\alpha}{n} \left[ I - (1 - \alpha) W'_n \right]^{-1} 1.$$
Influence Vector

Since none of $W_n$’s eigenvalues lie outside of the unit circle, it is possible to express $v_n$ in terms of a convergent power series:

$$v'_n = \frac{\alpha}{n} 1' \sum_{k=0}^{\infty} (1 - \alpha)^k W^k_n.$$

Or alternatively as:

$$v'_n = \frac{\alpha}{n} 1' + (1 - \alpha)v'_n W_n.$$  

The presence of higher-order interconnections can already be seen from this equation (the second term on the right-hand side).

Note that the influence vector related to the definition of the PageRank vector or the Bonacich centrality in the Internet search algorithms.
Alternative Interpretations

We could have alternatively consider a reduced-form model

$$\tilde{y} = (1 - \alpha) \tilde{W}_n \tilde{y} + \alpha \tilde{\epsilon}.$$ 

This could arise, for example, from:

1. Models in which $\epsilon_i$’s are not productivity shocks, but other shocks to sectoral or firm behavior.
2. Models in which units are firms rather than sectors (but then one needs to model “relationship-specific investments” and to some degree endogenize $\tilde{W}_n$).
3. Models of “strategic complementarities”.
Aggregate Volatility

- Recall aggregate output is given by \( y_n = v_n' \epsilon \).
- We focus on aggregate volatility, defined as the standard deviation of aggregate output.
- Aggregate volatility is clearly equal to:

\[
\sigma_{agg} \equiv (\text{var } y_n)^{1/2} = \sqrt{\sum_{i=1}^{n} \sigma_i^2 v_{n,i}^2}.
\]

Since \( \sigma_i \) is uniformly bounded, this implies that aggregate volatility scales with \( \|v_n\|_2 \), where \( \| \cdot \|_2 \) is the Euclidean (vector) norm.
- That is:

\[
\sigma_{agg} = \Theta(\|v_n\|_2).
\]

\( a_n = \Theta(b_n) \iff 0 < \liminf_{n \to \infty} a_n/b_n \leq \limsup_{n \to \infty} a_n/b_n < \infty. \)
Effect of Network Structure on Aggregate Volatility

Two strategies:

1. Compare two different networks of (fixed) size $n$;
2. Characterize impact of network properties on aggregate volatility as $n \to \infty$.

In this case, the question is equivalent to whether the Law of Large Numbers applies, and if so, how rapidly does $\sigma_n$ converge to zero.
Rings Are “Robust”

- We say that the network is regular if $d_i = d$ for each $i$.

- Suppose also that $\sigma_i = \sigma$ for each $i$.

**Proposition**

All regular networks achieve the lowest possible aggregate volatility

$$\sigma_{agg} = \frac{\sigma}{\sqrt{n}}.$$  

- Implication: complete graphs and rings are equally “robust”.
- Intuition: with the linear structure, shocks average out exactly provided that all sectors have the same degree.
Star Network Is “Fragile”

Proposition

*The highest level of aggregate volatility is generated by the star network and is equal to*

\[
\sigma_{\text{agg}} = \frac{\sigma}{\sqrt{1 - \left(\frac{n-1}{n}\right) \alpha (1 - \alpha)}}.
\]

- Intuition: the shock to the central sector of the star does not “wash out”.
- More general result: *unequal degrees—or asymmetric networks—create additional volatility.*
First-Order Interconnections

- To obtain sharper results, consider a sequence of economies with $n \to \infty$.
- Then the greatest degree of “robustness” (least systemic risk) corresponds to 
  \[ \sigma_{\text{agg}} \sim 1/\sqrt{n} \]
  (as in standard law of large numbers for independent variables).
- We capture the effect of first-order interconnections using the coefficient of variation of degrees.

Definition

Given an economy $\mathcal{E}_n$ with degree sequence $d^{(n)} = (d_1, d_2, \ldots, d_n)$, the coefficient of variation is

\[
\text{CV}(d^{(n)}) \equiv \frac{\text{STD}(d^{(n)})}{\bar{d}}
\]

where $\bar{d} \equiv \frac{1}{n} \sum_{i=1}^{n} d_i$ is the average degree, and

\[
\text{STD}(d^{(n)}) \equiv \left[ \sum_{i=1}^{n} (d_i - \bar{d})^2 / (n - 1) \right]^{1/2}
\]

is the standard deviation of the degree sequence.
Unequal Degrees Create Fragility

Theorem

For any sequence of economies, aggregate volatility satisfies

$$\sigma_{agg} = \Omega \left( 1 + \text{CV}(d(n)) \sqrt{n} \right).$$

$$a_n = \Omega(b_n) \iff \liminf_{n \to \infty} a_n / b_n > 0$$

Corollary

Law of large numbers fails for the star network as $\sigma_{agg} \not\to 0$. 
Power Law Degree Distributions and Aggregate Volatility

We say that the degree distribution for a sequence of economies has power law tail structure if, there exists $\beta > 1$ such that for each $n$ and for large $k$

$$P_n(k) \propto k^{-\beta},$$

where $P_n(k)$ is the counter-cumulative distribution of degrees and $\beta$ is the shape parameter.

Corollary

For a sequence of economies with power law tail structure with shape parameter $\beta \in (1, 2)$,

$$\sigma_{agg} = \Omega \left( n^{-\frac{\beta - 1}{\beta}} - \varepsilon \right)$$

where $\varepsilon > 0$ is arbitrary.

A smaller $\beta$ corresponds to a “thicker” tail and thus higher coefficient of variation, and greater fragility.
Higher-Order Interconnections

- The degree distribution does not capture the full extent of asymmetry of “connections”.
  
  - As a result, the extent of cascading in the examples so far is limited.
  
  - Two economies with the same degree distribution can have very different structure of connections and very different nature of volatility:

Example: It can be verified that $\|\hat{v}\|_2 = \Theta(d/n + 1/\sqrt{d})$ (e.g., when $d$ is of order $\Theta(\sqrt{n})$, then $\|\hat{v}\|_2 = \Theta(1/\sqrt[4]{n})$), whereas $\|v\|_2 = \Theta(1)$. 

![Diagram of interconnections]
Second-Order Interconnections

Definition

The second-order interconnectivity coefficient is defined as

\[ \tau_2(W_n) \equiv \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i,j} w_{ji}w_{ki}d_jd_k, \]

where \( d_j \) is the degree of sector \( j \).

- \( \tau_2 \) takes higher values when high degree sectors share “parents”.
Unequal Connections Create Fragility

Theorem

*Given a sequence of economies, the aggregate volatility satisfies*

\[
\sigma_{\text{agg}} = \Omega \left( \frac{1}{\sqrt{n}} + \frac{\text{CV}}{\sqrt{n}} + \frac{\sqrt{\tau_2(W_n)}}{n} \right)
\]

\[
\tau_2 = 0
\]

\[
\tau_2 \sim n^2
\]
Power Law Distribution of Second-Order Degrees

- Define second-order degree as
  \[ q_i \equiv \sum_{j=1}^{n} d_j w_{ji}. \]

**Corollary**

For a sequence of economies with power law tail structure for the second-order degree with shape parameter \( \zeta \in (1, 2) \)

\[ \sigma_{agg} = \Omega \left( n^{-\frac{\zeta-1}{\zeta}} - \varepsilon \right), \]

for any \( \varepsilon > 0 \).

- If both first and second-order degrees have power laws, then
  \[ \sigma_{agg} = \Omega \left( n^{-\frac{\zeta-1}{\zeta}} - \varepsilon + n^{-\frac{\beta-1}{\beta}} \right), \]
  i.e., dominant term: \( \min \{ \beta, \zeta \} \).
Higher-Order Interconnections and Volatility

Definition

Given an economy, the \((m+1)^{th}\)-order interconnectivity coefficient is defined as

\[
\tau_{m+1}(W_n) \equiv \sum_{i=1}^{n} \sum_{j_1, \ldots, j_m} (d_{j_1} d_{k_1}) \left( w_{j_m i} w_{k_m i} \right) \prod_{s=1}^{m-1} w_{j_s j_{s+1}} \prod_{r=1}^{m-1} w_{j_r j_{r+1}}
\]

\(\tau_m\) captures the extent to which large suppliers share suppliers \(m\) levels upstreams in the chain.

Theorem

Given a sequence of economies and for any \(m \in \mathbb{N}\), aggregate volatility satisfies

\[
\sigma_{agg} = \Omega \left( \frac{1}{\sqrt{n}} + \frac{CV}{\sqrt{n}} + \frac{\sqrt{\tau_2(W_n)}}{n} + \cdots + \frac{\sqrt{\tau_m(W_n)}}{n} \right).
\]
Roughly Equal Degrees Create Robustness

Definition
A sequence of economies is balanced if $\max_i d_i < c$ for some positive constant $c$ and all $n$.

Proposition
For any sequence of balanced economies, $\sigma_{agg} \sim 1/\sqrt{n}$.

Examples of robust structures:
The US input-output matrix (not disaggregated enough, but still useful).

423 sectors in the commodity-by-commodity direct requirements table.

(Bureau of Economic Analysis)

This gives us the equivalent of our $W_n$ matrix.

Includes sectors

- Semi-conductor and related device manufacturing,
- Wholesale trade,
- Retail trade,
- Real estate,
- Truck transportation,
- Advertising and related services.
First and Second-Order Degree Distributions

- Linear tail in the log-log scale (20% of the observations = 82 industries).
- Power law is a fairly good approximation to the tail of the distribution.

\[ \beta \approx 1.5 \]

\[ \zeta \approx 1.3 \]
Implied Behavior of Aggregate Volatility

- $\zeta < \beta$: second-order effects dominate first-order effects.

- Average (annual) standard deviation of the logarithm of value-added across 459 four-digit (SIC) manufacturing industries between 1958 and 2005 is 0.219. (NBER Manufacturing Productivity Database)

- Since manufacturing is about 20% of the economy, for the entire economy this corresponds to $5 \times 459 = 2295$ sectors at a comparable level of disaggregation.

- Had the structure been balanced: $\sigma_{\text{agg}} = 0.219 / \sqrt{2295} \approx 0.005$.

- But from the lower bound from the second-order degree distribution:

$$\sigma_{\text{agg}} \sim \sigma / n^{0.23} \approx 0.037$$
Do the insights above extend to financial networks?

Two major points of departure:

(a) The world of finance is not linear
(b) Dominated by a few large financial institutions
(no counter-part to disaggregation)
A Minimalist Model of Financial Networks

- $n$ risk-neutral financial institutions (banks)
- three dates: $t = 0, 1, 2$
- each bank has an initial capital $k$
- Banks lend to one another at $t = 0$ and write standard debt contracts in exchange to be repaid at $t = 1$.
- The face value of $j$’s debt to $i$ is equal to $y_{ij}$.
- This defines a financial network: a weighted directed graph on $n$ vertices:
  - Each vertex corresponds to a bank.
  - A directed edge (with weight $y_{ij}$) from $j$ to $i$ is present if bank $i$ is a creditor of bank $j$. 
A Minimalist Model of Financial Networks

- After borrowing, bank $i$ invests in a project with returns at $t = 1, 2$.
  - random return of $z_i$ at $t = 1$.
  - deterministic return of $A$ at $t = 2$ (if held to maturity).

- Bank $i$’s obligations:
  - interbank commitments $\{y_{ji}\}$
  - a more senior outside obligation of value $v > 0$.

- If the bank cannot meet its obligations, it defaults:
  - liquidates its project prematurely and gets $\zeta A$
  - costly liquidation: $\zeta \rightarrow 0$
Summary: Timing and Description of Events

- **$t = 0$:**
  - interbank lending happens
  - banks invest in projects

- **$t = 1$:**
  - short term returns $\{z_i\}$ are realized,
  - banks have to meet the interbank and outside obligations
  - any shortfall leads to default and forces costly liquidation

- **$t = 2$:**
  - remaining assets have their long-run returns realized.
Payment Equilibrium

- Focus on $t = 1$ with the financial network taken as given:
  - $z_j \in \{a, a - \epsilon\}$: short-term returns.
  - $\epsilon$: magnitude of negative shock
  - $y_j$: the total commitments of bank $j$ to all other banks, $y_j = \sum_{j \neq i} y_{ij}$.
  - $v$: outside commitments.

\[
x_{ij} = \begin{cases} 
  y_{ij} & \text{if } z_j + \sum_s x_{js} \geq v + y_j \\
  \frac{y_{ij}}{y_j} (z_j + \sum_s x_{js} - v) & \text{if } v \leq z_j + \sum_s x_{js} < v + y_j \\
  0 & \text{if } z_j + \sum_s x_{js} < v,
\end{cases}
\]

- Payment equilibrium: a fixed point $\{x_{ij}\}$ of the above set of equations.
Payment Equilibrium

Proposition

A payment equilibrium exists and is generically unique.

Idea:

- Define $Q = [q_{ij}] \in \mathbb{R}^{(n+1) \times (n+1)}$, where $q_{ij} = y_{ij} / y_j$ (fraction of obligations of agent $j \neq 0$ to agent $i$). Define also the continuous mapping

  $$\Phi(x) = [\min \{Qx + e, y\}]^+,$$

  where $e_j = z_j - v$ for all $j \in \{1, \ldots, n\}$.

- A fixed point of $\Phi$ yields a payment equilibrium.

- **Existence:** follows from Brouwer.

- **Uniqueness:** We show that two fixed points $x$ and $\hat{x}$ exist only if $(Qx)_i + e_i, (Q\hat{x})_i + e_i \in [0, y_i]$, implying

  $$Qx + e = x, \quad Q\hat{x} + e = \hat{x}.$$

- Possible only if $\sum_{i=1}^{n} e_i = 0.$
Notions of Fragility

- Focus on regular financial networks: all banks have identical interbank claims and liabilities: for all banks $i$,

$$\sum_{j \neq i} y_{ij} = \sum_{j \neq i} y_{ji} = y, \text{ for some } y.$$ 

**Lemma**

*With $m$ shocks, the social surplus in the economy is equal to*

$$W = na - me + (n - \#\text{defaults})A.$$ 

- Number of defaults in the presence of $m$ negative shocks
  - **Resilience**: maximum possible number of defaults
  - **Stability**: expected number of defaults
Small Shock Regime

Proposition

There exist $\epsilon^*$ and $y^*$ such that for all $\epsilon < \epsilon^*$ and $y > y^*$,

(a) the complete financial network is the most stable and resilient,

(b) the ring financial network is the least stable and resilient,

(c) the $\gamma$-convex combination of the ring and complete financial networks becomes more stable and resilient as $\gamma$ increases.
Intuition

- Sparsity $\rightarrow$ fragility
  Interconnectivity $\rightarrow$ resilience
  Similar to Allen and Gale (2000) and Freixas et al. (2000)

- **Intuition:** the complete network reduces the impact of a given bank’s failure on any other bank, whereas in the ring, all the losses are transferred to the next bank.

- In contrast to Acemoglu et al. (2012)
  A model of input-output economies with linear interactions.
  - with linear interactions, positive and negative shocks cancel out.
  - non-linearities of the debt contracts imply that defaults cannot be averaged out by “successes”.
Resilience and Stability

Large Shock Regime

Proposition

If $\epsilon > \epsilon^*$ and $y > y^*$, then

(a) complete and ring networks are the least resilient and stable networks.

(b) “weakly connected” networks are more stable and resilient than both.

Reminiscent of epidemics

Phase transition:
with large shocks, the complete is as fragile as the ring
Intuition

- Two absorption mechanisms:
  1. The excess liquidity of non-distressed banks $a - v > 0$.
  2. The senior creditors of the distressed banks with claims $v$.

- The complete network:
  - utilizes (i) very effectively, more than any other network.
  - utilizes (ii) less than any other network.
  - when shocks are small, (i) can absorb all the losses.

- Weakly connected networks:
  - do not utilize (i) that much.
  - utilize (ii) very effectively.
  - with large shocks, networks that utilize (ii) more effectively are more stable.
Summary and Future Work

A framework for studying the relationship between the structure of networks and the extent of contagion and cascading failures for different models.

Future Work:

- Network effect from fire sale externalities (and withdrawal of liquidity by a bank).
- Systemic risk in technological networks.
- Control (open and closed loop) to mitigate systemic risk and cascading failures.