Fall, 2013   CIT 592

Mathematical Foundations of Computer Science

Final - 2 hrs

Please read every question carefully. All 3 of us are here and you are welcome to ask us for any clarifications.

Not all questions are easy. Not all questions are hard. Please strategize appropriately.

There is one long question wherein we have supplied you with a bunch of hints and have basically written the outline of the proof. Please fill the empty spaces in that proof.

Feel free to rip out the formula sheet and use it for scratch paper.

Please use the reverse side of any sheet to continue a long answer or to redo a question if you feel you messed up. But do put the question number.

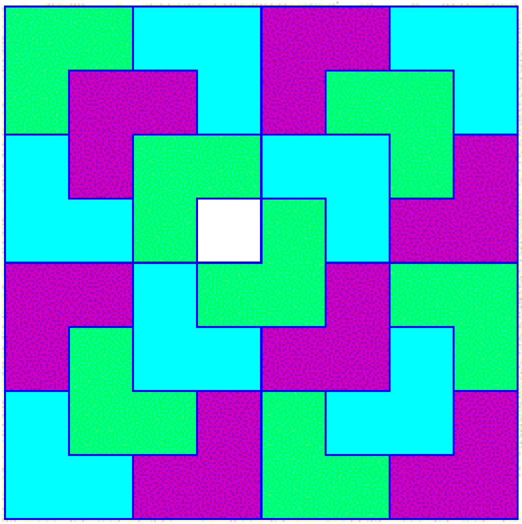
Good luck!

1. While watching a game of Champions League football in a cafe, you observe someone who is clearly supporting Manchester United in the game. What is the probability that person was actually born within 25 miles of Manchester ? Assume that

   - the probability that a randomly selected person in a typical cafe is born within 25 miles of Manchester is 1/20
   - the chance that a person born within 25 miles of Manchester actually supports United is 7/10 (well you do have some City fans.)
   - the probability that a person not born within 25 miles of Manchester supports United is 1/10

   Please define events in this question and DO NOT provide an answer without explanation. You are not required to do any arithmetic. (3 pts)

2. A right triomino is a 2-by-2 square minus one of the four squares. Show using induction that for any positive integer n, $2^n \times 2^n$ checkerboard/chessboard with any one square removed can be tiled using right triominoes. As an illustration, we have the tiling of the standard 8x8 chessboard with one square (the white square) taken out with triominiones(5 pts)



3. Let G be a connected graph that is not Eulerian. Prove that it is possible to add a single vertex to G, together with some edges from this new vertex to some old vertices such that the new graph is Eulerian. (4 pts)

4. What is the minium number of colours that you would need in order to colour K_n? What is the minimum number required for C_n? For this question it is ok to just give us the number. No explanation is needed.(2 pts)

5. A bipartite graph is defined as a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to a vertex in V.

   Show that a graph is bipartite if and only if it is 2-colourable.(4 pts)

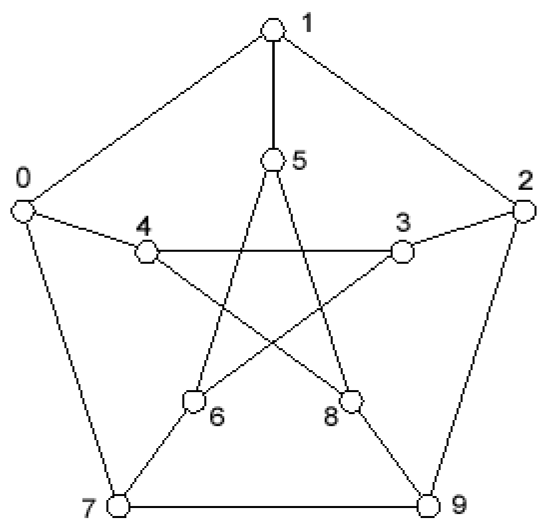
6. A graph G is said to be a forest if it can be partitioned into a bunch of trees. That is, G is a disjoint union of trees.

   Let G be a forest with n vertices and c connected components (in this case number of trees). Find and prove a formula for the number of edges of G.(2 pts)

7. Define a relation between points on the plane as two points being related if and only if the distance between them is less than 1. Is this relation reflexive? If so, then you don't need to write an explanation; but if not please provide a counter example. And then do the same for symmetric, antisymmetric, and transitive.(2pts)

8. In the graph shown below, answer the following questions (4 pts)

- What is the degree of each vertex of the graph?
- Is there an Eulerian tour in this graph? Is there an Eulerian trail from vertex 1 to vertex 9? Please mention what result you are using to say yes or no.
- Does the graph satisfy the conditions for Ore's theorem?
- Is there a Hamiltonian path (not cycle!)? If such a path exists, please list the vertices in the order in which I should visit them.



9. The degree sequence of an undirected graph is defined as the degrees of all the vertices of the graph being written out in a non-decreasing sequence. As an example, the degree sequence for P_3 would be $1, 1, 2$.

   Show that a degree sequence of length $n \geq 2$ can be the degree sequence of a tree if and only if

   (a) $d_i \geq 1$ for every i, that is there is no 0 in the sequence.

   (b) $\sum_{i=1}^{n} d_i = 2n - 2$

   For this particular question, since it is a little complicated we will you some hints. Fill in the blanks to get the solution.

   Also please note that the proof is spread out over 3 pages, so please be sure to read every page.

Proof

(1 pt) If the graph is a tree, then

Conversely, assume we are given a degree sequence that satifies those 2 properties (9a and 9b) and we want to show that the graph has to be a tree.

We proceed by induction on the length of the degree sequence.

(1 pt) Base case:

Induction hypothesis: If there is a degree sequence $e_1, e_2, \ldots, e_{n-1}$ that satisfies each $e_i \geq 1$ and $\sum_{i=1}^{n-1} e_i = 2(n-1) - 2$, then the graph corresponding to this degree sequence has to be a tree.

Now consider a degree sequence of length n that satisfies the two properties. That is we have a sequence $d_1, d_2, \ldots, d_n$ and each $d_i \geq 1$ and $\sum_{i=1}^{n} d_i = 2n - 2$.

Note that as a result of these two properties we get $d_1 = 1$. Fill the box below with why that is true. (1pt)

Now consider the sequence of degrees $d_2, d_3, \ldots, d_n$. It is of length $n - 1$, so we would like to apply the induction hypothesis. However when we sum the degrees, we get $2n - 3$.

Instead consider the sequence $d_2, d_3, \ldots, d_n - 1$.

This sequence satisfies the first property 9a because (1 pt)

This sequence satisfies the second property 9b because (1 pt)

Now apply the induction hypothesis and finish the proof of. Remember that you need to ‘put back’ the d_1 vertex.(2 pts)

10. One possible formation in soccer(football in most places) is to play 3-4-3, that is, 3 defenders, 4 midfielders and 3 forwards and of course, 1 goalkeeper. A team has 11 players.

    A team that plays in the 3-4-3 style lines up before the national anthem is played (that is in 1 straight line). The players who play the same position insist upon being lined up together. In how many ways can they be arranged.

    To clarify, an arrangement like M-M-F-F-M-D-G-D-D-M-F would not be ok with these players.

    When arranging the players in this manner, what is the probability that the goalkeeper is at one of the ends.

    (4 pts)

11. Using quantifiers and standard math notation, write down a quantified version of this statement

    'integers of the form 4n+3 cannot be written as a sum of squares of two integers' (2 pts)

12. There are n people in a room. We are going to draw something called the birthday graph for them. Put down n vertices for the people. If two people have the same birthday draw an edge between them. Assume that a person is equally likely to have his/her birthday on any day of the year. Also a year has 365 days. What is the expected number of edges?

Here are two hints that will help you in this question.

- How many pairs of people are there?
- For any pair, what is the probability that they have the same birthday?

(4 pts)

13. $T(n) = 16T(n/4) + n$ with $T(1) = 1$. Using the master theorem, what is the big-Oh running time of this algorithm. (2 pts)

The End
Thank you for making this course fun to teach.
Meet at 6pm outside Mad4Mex to go and celebrate the end of the semester!