Fall, 2013 CIT 592

Mathematical Foundations of Computer Science

Midterm2

- Please write your name and email on this first sheet.
- Please make at least one of them legible.
- Your answer needs to fit into the space provided. You are also allowed to use the back of the page to continue your solution.
- The formula sheet is the last sheet of the exam.
- You are allowed to leave your answers in the form of fractions. You are also allowed to retain things in the form $\binom{n}{r}$.
- When expressing a big-Oh result, remember to provide the answer in a 'simple' form.

Questions.

1. In the parlour game Nim, there are two players and two piles of matches. At each turn, a player removes some (non-zero) number of matches from one of the piles. The player who removes the last match wins. If the two piles contain the same number of matches at the start of the game, then the second player can always win. Figure out what the strategy for winning is and then prove this strategy does guarantee a win by using induction. (6 pts)

2. Solve $T(n) = 3T(n/2) + n^2$, given $T(1) = 1$ by drawing out a recursion tree. You can safely assume n is a power of 2. DO NOT APPLY THE MASTER THEOREM DIRECTLY. (4 pts)

3. A jigsaw puzzle consists of a number of pieces. Two or more pieces with matched boundaries can be put together to form a "big" piece. To be more precise, we use the term block to refer to either a single piece or a number of pieces with matched boundaries that are put together to form a "big" piece. Thus, we can simply say that blocks with matched boundaries can be put together to form another block. Finally, when all pieces are put together as one single block, the jigsaw puzzle is solved. Putting 2 blocks together with matched boundaries is called one move.

   Prove, using induction, that for a jigsaw puzzle with n pieces, it will always take $n-1$ moves to solve the puzzle. (3 pts)

4. Here's a recursive way to compute x^n. Consider a recursive function defined as $pow(x,n) = \begin{cases} pow(x, \frac{n}{2}) \times pow(x, \frac{n}{2}) & n \geq 2 \\ x & n = 1 \end{cases}$

   Basically we are doing this computation by observing $x^n = x^{n/2} \times x^{n/2}$. Let $T(n)$ represented the number of steps taken to compute this. Write a recurrence for $T(n)$. Now use the master theorem to solve this recurrence in big-theta. (2 pts)

5. Prove by structural induction that if you have a triangulated simple polygon with $n \geq 4, n \in \mathbb{N}$ vertices, it must have $n-3$ diagonals in the triangulation. To be clear, we are not counting the diagonals that are there in the original polygon. We are only counting the diagonals that contribute to this triangulation. (4 pts)

6. In a poker hand consisting of 5 cards (coming from the deck of 52 cards), find the probability of holding 3 aces. (2 pts)

7. Prove by induction that for any non-negative integer n, $(11)^{n+2} + (12)^{2n+1}$ is divisible by 133. (4 pts)

8. Arvind's pet chimpanzee has only learnt the letters 'a', 'b', 'c' and 'd'. If this chimpanzee now writes a 6-letter word using only these letters, what is the probability that this word contains at least one 'a', at least one 'b', at least one 'c', and at least one 'd'? Do not compute the exact value over here. Leave it in fraction form.(4 pts)

9. A company has 200 employees: 120 are women and 80 are men. Of the 120 female employees, 40 are classified as managers, while 20 of the 80 male employees are managers. Suppose that an employee is chosen at random.

   - Find the conditional probability that the employee is a manager given that the employee is female.
   - Find the conditional probability that the employee is female given that the employee is a manager. Compute this using the direct formula and then also compute it explicitly using Bayes Theorem.

   (6 pt)