EIGENVALUE PROBLEMS

Definition 1 (eigenvector and eigenvalue). Say matrix $A \in \mathbb{C}^{m \times m}$, then we call a nonzero vector $x \in \mathbb{C}^m$ an eigenvector, and $\lambda \in \mathbb{C}$ an eigenvalue, if $Ax = \lambda x$.

Lemma 1. Even if a matrix is real, some of its eigenvalues may be complex.

Lemma 2. Eigenvectors with different eigenvalues must be linearly independent.

Proof. Tip: show from the definition of eigenvalue and eigenvector.

Definition 2 (spectrum). The set of all eigenvalues of $A$ is the spectrum of $A$.

Definition 3 (eigenvalue decomposition). If we can write $A = X \Lambda X^{-1}$ where $\Lambda$ is diagonal and $X$ is invertible, then we call this the eigenvalue decomposition of $A$.

Note that eigenvalue decompositon does not always exist.

Definition 4 (eigenspace). The set of eigenvectors corresponding to a single eigenvalue, together with the $0$ vector, forms a subspace of $\mathbb{C}^m$, we call this an eigenspace $E_\lambda$.

Lemma 3. Each eigenvalue corresponds to $\text{dim}(E_\lambda)$ linearly independent eigenvectors.

Definition 5 (geometric multiplicity). The geometric multiplicity of an eigenvalue $\lambda$ is the dimension of its eigenspace $E_\lambda$. It also equals to $\text{dim}(\text{Null}(A - \lambda I))$.

Definition 6 (characteristic polynomial). The characteristic polynomial of $A \in \mathbb{C}^{m \times m}$ is (of degree $m$):

$$p(z) = \det(zI - A).$$

The coeff of its degree $m$ term is $1$.

Corollary 3.1. $\lambda$ is an eigenvalue $\iff p(\lambda) = 0$.

Definition 7 (algebraic multiplicity). Write $p(z) = (z - \lambda_1)(z - \lambda_2)...(z - \lambda_m)$ for some $\lambda_j \in \mathbb{C}$. If an eigenvalue appears more than once, call that the algebraic multiplicity of the eigenvalue. If the algebraic multiplicity is $1$, then this eigenvalue is called simple.

Corollary 3.2. All roots of $p(z)$ are simple $\iff A$ has $m$ distinct eigenvalues.

Definition 8 (similarity transformation). If $X \in \mathbb{C}^{m \times m}$ is nonsingular, then $A \rightarrow B = X^{-1}AX$ is called a similarity transformation of $A$, where $B$ and $A$ are similar.
Theorem 4. Similar matrices \((A \text{ and } X^{-1}AX)\) have the same
- characteristic polynomial
- eigenvalues
- algebraic multiplicity
- geometric multiplicity

Proof. Tip for geometric multiplicity: If \(E_\lambda\) is an eigenspace of \(A\), we can easily show \(X^{-1}E_\lambda\) is eigenspace of \(X^{-1}AX\).

Theorem 5. Algebraic multiplicity \(\geq\) geometric multiplicity for any \(\lambda\).

Proof. Let \(n\) be the geometric multiplicity of \(\lambda\) for \(A \in \mathbb{C}^{m \times m} (m \geq n)\). The eigenspace of \(\lambda\) is \(E_\lambda\) with \(\text{dim } n\). We can form an orthonormal basis for it, and extend these \(n\) basis vectors to \(m\) orthogonal vectors. That gives us a matrix \(V \in \mathbb{C}^{m \times m}\). Let’s define a matrix \(B := V^*AV = \begin{bmatrix} \lambda I_{n \times n} & C \\ 0 & D \end{bmatrix}\), then \(p_B(z) = (z - \text{lambda})^n \det(zI - D)\), therefore the algebraic multiplicity of \(\lambda\) for \(B\) is at least \(n\). Since \(A\) and \(B\) are similar, we can use theorem 4 to know that the algebraic multiplicity of \(\lambda\) for \(A\) is at least \(n\).

Definition 9 (defective eigenvalue). An eigenvalue whose algebraic multiplicity \(>\) geometric multiplicity is a **defective eigenvalue**. The corresponding matrix is a **defective matrix**.

Corollary 5.1. Diagonal matrix is nondefective.

Theorem 6. \(A^{m \times m}\) is **nondefective**. \(\iff\) \(A\) has \(m\) linearly independent eigenvectors. \(\iff\) \(A\) is **diagonalizable** \((A = X^{-1}AX)\).

Proof. The first \(\rightarrow\) can be shown using Lemma 3 and Lemma 2.

Lemma 7. Determinant and trace are related to eigenvalues only.

Definition 10 (unitary matrix). Matrix \(Q\) is unitary if \(Q^*Q = QQ^* = I\).

Definition 11 (unitary diagonizable). Matrix \(A\) is unitary diagonizable if \(A = Q\Lambda Q^*\) for some unitary matrix \(Q\).

Definition 12 (Hermitian matrix). Matrix \(A\) is Hermitian if \(A = A^*\).

Definition 13 (normal matrix). Matrix \(A\) is normal if \(A^*A = AA^*\).
Lemma 8. A Hermitian matrix is a normal matrix.

Lemma 9. A is unitarily diagonalizable. ⇔ A is normal.

Lemma 10 (Schur factorization). Every square matrix $A \in \mathbb{C}^{n \times n}$ has a Schur factorization, meaning $A = QTQ^*$ for a unitary $Q$ and an upper-triangular $T$.

Proof. We can prove this using induction. The lemma is trivially true for $n = 1$. Now suppose it is true for all matrices with size $< n$. Now we are looking at $A \in \mathbb{C}^{n \times n}$ where $n > 1$. Since every matrix has eigenvalues over the complex domain (through the polynomial root), we can take one eigenvalue $\lambda$ and an eigenvector $u_1 \in \mathbb{C}^n$. Now we use Gram-Schmidt to expand $u_1$ to an orthonormal basis of $\mathbb{C}^n$, let’s call the resulting unitary matrix $U_1 = [u_1, u_2, ..., u_n]$.

We have

$$U_1^*AU_1 = \begin{bmatrix} \lambda & * \\ \vdots & \vdots \\ \lambda u_1^* & * \\ u_n^* \\ \end{bmatrix} = \begin{bmatrix} \lambda & * \\ 0 & A_1 \\ \end{bmatrix}.$$

Note that $A_1$ is $(n - 1) \times (n - 1)$. Now through deduction assumption we know there exists upper-triangular $T_1 = U_2^*A_1U_2$.

We define $U_3 = \begin{bmatrix} 1 & 0^T \\ 0 & U_2 \end{bmatrix}$, and $U = U_1U_3$, then we can show $U^*AU$ has an upper-triangular form.

Theorem 11. A Hermitian matrix is unitarily diagonalizable, and its eigenvalues are real. Particulaly, all real symmetric matrices are orthogonal diagonalizable with real eigenvalues.

Proof. Use Lemma 10 we can write

$$Q^*AQ = T.$$

Take its conjugate we get $Q^*A^*Q = T^*$, since $A = A^*$ we know $T = T^*$. Note that $T$ is upper triangular and complex. $T = T^*$ means $T$ has to be diagonal and real.