Deformation Gradient

\[ F \, dX = dx \]

\[ F \, dX_j \, E_j = dx_k \, e_k \]

\[ e_i \cdot (F \, dX_j \, E_j) = e_i \cdot dx_k \, e_k \]

\[ e_i \cdot F E_j \, dX_j = dx_i \]

Let's call this \( F_{ij} \), then \( dx_i = F_{ij} \, dX_j \).

Further more \( dx = dx_i \, e_i \)

\[ = F_{ij} \, dX_j \, e_i \]

\[ = F_{ij} \, (dX \cdot E_j) \, e_i \]

\[ = F_{ij} \, (e_i \otimes E_j) \, dX \]

Recall \( dx = F \, dX \)

Therefore \( F = F_{ij} \, (e_i \otimes E_j) \), \( F_{ij} = \frac{\partial X_i}{\partial X_j} \)

"two point tensor!"
Now make things easier by choosing \( \{ E_i \} = \{ e_i \} \).

- Examples of deformation gradient

1. Translation.
   \[
   x = X + t v \ N
   \]
   \[
   F = \frac{\partial x}{\partial X} = \begin{pmatrix}
   \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\
   \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2}
   \end{pmatrix} = \begin{pmatrix}
   1 & 0 \\
   0 & 1
   \end{pmatrix} = I
   \]

2. Rotation.
   \[
   x = R \ X + b
   \]
   \[
   F = \begin{pmatrix}
   \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\
   \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2}
   \end{pmatrix} x_i = R_{ij} x_j + b_j
   \]

   \( \text{is this clear to everyone?} \)

   \( \text{① Way 1. Explicit write down all.} \)
   \[
   x_1 = R_{11} x_1 + R_{12} x_2 + b_1,
   x_2 = R_{21} x_1 + R_{22} x_2 + b_2
   \]
   \( \Rightarrow \)
   \[
   F = \begin{pmatrix}
   R_{11} & R_{12} \\
   R_{21} & R_{22}
   \end{pmatrix} = R
   \]

   \( \text{② Way 2. purely index} \)
   \[
   x_i = R_{ij} x_j + b_i : \text{ all scalars.}
   \]
   \[
   \frac{\partial x_i}{\partial X_j} = R_{ij} \quad \text{easy!}
   \]

   \( \text{Powerfulness of implicit summation.} \)
3. \[ \chi_1 = \frac{1}{4} \left( 18 + 4X_1 + 6X_2 \right) \]

\[ \chi_2 = \frac{1}{4} \left( 4 + 6X_2 \right) \]

\( X_2, \chi_2 \)

\[ F = \begin{pmatrix} 1 & 3/2 \\ 0 & 3/2 \end{pmatrix} \]
Let's look at the determinant.

1. $\det(E) = 1$
2. $\det(\mathbb{R}) = 1$
3. $\det(F) = \det\left(\begin{array}{ccc} 1 & \frac{3}{2} \\ 0 & \frac{3}{2} \end{array}\right) = \frac{3}{2} > 1$

What does $\det(F)$ tell us?
The answer is volume change!

Let $J = \det(F)$

Then only small volume.

2D, let's take a small triangle.

$a = FA, \quad b = FB, \quad [a, b] = F[A, B]$

$\det([a, b]) = J \det([A, B])$

Note \( \text{triangle volume} = \frac{1}{2} |A \times B| \)

$= \frac{1}{2} (A_x B_y - A_y B_x)$

$= \frac{1}{2} \det([A, B])$

So volume change is $dv = J \, dV$.
Continuum Motion, Velocity and Acceleration.

- \[ x = \phi (x, t) = \mathcal{X} (x, t) \quad x \in \mathcal{X}, \quad t \in T. \]
- \[ V (x, t) : The \ velocity \ of \ the \ material \ particle \ labeled \ by \ x \ in \ \mathcal{X}, \ at \ time \ t. \]

It is a Lagrangian quantity!

\[
\begin{array}{c}
\text{4} \\
\text{7} \\
\text{A} \\
\end{array}
\hspace{2cm}
\begin{array}{c}
\text{14} \\
\text{D} \\
\end{array}
\]

: a rotating and translating box.

Now what is \[ V \left( (7^t), t \right) \]?

A: \[ \]
B: \[ \]
C: \( (0) \)

Now recall we had a Lagrangian vs. Eulerian.

\[ V \left( (x, t) \right) \quad \text{b. vs.} \quad V \left( (X, t) \right) \]

What does this mean?

What is \[ V \left( (7^t), t \right) \]?
A: \[ \]
B: \[ \]
C: \( (0) \)

What is \[ V \left( (12^t), t \right) \]?
A: \[ \]
B: \[ \]
C: \( (0) \)

What is \[ V \left( (14^t), t \right) \]?
A: \[ \]
B: \[ \]
C: \( (0) \)
we observe \( V\left(\begin{pmatrix}4 \\ 1 \end{pmatrix}, t\right) = V\left(\begin{pmatrix}12 \\ 14 \end{pmatrix}, t\right) \)

Recall: \( x = (X, t) \)
\[
R(x, t) = R\left(\phi(X, t)\right, t\right)
\]
\[
P(x, t) = R\left(\phi^{-1}(x, t)\right, t\right)
\]
something is going on here!

Yes, it is \( V(x, t) = V\left(\phi(x, t), t\right) \)!

push forward!

Look backwards... \( V\left(\begin{pmatrix}12 \\ 14 \end{pmatrix}, t\right) = V\left(\begin{pmatrix}4 \\ 7 \end{pmatrix}, t\right) \), this is just
\[
V(x, t) = V\left(\phi^{-1}(x, t), t\right)
\]
pull back!

Now we know what \( V(x, t) \) means.

To compute it, \( V(x, t) = \frac{\partial\phi}{\partial t}(X, t) \)

E.g. Translation \( x = \phi(X, t) = X + tvn \)
\[
V(x, t) = \frac{\partial\phi}{\partial t} = v n
\]

Scaling \( x = \phi(X, t) = (t + 1)X \)
\[
V = \frac{\partial\phi}{\partial t} = X
\]
Acceleration:
\[
A(x, t) = \frac{\partial V}{\partial t}(x, t) = \frac{\partial^2 \phi}{\partial t^2}(x, t)
\]

Lagrangian quantities:
- \( x = \phi(x, t) \)
- \( v = \frac{\partial \phi}{\partial t}(x, t) \)
- \( a = \frac{\partial v}{\partial t}(x, t) \)

\[
F = \frac{\partial \phi}{\partial x}(x, t)
\]

is a Lagrangian quantity.

Therefore we say "The deformation gradient of a material particle".

Lagrangian \( v \) and \( a \) are simple. How about Eulerian?

1. \( v(x, t) = \frac{\partial \phi}{\partial t}(x, t) \)
   
   plug in
   \[
v(a, t) = v(\phi^{-1}(a, t), t)
   \]
   
   \[
   = \frac{\partial \phi}{\partial t}(\phi^{-1}(a, t), t)
   \]
   
   still ok.

2. \( a : a(x, t) = \frac{\partial v}{\partial t}(x, t) \)

   \[
a(x, t) = \frac{\partial v}{\partial t}(x, t) \]

   where
   \[
   \frac{\partial v}{\partial t} = \frac{\partial^2 \phi}{\partial t^2}(x, t)
   \]
   
   using \( v(x, t) = v(x, t) \)

   we have
   \[
   = \frac{\partial v}{\partial t}(\phi(x, t), t) + \frac{\partial v}{\partial a}(\phi(x, t), t) \frac{\partial \phi}{\partial t}(x, t)
   \]

   Therefore
   \[
a(x, t) = \frac{\partial v}{\partial t}(\phi(x, t), t) + \frac{\partial v}{\partial a}(\phi(x, t), t) \frac{\partial \phi}{\partial t}(x, t)
   \]

   \[
   = \frac{\partial v}{\partial t}(a, t) + \frac{\partial v}{\partial a}(a, t) \frac{\partial \phi}{\partial t}(a, t)
   \]

   Note \( a(x, t) \neq \frac{\partial v}{\partial a}(x, t) \)!
Material derivative

\[ a (x, t) = \frac{\partial x}{\partial t} (x, t) + \frac{\partial v}{\partial x} (x, t) \cdot v(x, t) \]

\[ =: \frac{Dv}{Dt} (x, t) \]

In general, for any Eulerian function \( f(\cdot, t) : \mathbb{R}^n \rightarrow \mathbb{R} \)

\[ \frac{Df}{Dt} (x, t) = \frac{\partial f}{\partial t} (x, t) + \frac{\partial f}{\partial x} (x, t) \cdot v(x, t) \]

Observe:

\[ \begin{cases} 
A(x, t) = \frac{\partial x}{\partial t} (x, t) \\
a(x, t) = \frac{Dv}{Dt} (x, t) 
\end{cases} \]

Therefore, \( \frac{D}{Dt} (x, t) \) is the push-forward of \( \frac{\partial}{\partial t} (x, t) \)!