Data Structures and Algorithms
Homework Assignment 2

Given: January 18, 2016
Due: January 28, 2016

Note: The homework is due electronically on the course website on Thursday, January 28 by 10:30 am, the beginning of the class. For late submissions, please refer to the Late Submission Policy on the course webpage: http://www.seas.upenn.edu/~cis121

You must use the hw121.cls template provided on the course website.

Please write concise and clear solutions; you will get only a partial credit for correct solutions that are either unnecessarily long or not clear.

You are allowed to discuss ideas for solving homework problems in groups of up to 3 people but you must write your solutions independently. Also, you must write on your homework the names of the people with whom you discussed.

Finally, you are not allowed to use any material outside of the class notes and the textbook. Any violation of this policy may seriously affect your grade in the class.

1. The very famous rock band, the Pigeonhole Principle Players of Philadelphia (PPPP) needs to ship their equipment to \( n \) different cities for their nationwide tour. They have \( n \) buses that they can use. Each of the buses has a schedule that says, for each day of the month, which city it is currently at, or whether it is in transit. (You can assume that the “month” here has \( m \) days for some \( m > n \)). Each bus stops at each city for exactly one day during the month. In order to stop crazed fans from swarming the buses, the PPPP operation wants to stay covert as they visit cities, so they have the following requirement of the schedule:

\[(*) \text{ No two buses can be in the same city on the same day} \]

Due to the extensive touring the band has done, the buses require some maintenance this month. The maintenance will be performed via the following scheme. Each city has a maintenance station, so they decide that they can modify the schedule by having each bus \( B_i \) receive service in a specific city. That is, for each bus \( B_i \), there will be some day when it delivers its supplies to the scheduled city, say \( C_j \), and then remains in that city for the rest of the month. This bus \( B_i \) will not visit the remaining cities for that month, but it will have the same original schedule up to the point where they reach the city \( C_j \).

The fate of the PPPP’s US tour depends on you! Given the schedule for each bus, can you modify the schedule for maintenance but maintain the requirement \((*)\)? (If two buses are in the same city, the fans will surely discover them, and there could be a riot!) That is, for each bus \( B_i \), find a city \( C_j \) where the bus will receive maintenance for the rest of the month such that the above condition is met.
Show that this modification can always be found, and give an algorithm to find it. Prove the correctness of your algorithm.

Hint: try reducing this problem to the stable matching problem!

2. The Pigeonhole Principle Players of Philadelphia (PPPP), are playing a concert. As the PPPP’s biggest fans, a group of CIS 160 and CIS 121 TAs are going together. There are $n$ TAs from 121 and $n$ TAs from 160 who attend. At this concert, the TA’s want to dance together. The TAs match themselves with someone from a different class than their own. Consider two possible stable matches for the dances, $X$ and $Y$. The third dance comes on, and the TAs need to figure out how to construct another stable matching $Z$. For each 121 TA $a$, if $X$ pairs $a$ with the 160 TA $b_x^a$ and $Y$ pairs $a$ with $b_y^a$, then in $Z$, $a$ is paired with the 160 TA that $a$ prefers most among $b_x^a$ and $b_y^a$. Note that $b_x^a$ and $b_y^a$ could be the same TA. Prove or disprove that $Z$ is a stable matching.

Now consider a pairing $Z'$ in which each 121 TA $a$ is paired with the 160 TA that $a$ prefers the least among $b_x^a$ and $b_y^a$. Prove or disprove that $Z'$ is a stable matching.

3. There are $n$ players playing a ping-pong tournament. Each pair of players plays some non-zero number of matches. For any two distinct players $a$ and $b$, if you wanted to know if $a$ had won a match against player $b$, you can find out by asking the yes/no question: “Has $a$ won any match against $b$?” This is the only type of question that you are allowed to ask. Thus, if you wanted to know if $a$ won a match against $b$ and won a match against $c$ then you can find out this information by asking two questions: “Has $a$ won any match against $b$?” and “Has $a$ won any match against $c$?”

Give an algorithm to determine if there is a player who has won all his matches and if so, your algorithm must output the player. Your algorithm should use $O(n)$ questions. Justify your answer.

4. A round robin badminton tournament has $2n$ participating players. Two rounds have been played so far. Prove that we can still split the players into two groups of $n$ players each so that no players in the same group have played each other yet.

5. Let’s return to the dancing scenario in problem 2. The 160 TAs are a little peeved that they didn’t get to do the asking when the stable matches were formed. Alex, one particularly clever 160 TA thinks that he can outsmart those pesky 121 TAs by misrepresenting his preferences. Suppose Alex prefers the 121 TA $a$ to $a'$, but both $a$ and $a'$ are low on his list of preferences. Can it be the case that if he misrepresents his preferences and switches the order of $a$ and $a'$ on his list of preferences when the algorithm is run, Alex will get to dance with a 121 TA he truly prefers to both $a$ and $a'$?

You should prove this by either:

a. Give a proof that, for any instance of the Stable matching problem, no 160 TA can improve their partner by switching the order of a pair on their list; or

b. Given an example of a set of preference lists for which Alex can improve his partner by switching the ordering of a pair on his list.