Graph Representations

Let $G = (V, E)$ with $|V| = n$, $|E| = m$.

Adjacency Matrix

One way to represent $G$ is with an $n \times n$ matrix $A$ where $A[i][j] = 1$ if there is an edge from vertex $i$ to vertex $j$ and 0 otherwise. The primary advantage of this approach is that you can check whether or not there is an edge connecting two vertices in $O(1)$ time. The disadvantage, however, is that this representation takes up $O(n^2)$ space. When $n$ is large, this might become untenable.

Two things worth noting:

- If $G$ is undirected, then its adjacency matrix is symmetric.
- Entries along the diagonal of an adjacency matrix (technically representing the presence of edges from vertices to themselves) are 0 by convention, as our graphs are simple.

Adjacency List

Another way to represent $G$ is to use an adjacency list. Each vertex $v$ is associated to a list $\text{neighbors}(v)$ which contains the nodes $u$ such that $(v, u) \in E$. The advantage of this representation is that we use less space, $O(n + m)$, which is better than $O(n^2)$ of adjacency matrices as long as $m \ll n^2$. The disadvantage, though, is that checking whether $(u, v) \in E$ takes (potentially) linear time.

Graph Traversals

We now look at two ways to traverse a graph.

BFS (Breadth First Search)

In BFS, we begin at a node $v$ (level 0) and explore the graph in “layers.” First we would explore all children of $v$ (level 1), then the children of the nodes in level 1 (these would make up level 2), etc. The key point here is that we explore all nodes at level $i$ before exploring any nodes at level $i + 1$. The output of BFS is called a BFS tree. We typically use a queue to implement this algorithm. For implementation details, see https://en.wikipedia.org/wiki/Breadth-first_search.

The running time of BFS is $O(n + m)$, because each vertex is added and removed from the queue once and, in the worst case, we need to traverse every edge to visit each node.

DFS (Depth First Search)

In DFS, we begin at a node $v$ and examine its neighbors. As soon as we encounter a neighbor that hasn’t been visited, visit it. Once we arrive at a node for which all of its neighbors have been visited, we “backtrack” until we reach a node that has still unvisited neighbors (in the form of returning from recursive visit calls). For more information, see https://en.wikipedia.org/wiki/Depth-first_search.

The running time analysis for DFS is similar to that of BFS, giving a running time of $O(n + m)$. 
Problem 1: Cycle Detection

Design an algorithm to determine whether or not a graph has a cycle.

Problem 2: Bipartite Graph Detection

Design an algorithm to determine whether or not a graph is bipartite (that there is a way to partition $V$ into sets $V_1$ and $V_2$ such that if $(u, v) \in E$, then $u, v$ cannot both be in $V_1$ or $V_2$).

Problem 3: Bipartite Graph Detection

Design an algorithm to find the shortest path between nodes $u$ and $v$ in a connected, unweighted graph.