Learning Goals

During this lab, you will:

1. Review stacks and queues.
2. Review amortized runtime analysis and strengthen intuition for applying it to new problems.
3. Practice using stacks and queues to accomplish a variety of tasks.

Stacks and Queues

Recall the stack and queue ADTs (abstract data types) from lecture. Each is characterized by a specific way of removing elements and has a set of supported operations.

<table>
<thead>
<tr>
<th>Stack</th>
<th>Queue</th>
</tr>
</thead>
<tbody>
<tr>
<td>• LIFO (last-in-first-out) - the most recent element that has been added to the stack will be removed first.</td>
<td>• FIFO (first-in-first-out) - the least recent element that has been added to the queue will be removed first.</td>
</tr>
<tr>
<td>• Supported operations:</td>
<td>• Supported operations:</td>
</tr>
<tr>
<td>– push</td>
<td>– enqueue</td>
</tr>
<tr>
<td>– pop</td>
<td>– dequeue</td>
</tr>
<tr>
<td>– peek</td>
<td>– peek</td>
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<tr>
<td>– isEmpty</td>
<td>– isEmpty</td>
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<tr>
<td>– size</td>
<td>– size</td>
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Implementation Details

Stacks and queues can be implemented “under the hood” with almost any data structure. In this course, we will implement stacks and queues using expandable arrays. The rules we will use for increasing or decreasing the size of a stack or queue’s underlying array are as follows:

1. If the array of size \( n \) is full, create a new array of size \( 2n \), and copy all elements into the new array.
2. If the array of size \( n \) has \( \frac{3}{4} \) elements in it, create a new array of size \( \frac{5}{2} \), and copy all elements into the new array.
Amortized Analysis

*Amortized analysis* refers to finding the time-averaged cost for a sequence of operations. In other words, it is the time required to perform a sequence of operations averaged over all the operations performed.\(^1\)

Since amortized analysis for the stack push operation was covered in lecture, we are going to take a closer look at the stack pop operation.\(^2\)

The worst case runtime for a single pop operation is \(O(n)\), since we may need to resize the array and copy the elements into it. Based on this runtime, we might conclude that a tight bound for the worst case runtime for \(n\) pop operations is \(O(n^2)\), since there are \(n\) operations and each operation takes worst case \(O(n)\) time; however, we can find a tighter bound through some careful analysis.

If we start from a full stack of size \(n\), what is the total cost of a sequence of \(n\) pop operations?

Initially, the array is of size \(n\) and contains \(n\) elements. To make our analysis simpler, let’s immediately pop the first \(\frac{n}{2}\) elements. Each of these pops takes \(O(1)\) time. Now our array is of size \(n\) but contains only \(\frac{n}{2}\) elements.

In accordance with our rules, we can pop \(\frac{n}{2}\) more elements before resizing the array. Each of these pops takes \(O(1)\) time. Once we have pop’d those elements (leaving us with \(\frac{n}{4}\) elements in our array), we must reduce the size of our array to \(\frac{n}{2}\), and copy the remaining \(\frac{n}{2}\) elements into the new array. Thus, the total cost for the first \(\frac{n}{2}\) pop operations is \(T(\frac{n}{2}) = \frac{n}{2} + (\frac{n}{4} + \frac{n}{2} + \frac{n}{4})\).

We can apply identical analysis to the new array of size \(\frac{n}{2}\) that contains \(\frac{n}{4}\) elements. We get \(\frac{n}{8}\) pops “for free”, after which we resize the array to be of size \(\frac{n}{4}\), and copy the remaining \(\frac{n}{4}\) elements into the smaller array. Thus, the total cost for the first \(\frac{7n}{8}\) pop operations is \(T(\frac{7n}{8}) = \frac{n}{2} + (\frac{n}{4} + \frac{n}{2} + \frac{n}{4} + \frac{n}{8})\).

Are you noticing a pattern?

Let’s rewrite the expression slightly and continue to expand it:

\[
T(n) = \frac{n}{2} + \left(\frac{1}{4} \left(\frac{n}{20}\right) + \frac{1}{2} \left(\frac{n}{25}\right) + \frac{1}{4} \left(\frac{n}{20}\right)\right)
+ \left(\frac{1}{4} \left(\frac{n}{27}\right) + \frac{1}{2} \left(\frac{n}{27}\right) + \frac{1}{4} \left(\frac{n}{27}\right)\right)
+ \left(\frac{1}{4} \left(\frac{n}{27}\right) + \frac{1}{2} \left(\frac{n}{22}\right) + \frac{1}{4} \left(\frac{n}{22}\right)\right)
+ \ldots
+ \left(\frac{1}{4} \left(\frac{n}{4}\right) + \frac{1}{2} \left(\frac{n}{4}\right) + \frac{1}{4} \left(\frac{n}{4}\right)\right)
\]

We can now calculate the total cost of \(n\) pop operations:

\[
T(n) \leq \frac{n}{2} + \sum_{i=0}^{\infty} \left(\frac{1}{4} \left(\frac{n}{2i}\right) + \frac{1}{2} \left(\frac{n}{2i}\right) + \frac{1}{4} \left(\frac{n}{2i}\right)\right)
= \frac{n}{2} + n \sum_{i=0}^{\infty} \frac{1}{2i}
= \frac{n}{2} + 2n
\leq 3n
= O(n)
\]

(The first term in the summation is the *cost of the initial pops*, the second term is the *cost of allocating* a new array, and the third term is the *cost of copying* the remaining elements into the new array.)

Thus, the *amortized* time complexity of a pop operation is \(3 = O(1)\), even though the worst case time complexity of a single pop operation is \(O(n)\).

\(^1\)http://www.seas.upenn.edu/~cis121/current/lectures/stacksQueues.pdf

\(^2\)The analysis for enqueue and dequeue is similar to that of push and pop, respectively.
Problems

Problem 1: Sorting Using Stacks

Given: A full stack $S_1$ of size $n$ and an empty stack $S_2$ of size $n$.

Objective: Sort the $n$ elements in ascending order in $S_2$. You may only use the given 2 stacks $S_1$ and $S_2$ (each of size $n$) and $O(1)$ additional space. What is the runtime of your sorting procedure?

Example:

```
4 3 1 5
2
```

Hint: Start with a simpler example:

```
3 2 1
```

Problem 2: Spiral Order Tree Traversal

Given: A binary tree $T$.

Objective: Print the spiral order traversal of the tree $T$.

Example:

```
Figure 1: For this tree, your function should print 1, 2, 3, 4, 5, 6, 7.
```

Hint: Try using 2 stacks.

Problem 3: Stack With Two Queues

Given: Two queues $Q_1$ and $Q_2$, each of size $n$.

Objective: Implement a stack using $Q_1$ and $Q_2$. Your stack’s `push` and `pop` methods should be implemented using only your queues’ `enqueue`, `dequeue`, and/or `peek` methods.

Problem 4: Queue With Two Stacks

Given: Two stacks $S_1$ and $S_2$, each of size $n$.

Objective: Implement a queue using $S_1$ and $S_2$. Your queue’s `enqueue` and `dequeue` methods should be implemented using only your stacks’ `push`, `pop`, and/or `peek` methods. What are the runtimes of your new queue’s `enqueue` and `dequeue` methods?