Learning Goals

During this lab, you will:

• Review stacks and queues.
• Review amortized running time analysis and strengthen intuition for applying it to new problems.
• Practice using stacks and queues to accomplish a variety of tasks.

Stacks and Queues

Recall the stack and queue ADTs (abstract data types) from lecture. Each is characterized by a specific way of removing elements and has a set of supported operations.

<table>
<thead>
<tr>
<th>Stack</th>
<th>Queue</th>
</tr>
</thead>
<tbody>
<tr>
<td>• LIFO (last-in-first-out)—the most recent element that has been added to the stack will be removed first.</td>
<td>• FIFO (first-in-first-out)—the least recent element that has been added to the queue will be removed first.</td>
</tr>
<tr>
<td>• Supported operations:</td>
<td>• Supported operations:</td>
</tr>
<tr>
<td>– push</td>
<td>– enqueue</td>
</tr>
<tr>
<td>– pop</td>
<td>– dequeue</td>
</tr>
<tr>
<td>– peek</td>
<td>– peek</td>
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<tr>
<td>– isEmpty</td>
<td>– isEmpty</td>
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<tr>
<td>– size</td>
<td>– size</td>
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</table>

Implementation Details

Stacks and queues can be implemented “under the hood” with almost any data structure. In this course, we will implement stacks and queues using expandable arrays. The rules we will use for increasing or decreasing the size of a stack or queue’s underlying array are as follows:

1. If the array of size $n$ is full, create a new array of size $2n$, and copy all elements into the new array.
2. If the array of size $n$ has $\frac{n}{4}$ elements in it, create a new array of size $\frac{n}{2}$, and copy all elements into the new array.

Amortized Analysis

Amortized analysis refers to finding the time-averaged cost for a sequence of operations. In other words, it is the time required to perform a sequence of operations averaged over all the operations performed.

1 http://www.seas.upenn.edu/~cis121/current/lectures/stacksQueues.pdf
Since amortized analysis for the stack `push` operation was covered in lecture, we are going to take a closer look at the stack `pop` operation.\footnote{The analysis for `enqueue` and `dequeue` is similar to that of `push` and `pop`, respectively.}

The worst case running time for a single `pop` operation is \(O(n)\), since we may need to resize the array and copy the elements into it. Based on this running time, we might conclude that a tight bound for the worst case running time for \(n\) `pop` operations is \(O(n^2)\), since there are \(n\) operations and each operation takes worst case \(O(n)\) time; however, we can find a tighter bound through some careful analysis.

If we start from a full stack of size \(n\), what is the total cost of a sequence of \(n\) `pop` operations?

Initially, the array is of size \(n\) and contains \(n\) elements. To make our analysis simpler, let’s immediately `pop` the first \(\frac{n}{2}\) elements. Each of these `pops` takes \(O(1)\) time. Now our array is of size \(n\) but contains only \(\frac{n}{2}\) elements.

In accordance with our rules, we can `pop` \(\frac{n}{4}\) more elements before resizing the array. Each of these `pops` takes \(O(1)\) time. Once we have `pop`’d those elements (leaving us with \(\frac{n}{4}\) elements in our array), we must reduce the size of our array to \(\frac{n}{2}\), and copy the remaining \(\frac{n}{4}\) elements into the new array. Thus, the total cost for the first \(\frac{3n}{4}\) `pop` operations is \(T(\frac{3n}{4}) = \frac{n}{2} + (\frac{n}{2} + \frac{n}{2} + \frac{n}{4})\).

We can apply identical analysis to the new array of size \(\frac{n}{2}\) that contains \(\frac{n}{4}\) elements. We get \(\frac{1}{4} (\frac{n}{2}) = \frac{n}{8}\) `pops” for free”, after which we resize the array to be of size \(\frac{1}{2} (\frac{n}{2}) = \frac{n}{4}\) and copy the remaining \(\frac{n}{2}\) elements into the smaller array. Thus, the total cost for the first \(\frac{2n}{5}\) `pop` operations is \(T(\frac{2n}{5}) = \frac{n}{2} + (\frac{n}{4} + \frac{n}{2} + \frac{n}{4}) + (\frac{n}{4} + \frac{n}{2} + \frac{n}{2})\).

Are you noticing a pattern?

Let’s rewrite the expression slightly and continue to expand it:

\[
T(n) = \frac{n}{2} + \left(\frac{1}{4} \left(\frac{n}{2^0}\right) + \frac{1}{2} \left(\frac{n}{2^0}\right) + \frac{1}{4} \left(\frac{n}{2^0}\right)\right) + \left(\frac{1}{4} \left(\frac{n}{2^1}\right) + \frac{1}{2} \left(\frac{n}{2^1}\right) + \frac{1}{4} \left(\frac{n}{2^1}\right)\right) + \\
\quad + \left(\frac{1}{4} \left(\frac{n}{2^2}\right) + \frac{1}{2} \left(\frac{n}{2^2}\right) + \frac{1}{4} \left(\frac{n}{2^2}\right)\right) + \cdots + \left(\frac{1}{4} (4) + \frac{1}{2} (4) + \frac{1}{4} (4)\right)
\]

We can now calculate the total cost of \(n\) `pop` operations:

\[
T(n) \leq \frac{n}{2} + \sum_{i=0}^{\infty} \left(\frac{1}{4} \left(\frac{n}{2^{i}}\right) + \frac{1}{2} \left(\frac{n}{2^{i}}\right) + \frac{1}{4} \left(\frac{n}{2^{i}}\right)\right)
\]
\[
= \frac{n}{2} + n \sum_{i=0}^{\infty} \frac{1}{2^{i}}
\]
\[
= \frac{n}{2} + 2n
\]
\[
\leq 3n
\]
\[
= O(n)
\]

(The first term in the summation is the \textit{cost of the initial pops}, the second term is the \textit{cost of allocating} a new array, and the third term is the \textit{cost of copying} the remaining elements into the new array.)

Thus, the \textit{amortized} time complexity of a single `pop` operation is \(3 = O(1)\), even though the worst case time complexity of a single `pop` operation is \(O(n)\).

\section*{Problems}

\subsection*{Problem 1: Sorting Using Stacks}

\textit{Given}: A full stack \(S_1\) of size \(n\) and an empty stack \(S_2\) of size \(n\).

\textit{Objective}: Sort the \(n\) elements in ascending order in \(S_2\). You may only use the given 2 stacks \(S_1\) and \(S_2\) (each of size \(n\)) and \(O(1)\) additional space. What is the running time of your sorting procedure?

\textit{Example}:
Problem 2: Spiral Order Tree Traversal

Given: A binary tree \( T \).

Objective: Print the spiral order traversal of the tree \( T \).

Example:

![Spiral Order Tree Traversal Example](image)

Hint: Try using 2 stacks.

Problem 3: Stack With Two Queues

Given: Two queues \( Q_1 \) and \( Q_2 \), each of size \( n \).

Objective: Implement a stack using \( Q_1 \) and \( Q_2 \). Your stack’s push and pop methods should be implemented using only your queues’ enqueue, dequeue, and/or peek methods.

Problem 4: Queue With Two Stacks

Given: Two stacks \( S_1 \) and \( S_2 \), each of size \( n \).

Objective: Implement a queue using \( S_1 \) and \( S_2 \). Your queue’s enqueue and dequeue methods should be implemented using only your stacks’ push, pop, and/or peek methods. What are the running times of your new queue’s enqueue and dequeue methods?