Properties of Tries

(Normal) Tries

1. A trie is a special kind of rooted tree used to store strings composed over some alphabet $\Sigma$. When a string such as \textit{algorithms} is input into the trie, a path from the root is constructed corresponding to the sequence of characters. Once the path $a \to l \to g \cdots$ has been constructed, a special node is constructed with character $\$ \$ to denote the end of a valid word.

2. The root of a trie represents an empty substring. Each node in a trie can have $|\Sigma|$ children. Usually, these are stored in an array, as characters have a direct mapping to integers. Further, the descendants of a node $v$ are all inserted words reachable on a path from $v$ towards its children.

3. Tries implement basic functionality \texttt{insert(x)}, \texttt{find(x)}, and \texttt{delete(x)}. Time complexity of Trie insertion is usually defined over $m$, the total string size of all inserted strings. With the algorithms we’ve discussed in class, trie construction runs in $O(m)$ time. \texttt{find(x)} and \texttt{delete(x)} both run in $O(l)$ time, where $l$ is the length of the string being inserted or deleted.

Patricia Tries

Also known as compact tries, these data structures offer space usage improvements over simple tries. As we’ve discussed, a trie’s space usage is $O(m)$ where $m$ is the total size of all strings input. What property in a sequence of insertions fulfills this upper bound?

However, a simple motivation leads to an optimization. If a sequence of nodes all have one children, why have multiple nodes? In a compact trie, all such sequences are compacted into one node. A compact trie has $O(s)$ nodes, where $s$ is the total number of strings inserted into the dictionary. This can be proven as follows:

Upon insertion of a string, how many nodes are created? As you insert, you walk down the existing structure of the trie until you “fall off”, meaning no prior insertion of a string has had the same prefix. At this point, a new node is created, and this node stores the entirety of the rest of the string to be inserted. Insertion finishes, and no further nodes are created.

A final optimization is as follows: instead of storing strings at the nodes of our trie, we store every inserted string \textit{exactly once} in an auxiliary array. Nodes then represent their character sequences by storing indices into the strings. Thus, if \textit{algorithms} is the first string I insert into my trie, I’ll have exactly 1 non-root node, with the data (1, 1, 10). The first 1 points to the first entry in the auxiliary array. The 1, 10 mean that the node stores the substring from index 1 to index 10 (i.e. the whole string, in this case.)

Suffix Trees

A suffix tree is a trie whose stored strings are all \textit{suffixes} of one long input sting. For the input string /textit{ENIAC}, a suffix trie holds

$$\{C\}, \{A \to C\}, \{I \to A \to C\}, \{N \to I \to A \to C\}, \{E \to N \to I \to A \to C\}$$

Suffix trees are often used to search for substrings in a given string. You’ve probably heard the phrase “every substring is a prefix of some suffix.” This can be a little hard to grok at first. If a string is a sequence of characters $S[1, \ldots m]$, then any substring is defined by some start index greater than or equal to 1, and some end index less than or equal to m. We have m suffixes in our suffix tree, each one corresponding to a different start index. How many nodes we travel down the path of whichever suffix we’ve chosen determines the end index. Thus, by traversing down a portion of some suffix, we’re reading “a prefix of the suffix.”
A traditional suffix trie uses $O(m^2)$ space (consider storing a sequence of all unique characters.) A compact suffix trie uses just $O(m)$ space. The same space usage argument given for traditional tries, above, suffices for this.

## Writing an Autocompletion Program

### Formalizing the Problem

As you type characters into your smartphone, its keyboard program predicts the word you’re writing. It does this by keeping track of the characters you’ve already typed in the current word, and making a guess at what words are most likely to complete the sequence so far.

Let alphabet $\Sigma$ be an ordered set of characters. Let $M$ be a vocabulary (a set of words) over $\Sigma$, with $M >> \Sigma$. Let $S$ be an unbounded sequence $(s_1, s_2, \ldots)$ of characters, with each $s_i \in \Sigma \cup \{\text{space}\}$ (Words can’t have a space character, but the sequence can.). By unbounded, we mean you receive the elements of $S$ in order, but don’t know any information past the last $s_i$ that you’ve received so far. Let the maximum number of words to present to the user at any time be $k$. This is usually bounded by screen size, and utility to the user. (If you’ve typed in nothing, do you want the whole English vocabulary given as a helpful suggestion?)

**Goal:** In a few steps, construct a data structure that permits the efficient and accurate prediction of words as sequence $S$ is received. Once there are $k$ or fewer possible word completions, keep track of the top $k$ as each character $s_i$ is received.

### Step 1: Adding Size Information to Tries

Let’s say you have $M$ stored in a text file, one word per line, like a really bare dictionary. Modify trie insertion to keep track of an extra bit of information: for each node $v$, let $\text{size}$ be the number of unique words that are children of $v$. What do you do upon insertion? What do you do upon deletion?

### Step 2: Extending to probabilities

The $\text{size}$ field for each node is helpful in determining when there are $k$ or fewer predictions left. However, how do we know which of the $k$ words are the most likely completions? This kind of information is usually estimated by reading a large corpus, or body of text. Our method will be to read a corpus, and store the number of times each word is seen with its node in the trie. Thus, consider a corpus to be a sequence of insertion operations. Modify trie insertion again, this time keeping track of both $\text{size}$, and $\text{frequency}$, the number of times each word has been inserted into the trie. How can we make sure that we don’t increase the $\text{size}$ field on nodes when inserting non-distinct words?

### Step 3: Putting it all together

We now have a trie that stores the words in our vocabulary, as well as information at each node about the number of unique words “beneath” it. Further, we have frequency information in the form of integers associated with each leaf node (unique word). We now have the information we need to do probabilistic autocompletion. We now consider sequence $S$. Recall that this is the sequence of characters typed in by the user. Every time a character $s_i$ is keyed in, we receive that character, and decide what to do. Design a function $\text{onChar}(c)$ that takes character $c$ and returns a list $L$ of potential autocompletions with the following properties:

1. The sequence is size $k$ or smaller.
2. The sequence is ordered – that is, the first element is the most-likely word for autocompletion. Note that we’ll be estimating the “probability” as the frequency of each word divided by the total frequency of all possible words given the sequence so far.
3. $\text{onChar}(c)$ runs in $O(k \log(k))$ time.
Note that as we get more information (as we receive more characters in $S$,) our set of candidate completions become more accurate predictions. This is because we are effectively conditioning our probability calculations on the sequence so far, and more information $\rightarrow$ better predictions.

**Testing your Understanding**

**Problem 1.** Given some arbitrarily long string, how can we find the longest repeated substring? What about if we want the longest that is repeated $k$ times?

**Problem 2.** Given a set of $N$ strings, how can we find the longest common prefix between any two strings? Analyze the running time of your algorithm and give a short proof of correctness.