Data Structures and Algorithms
Homework Assignment 1

Given: January 14, 2018 **Due:** January 22, 2018

Note: The homework is due **electronically on Gradescope and Canvas** on Monday, January 22 by 11:59 pm. For late submissions, please refer to the Late Submission Policy on the course webpage: `http://www.seas.upenn.edu/~cis121`

You must use the `hw121.cls` template provided on the course website. You can find the full template on the Piazza post outlining the Written Homework Guidelines

Please write concise and clear solutions; you will get only a partial credit for correct solutions that are either unnecessarily long or not clear. Please refer to the Written Homework Guidelines for all the requirements: `http://bit.ly/2DrYm3A`

You are allowed to discuss ideas for solving homework problems in groups of up to 3 people but *you must write your solutions independently*. Also, you must write on your homework the names of the people with whom you discussed.

Finally, you are not allowed to use *any* material outside of the class notes and the textbook. Any violation of this policy may seriously affect your grade in the class.

[**Hiking & Bonding**] **1.** Some CIS TAs enjoy doing hiking trips. In fact, they decided to start a series of weekly hiking trips during the Spring semester. The requirement for a hiking trip to happen is that n CIS-120 TAs and n CIS-121 TAs show up. To make the hikes enjoyable, the TAs implemented a buddy system: each TA is paired with another TA. Since there are a lot of new TAs, they also decide that each TA should pair up with a new TA during each hike. This means that on every hike, each TA is paired up with a TA that they have never been buddies with on a previous hike. Note that a CIS-121 TA does not mind hiking with a CIS-120 TA and vice-versa.

After the second hike of the semester, Shirali wants to organize a huge bonding activity that requires two groups. Her goal is for all the TAs to bond with someone new, so she wants to make sure that no two TAs that have been buddies during the hikes are in the same group. Prove that Shirali can create these two equal size groups such that no TA is in the same group as one of their former hiking buddies.

[**S&S Gamble**] **2.** After amassing huge fortunes out of their real-estate ventures, Steven and Sadat disbanded their company *S&S Cash* and parted ways. However, being long-time friends, they still meet occasionally. This week, they decided to meet at a horse racing competition to settle their gambling feud once and for all.

Steven placed a bet with Sadat that if any horse wins all its races, then Steven wins \$121,000 and a trophy with that horses name on it. There are h horses, but unlike most horse races only 2 horses can race at a time, and each pair of horses races some non-zero number of times.

Knowing that his luck is not something to be proud of, Sadat decides that Steven cannot watch the horse race. Instead, Steven can only find out if he won the bet by asking him the following yes/no question: "Has a won any race against b?". This is the only type of question that he is allowed to ask. Thus, if he wanted to know if a won a race against b and won a race against c then he can find out this information by asking two questions: "Has a won any race against b?" and "Has a won any race against c?"

Give Steven and algorithm to determine if he won the bet using at most $3h$ questions. If he won, then also tell him the name of the horse that caused him to win so Sadat can order the coveted trophy. Justify the correctness of your algorithm.

[**Pierogi Fest**] **3.** After her winter break in Europe, Jess decided to open a pierogi restaurant. She came up with n different exciting flavors of pierogis and decided to invite all n CIS-121 TA's for a taste testing. To ensure that all the TAs will try each of the n flavors, she set up a tasting booth for each pierogi flavor and set up a schedule of m time intervals for each TA such that for each time interval, the TA is either eating a pierogi or taking a break from the food. You can assume that $m > n$. Each TA only needs to spend one time interval at each pierogi station and no two TAs can be at the same station at once.

However, the night before the tasting session, all the 121 TAs stayed up to grade so they're all really sleepy and will not be able to resist falling asleep from the food coma. Taking that into account, Jess decides to modify the tasting booths such that a TA can nap in it. The piergoi station in which each TA sleeps may be different. Having taken 121, she decided that it is possible to come up with an algorithm to modify her original schedule to allow every TA to nap under the following scheme. Each TA t_i will follow their original schedule until they hit a tasting station S_j and will nap in that station until the end of the tasting session. This TA t_i will not visit any more stations. While the TAs are napping, the station they are in is considered occupied and no other TAs can visit that station anymore. The original requirement of having at most one TA at each station still stands. To prevent the TAs from not eating the different flavors that they missed when they were napping, Jess will give them a takeout box of the rest of the pierogis that they missed.

However, Jess took 121 so long ago that she can't think of a valid algorithm. Help her out by showing that such a modification can always be found and give an algorithm to find it. Prove the correctness of your algorithm.

Hint: Try to reduce this problem to the stable matching problem!

[**Twenty-One Pigeons: The Making**] **4.** Ziad manages an entertainment studio, and is thinking of creating a music band under the name *Twenty-One Pigeons*. To choose the members, he decides to organize a local duet competition, where participants sing in pairs of opposite genders. Surprisingly, only CIS-121 TAs show up to the competition. In fact, an even number of TAs show up. To be more specific, there are n male TAs, and n female TAs.

The competition structure Ziad chose is unusual: he divided it into three distinct rounds, such that in each round, each man will sing with every woman and all these possible pairs will sing the same song. The three songs Ziad chose are: **Highway to Hole** in round one, **Free Pigeon** in round two, and finally **Sweet Pigeon O'Mine**.

In order to make the tournament fair, he asked each TA to rank all other TAs of the opposite gender based on how well he/she thought each other TAs can sing. He then decided to make three possibly different stable matchings that correspond to each round. In particular, each matching matches a male TA to a female TA. Ziad successfully made a stable matching for the first two rounds, but failed to come up with a clever way to match the TAs for the last round. Krishna suggested the following two options:

(A) For each male TA x, if the **Highway to Hole** matching pairs TA y with x, and the **Free Pigeon** matching pairs TA z with x, then the **Sweet Pigeon O'Mine** matching should pair x with the TA that x thinks has the best singing ability out of the first two rounds: either y or z. Note that y and z can be the same person.

(B) The same approach, but instead of pairing x with TA that x thinks has the best singing ability, the third matching should pair x with the TA that x thinks has the worst singing ability (again from y and z only). Krishna argued that this might be a better learning experience for everyone.

To make sure that the third round is fair for everyone, Ziad asks you to prove or disprove, for each option A and B, if the resulting matching is a stable matching.

[Sneha's Pet Mischief] 5. The 121 TAs are starting to get lonely, so they decide to go to a pet shop to buy furry friends to keep them company while they code. When they arrive at the pet shop, they realize that there are exactly the same number of pets available as there are TAs, so the TAs begin fighting over who gets to take which pet. The pet shop owner decides to resolve this problem by creating a system to match each TA with their own pet. Each TA creates a list of their preferences for the pets, and the pet shop owner creates preference lists for each pet for which TAs he thinks will be the best owners for that pet. Then, the pet shop owner uses the Gale-Shapley algorithm to produce a stable matching, where each pet goes through their preference list until they are matched with a TA who accepts the pet, and this process repeats until all pets and TAs are matched.

After learning of this system, Sneha thinks she might be able to fool the pet shop owner into getting the pet she wants by misrepresenting her preferences. Suppose Sneha prefers pet p to p', but both p and p' are low on her list of preferences. Can it be the case that if she misrepresents her preferences and switches the order of p and p' on her list of preferences, when the algorithm is run, Sneha will be matched with the pet she truly wants over both p and p'?

You should prove this by either:

a. Give a proof that, for any instance of the stable matching problem, no TA can improve their pet by switching the order of a pair of pets on their list; or

b. Give an example of a set of preference lists for which Sneha can improve her pet by switching the ordering of a pair of pets on her list.