total of at most $cn^2/2$, again plus the time in subsequent recursive calls. At the third level, we have four problems each of size $n/4$, each taking time at most $c(n/4)^2 = cn^2/16$, for a total of at most $cn^2/4$. Already we see that something is different from our solution to the analogous recurrence (5.1); whereas the total amount of work per level remained the same in that case, here it's decreasing.

- *Identifying a pattern:* At an arbitrary level j of the recursion, there are 2^j subproblems, each of size $n/2^j$, and hence the total work at this level is bounded by $2^j c(\frac{n}{2^j})^2 = cn^2/2^j$.
- *Summing over all levels of recursion:* Having gotten this far in the calculation, we've arrived at almost exactly the same sum that we had for the case $q = 1$ in the previous recurrence. We have

$$T(n) \le \sum_{j=0}^{\log_2 n-1} \frac{cn^2}{2^j} = cn^2 \sum_{j=0}^{\log_2 n-1} \left(\frac{1}{2^j}\right) \le 2cn^2 = O(n^2),$$

where the second inequality follows from the fact that we have a convergent geometric sum.

In retrospect, our initial guess of $T(n) = O(n^2 \log n)$, based on the analogy to (5.1), was an overestimate because of how quickly n^2 decreases as we replace it with $(\frac{n}{2})^2$, $(\frac{n}{4})^2$, $(\frac{n}{8})^2$, and so forth in the unrolling of the recurrence. This means that we get a geometric sum, rather than one that grows by a fixed amount over all n levels (as in the solution to (5.1)).

5.3 Counting Inversions

We've spent some time discussing approaches to solving a number of common recurrences. The remainder of the chapter will illustrate the application of divide-and-conquer to problems from a number of different domains; we will use what we've seen in the previous sections to bound the running times of these algorithms. We begin by showing how a variant of the Mergesort technique can be used to solve a problem that is not directly related to sorting numbers.

The Problem

We will consider a problem that arises in the analysis of *rankings*, which are becoming important to a number of current applications. For example, a number of sites on the Web make use of a technique known as *collaborative filtering*, in which they try to match your preferences (for books, movies, restaurants) with those of other people out on the Internet. Once the Web site has identified people with "similar" tastes to yours—based on a comparison of how you and they rate various things—it can recommend new things that these other people have liked. Another application arises in *meta-search tools* on the Web, which execute the same query on many different search engines and then try to synthesize the results by looking for similarities and differences among the various rankings that the search engines return.

A core issue in applications like this is the problem of comparing two rankings. You rank a set of n movies, and then a collaborative filtering system consults its database to look for other people who had "similar" rankings. But what's a good way to measure, numerically, how similar two people's rankings are? Clearly an identical ranking is very similar, and a completely reversed ranking is very different; we want something that interpolates through the middle region.

Let's consider comparing your ranking and a stranger's ranking of the same set of n movies. A natural method would be to label the movies from 1 to n according to your ranking, then order these labels according to the stranger's ranking, and see how many pairs are "out of order." More concretely, we will consider the following problem. We are given a sequence of n numbers $a_1, \ldots, a_n$; we will assume that all the numbers are distinct. We want to define a measure that tells us how far this list is from being in ascending order; the value of the measure should be 0 if $a_1 < a_2 < \ldots < a_n$, and should increase as the numbers become more scrambled.

A natural way to quantify this notion is by counting the number of *inversions*. We say that two indices $i < j$ form an inversion if $a_i > a_j$, that is, if the two elements a_i and a_j are "out of order." We will seek to determine the number of inversions in the sequence $a_1, \ldots, a_n$.

Just to pin down this definition, consider an example in which the sequence is $2, 4, 1, 3, 5$. There are three inversions in this sequence: $(2, 1)$, $(4, 1)$, and $(4, 3)$. There is also an appealing geometric way to visualize the inversions, pictured in Figure 5.4: we draw the sequence of input numbers in the order they're provided, and below that in ascending order. We then draw a line segment between each number in the top list and its copy in the lower list. Each crossing pair of line segments corresponds to one pair that is in the opposite order in the two lists—in other words, an inversion.

Note how the number of inversions is a measure that smoothly interpolates between complete agreement (when the sequence is in ascending order, then there are no inversions) and complete disagreement (if the sequence is in descending order, then every pair forms an inversion, and so there are $\binom{n}{2}$ of them).

Designing and Analyzing the Algorithm

What is the simplest algorithm to count inversions? Clearly, we could look at every pair of numbers (a_i, a_j) and determine whether they constitute an inversion; this would take $O(n^2)$ time.

We now show how to count the number of inversions much more quickly, in $O(n \log n)$ time. Note that since there can be a quadratic number of inversions, such an algorithm must be able to compute the total number without ever looking at each inversion individually. The basic idea is to follow the strategy (†) defined in Section 5.1. We set $m = \lceil n/2 \rceil$ and divide the list into the two pieces $a_1, \ldots, a_m$ and $a_{m+1}, \ldots, a_n$. We first count the number of inversions in each of these two halves separately. Then we count the number of inversions (a_i, a_j), where the two numbers belong to different halves; the trick is that we must do this part in $O(n)$ time, if we want to apply (5.2). Note that these first-half/second-half inversions have a particularly nice form: they are precisely the pairs (a_i, a_j), where a_i is in the first half, a_j is in the second half, and $a_i > a_j$.

To help with counting the number of inversions between the two halves, we will make the algorithm recursively sort the numbers in the two halves as well. Having the recursive step do a bit more work (sorting as well as counting inversions) will make the "combining" portion of the algorithm easier.

So the crucial routine in this process is `Merge-and-Count`. Suppose we have recursively sorted the first and second halves of the list and counted the inversions in each. We now have two sorted lists A and B, containing the first and second halves, respectively. We want to produce a single sorted list C from their union, while also counting the number of pairs (a, b) with $a \in A$, $b \in B$, and $a > b$. By our previous discussion, this is precisely what we will need for the "combining" step that computes the number of first-half/second-half inversions.

This is closely related to the simpler problem we discussed in Chapter 2, which formed the corresponding "combining" step for Mergesort: there we had two sorted lists A and B, and we wanted to merge them into a single sorted list in $O(n)$ time. The difference here is that we want to do something extra: not only should we produce a single sorted list from A and B, but we should also count the number of "inverted pairs" (a, b) where $a \in A$, $b \in B$, and $a > b$.

It turns out that we will be able to do this in very much the same style that we used for merging. Our `Merge-and-Count` routine will walk through the sorted lists A and B, removing elements from the front and appending them to the sorted list C. In a given step, we have a *Current* pointer into each list, showing our current position. Suppose that these pointers are currently

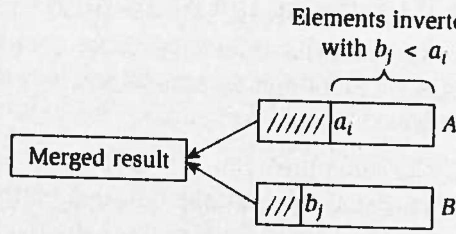

Figure 5.5 Merging two sorted lists while also counting the number of inversions between them.

at elements a_i and b_j. In one step, we compare the elements a_i and b_j being pointed to in each list, remove the smaller one from its list, and append it to the end of list C.

This takes care of merging. How do we also count the number of inversions? Because A and B are sorted, it is actually very easy to keep track of the number of inversions we encounter. Every time the element a_i is appended to C, no new inversions are encountered, since a_i is smaller than everything left in list B, and it comes before all of them. On the other hand, if b_j is appended to list C, then it is smaller than all the remaining items in A, and it comes after all of them, so we increase our count of the number of inversions by the number of elements remaining in A. This is the crucial idea: in constant time, we have accounted for a potentially large number of inversions. See Figure 5.5 for an illustration of this process.

To summarize, we have the following algorithm.

```
Merge-and-Count(A,B)
  Maintain a Current pointer into each list, initialized to
    point to the front elements
  Maintain a variable Count for the number of inversions,
    initialized to 0
  While both lists are nonempty:
    Let a_i and b_j be the elements pointed to by the Current pointer
    Append the smaller of these two to the output list
    If b_j is the smaller element then
      Increment Count by the number of elements remaining in A
    Endif
    Advance the Current pointer in the list from which the
      smaller element was selected.
  EndWhile
```

```
    Once one list is empty, append the remainder of the other list
        to the output
    Return Count and the merged list
```

The running time of Merge-and-Count can be bounded by the analogue of the argument we used for the original merging algorithm at the heart of Mergesort: each iteration of the While loop takes constant time, and in each iteration we add some element to the output that will never be seen again. Thus the number of iterations can be at most the sum of the initial lengths of A and B, and so the total running time is $O(n)$.

We use this Merge-and-Count routine in a recursive procedure that simultaneously sorts and counts the number of inversions in a list L.

```
Sort-and-Count(L)
  If the list has one element then
      there are no inversions
  Else
      Divide the list into two halves:
         A contains the first ⌈n/2⌉ elements
         B contains the remaining ⌊n/2⌋ elements
      (r_A, A) = Sort-and-Count(A)
      (r_B, B) = Sort-and-Count(B)
      (r, L) = Merge-and-Count(A, B)
  Endif
  Return r = r_A + r_B + r, and the sorted list L
```

Since our Merge-and-Count procedure takes $O(n)$ time, the running time $T(n)$ of the full Sort-and-Count procedure satisfies the recurrence (5.1). By (5.2), we have

(5.7) *The Sort-and-Count algorithm correctly sorts the input list and counts the number of inversions; it runs in $O(n \log n)$ time for a list with n elements.*

5.4 Finding the Closest Pair of Points

We now describe another problem that can be solved by an algorithm in the style we've been discussing; but finding the right way to "merge" the solutions to the two subproblems it generates requires quite a bit of ingenuity.