Data Structures and Algorithms
Recurrences
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Example. The running time of Karatsuba's algorithm for integer multiplication is given by the following recurrence. Solve the recurrence. Assume that n is a power of 2.

$$T(n) = \begin{cases} 3T(n/2) + n, & n \geq 2 \\ 1, & \text{n=1} \end{cases}$$

Solution.

$$\begin{aligned} T(n) &= 3T(n/2) + n \\ &= 3^2 T(n/2^2) + 3n/2 + n \\ &= 3^3 T(n/2^3) + (3/2)^2 n + 3n/2 + n \\ & \ldots \\ & \ldots \\ &= 3^k T(n/2^k) + n \sum_{i=0}^{k-1} (3/2)^i \end{aligned}$$

The recursion bottoms out when $n/2^k = 1$, i.e., $k = \lg n$. Thus, we get

$$\begin{aligned} T(n) &= 3^{\log_2 n} T(1) + n \sum_{i=0}^{\lg n - 1} (3/2)^i \\ &= 3^{\log_2 n} + n \left(\frac{(3/2)^{\lg n} - 1}{3/2 - 1} \right) \\ &= n^{\frac{\log_3 n}{\log_3 2}} + 2n((3/2)^{\log_{3/2} n / \log_{3/2} 2} - 1) \\ &= n^{\log_2 3} + 2n(n^{\lg(3/2)} - 1) \\ &= n^{\log_2 3} + 2n(n^{\lg 3 - \lg 2)} - 1) \\ &= n^{\log_2 3} + 2n(n^{\lg 3 - 1}) - 2n \\ &= n^{\log_2 3} + 2n^{\lg 3} - 2n \\ &= 3n^{\log_2 3} - 2n \\ &= \Theta(n^{\lg 3}) \qquad \text{(can be argued by setting } c = 3 \text{ and } n_0 = 1) \end{aligned}$$

Example. Find the running time expressed by the following recurrence. Assume that n is a power of 3.

$$T(n) = \begin{cases} 5T(n/3) + n^2, & n \geq 2 \\ 1, & \text{n=1} \end{cases}$$

$$
\begin{aligned}
T(n) &= 5T(n/3) + n^2 \\
&= 5^2T(n/3^2) + 5(n/3)^2 + n^2 \\
&= 5^3T(n/3^3) + 5^2(n/3^2)^2 + 5n^2/3^2 + n^2 \\
&\dots \\
&\dots \\
&= 5^kT(n/3^k) + n^2\left(1 + \frac{5}{9} + \left(\frac{5}{9}\right)^2 + \dots + \left(\frac{5}{9}\right)^{k-1}\right)
\end{aligned}
$$

The recursion bottoms out when $n/3^k = 1$, i.e., $k = \log_3 n$. Thus, we get

$$
\begin{aligned}
T(n) &= 5^{\log_3 n}T(1) + n^2\sum_{i=0}^{k-1}(5/9)^i \\
&= 5^{\log_3 n} + n^2\left(\frac{(5/9)^{\log_3 n} - 1}{(5/9) - 1}\right) \\
&= 5^{\log_5 n/\log_5 3} + \frac{9n^2}{4}\left(1 - (n^{\log_3 5}/n^2)\right) \\
&= n^{\log_3 5} + \frac{9n^2}{4} - \frac{9n^{\log_3 5}}{4} \\
&= \frac{9n^2}{4} - \frac{5n^{\log_3 5}}{4} \\
&= \Theta(n^2)
\end{aligned}
$$

Simplified Master Theorem. Let $a \geq 1$, $b > 1$ be constants and let $T(n)$ be the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^k)$$

defined for $n \geq 0$ (we assume that n is a power of b, though this does not make a difference in asymptotic analysis). The base case, $T(1)$ can be any constant value. Then

Case 1: if $a > b^k$, then $T(n) \in \Theta(n^{\log_b a})$.
Case 2: if $a = b^k$, then $T(n) \in \Theta(n^k \log_b n)$.
Case 3: if $a < b^k$, then $T(n) \in \Theta(n^k)$.

Example. Solve the following recurrences using the Master method. Assume that n is a power of 2 and $T(1) = c$, for some constant c.

a. $T(n) = 4T(n/2) + n$

b. $T(n) = T(n/3) + n$

c. $T(n) = 9T(n/3) + n^{2.5}$

d. $T(n) = 8T(n/2) + n^3$

Solution.

a. $a = 4, b = 2$, and $k = 1$. Thus case 1 of the Simplified Master Theorem applies and hence $T(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)$.

b. $a = 1, b = 3$, and $k = 1$. Thus case 3 of the Simplified Master Theorem applies and hence $T(n) = \Theta(n)$.

c. $a = 9, b = 3$, and $k = 2.5$. Thus case 3 of the Simplified Master Theorem applies and hence $T(n) = \Theta(n^{2.5})$.

d. $a = 8, b = 2$, and $k = 3$. Thus case 2 of the Simplified Master Theorem applies and hence $T(n) = \Theta(n^3 \log_2 n)$.