Example. The running time of Karatsuba’s algorithm for integer multiplication is given by the following recurrence. Solve the recurrence. Assume that $n$ is a power of 2.

$$T(n) = \begin{cases} 
3T(n/2) + n, & n \geq 2 \\
1, & n=1 
\end{cases}$$

Solution.

$$T(n) = 3T(n/2) + n$$
$$= 3^2T(n/2^2) + 3n/2 + n$$
$$= 3^3T(n/2^3) + (3/2)^2 n + 3n/2 + n$$
$$\ldots$$
$$\ldots$$
$$= 3^kT(n/2^k) + n \sum_{i=0}^{k-1} (3/2)^i$$

The recursion bottoms out when $n/2^k = 1$, i.e., $k = \lg n$. Thus, we get

$$T(n) = 3^{\log_2 n} T(1) + n \sum_{i=0}^{\log_3 n-1} (3/2)^i$$

$$= 3^{\log_2 n} + n \left( \frac{(3/2)^{\log_3 n} - 1}{3/2 - 1} \right)$$
$$= n^{\log_3 n} + 2n \left( (3/2)^{\log_3 n/\log_3 2} - 1 \right)$$
$$= n^{\log_2 3} + 2n^{\log_3 3 - \log_2 2} - 1$$
$$= n^{\log_2 3} + 2n^{\log_3 3 - 1} - 2n$$
$$= n^{\log_2 3} + 2n^{\log_2 3} - 2n$$
$$= 3n^{\log_2 3} - 2n$$
$$= \Theta(n^{\log_2 3}) \quad \text{(can be argued by setting } c = 3 \text{ and } n_0 = 1)$$

Example. Find the running time expressed by the following recurrence. Assume that $n$ is a power of 3.

$$T(n) = \begin{cases} 
5T(n/3) + n^2, & n \geq 2 \\
1, & n=1 
\end{cases}$$
\[ T(n) = 5T(n/3) + n^2 \]
\[ = 5^2T(n/3^2) + 5(n/3)^2 + n^2 \]
\[ = 5^3T(n/3^3) + 5^2(n/3^2)^2 + 5n^2/3^2 + n^2 \]
\[ \ldots \]
\[ \ldots \]
\[ = 5^kT(n/3^k) + n^2 \left( 1 + \frac{5}{9} + \left(\frac{5}{9}\right)^2 + \ldots + \left(\frac{5}{9}\right)^{k-1} \right) \]

The recursion bottoms out when \( n/3^k = 1 \), i.e., \( k = \log_3 n \). Thus, we get
\[ T(n) = 5^{\log_3 n}T(1) + n^2 \sum_{i=0}^{k-1} \left(\frac{5}{9}\right)^i \]
\[ = 5^{\log_3 n} + n^2 \left( \frac{(5/9)^{\log_3 n} - 1}{(5/9) - 1} \right) \]
\[ = 5^{\log_3 n} + \frac{9n^2}{4} \left( 1 - \left( n^{\log_3 5}/n^2 \right) \right) \]
\[ = n^{\log_3 5} + \frac{9n^2}{4} - \frac{9n^{\log_3 5}}{4} \]
\[ = \Theta(n^2) \]

**Simplified Master Theorem.** Let \( a \geq 1, b > 1 \) be constants and let \( T(n) \) be the recurrence
\[ T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^k) \]
defined for \( n \geq 0 \) (we assume that \( n \) is a power of \( b \), though this does not make a difference in asymptotic analysis). The base case, \( T(1) \) can be any constant value. Then

**Case 1:** if \( a > b^k \), then \( T(n) \in \Theta(n^{\log_b a}) \).
**Case 2:** if \( a = b^k \), then \( T(n) \in \Theta(n^{k \log_b n}) \).
**Case 3:** if \( a < b^k \), then \( T(n) \in \Theta(n^k) \).

**Example.** Solve the following recurrences using the Master method. Assume that \( n \) is a power of 2 and \( T(1) = c \), for some constant \( c \).

a. \( T(n) = 4T(n/2) + n \)
b. \( T(n) = T(n/3) + n \)
c. \( T(n) = 9T(n/3) + n^{2.5} \)
d. \( T(n) = 8T(n/2) + n^3 \)
Solution.

a. $a = 4, b = 2, \text{ and } k = 1$. Thus case 1 of the Simplified Master Theorem applies and hence $T(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)$.

b. $a = 1, b = 3, \text{ and } k = 1$. Thus case 3 of the Simplified Master Theorem applies and hence $T(n) = \Theta(n)$.

c. $a = 9, b = 3, \text{ and } k = 2.5$. Thus case 3 of the Simplified Master Theorem applies and hence $T(n) = \Theta(n^{2.5})$.

d. $a = 8, b = 2, \text{ and } k = 3$. Thus case 2 of the Simplified Master Theorem applies and hence $T(n) = \Theta(n^3 \log_2 n)$. 