Types of analyses

**Best case.** Lower bound on cost.
- Determined by “easiest” input.
- Provides a goal for all inputs.

**Worst case.** Upper bound on cost.
- Determined by “most difficult” input.
- Provides a guarantee for all inputs.

**Average case.** Expected cost for random input.
- Need a model for “random” input.
- Provides a way to predict performance.

**Ex 1.** Array accesses for brute-force 3-SUM.
- Best: \( \sim \frac{1}{2} N^3 \)
- Average: \( \sim \frac{1}{2} N^3 \)
- Worst: \( \sim \frac{1}{2} N^3 \)

**Ex 2.** Compares for binary search.
- Best: \( \sim 1 \)
- Average: \( \sim \log N \)
- Worst: \( \sim \log N \)
Types of analyses

Best case. Lower bound on cost.
Worst case. Upper bound on cost.
Average case. “Expected” cost.

Actual data might not match input model?
- Need to understand input to effectively process it.
- Approach 1: design for the worst case.
- Approach 2: randomize, depend on probabilistic guarantee.
Theory of algorithms

Goals.

- Establish “difficulty” of a problem.
- Develop “optimal” algorithms.

Approach.

- Suppress details in analysis: analyze “to within a constant factor.”
- Eliminate variability in input model: focus on the worst case.

Upper bound. Performance guarantee of algorithm for any input.

Lower bound. Proof that no algorithm can do better.

Optimal algorithm. Lower bound = upper bound (to within a constant factor).
## Commonly-used notations in the theory of algorithms

<table>
<thead>
<tr>
<th>notation</th>
<th>provides</th>
<th>example</th>
<th>shorthand for</th>
<th>used to</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Big Theta</strong></td>
<td>asymptotic order of growth</td>
<td>$\Theta(N^2)$</td>
<td>$\frac{1}{2}N^2$</td>
<td>classify algorithms</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$10N^2$</td>
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<td></td>
<td>$5N^2 + 22N\log N + 3N$</td>
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<td>$\vdots$</td>
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</tr>
<tr>
<td><strong>Big O</strong></td>
<td>$\Theta(N^2)$ and smaller</td>
<td>$O(N^2)$</td>
<td>$10N^2$</td>
<td>develop upper bounds</td>
</tr>
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<td>$100N$</td>
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<td></td>
<td></td>
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<tr>
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<td></td>
<td>$N^5$</td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\vdots$</td>
<td></td>
</tr>
</tbody>
</table>
Theory of algorithms: example 1

Goals.
- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 1-SUM = “Is there a 0 in the array?”

Upper bound. A specific algorithm.
- Running time of the optimal algorithm for 1-SUM is $O(N)$.

Lower bound. Proof that no algorithm can do better.
- Ex. Have to examine all $N$ entries (any unexamined one might be 0).
- Running time of the optimal algorithm for 1-SUM is $\Omega(N)$.

Optimal algorithm.
- Lower bound equals upper bound (to within a constant factor).
- Ex. Brute-force algorithm for 1-SUM is optimal: its running time is $\Theta(N)$.
Goals.

- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 3-SUM.

Upper bound. A specific algorithm.

- Ex. Brute-force algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is $O(N^3)$. 
Goals.
- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 3-SUM.

Upper bound. A specific algorithm.
- Ex. Improved algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is $O(N^2 \log N)$.

Lower bound. Proof that no algorithm can do better.
- Ex. Have to examine all $N$ entries to solve 3-SUM.
- Running time of the optimal algorithm for solving 3-SUM is $\Omega(N)$.

Open problems.
- Optimal algorithm for 3-SUM?
- Subquadratic algorithm for 3-SUM?
- Quadratic lower bound for 3-SUM?
Algorithm design approach

Start.
- Develop an algorithm.
- Prove a lower bound.

Gap?
- Lower the upper bound (discover a new algorithm).
- Raise the lower bound (more difficult).

Golden Age of Algorithm Design.
- 1970s–.
  - Steadily decreasing upper bounds for many important problems.
  - Many known optimal algorithms.

Caveats.
- Overly pessimistic to focus on worst case?
- Need better than “to within a constant factor” to predict performance.
Commonly-used notations in the theory of algorithms

<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>Tilde</td>
<td>leading term</td>
<td>( \sim 10 N^2 )</td>
<td>( 10 N^2 )</td>
<td>provide approximate model</td>
</tr>
<tr>
<td>Big Theta</td>
<td>asymptotic order of growth</td>
<td>( \Theta(N^2) )</td>
<td>( \frac{1}{2} N^2 )</td>
<td>classify algorithms</td>
</tr>
<tr>
<td>Big Oh</td>
<td>( \Theta(N^2) ) and smaller</td>
<td>( O(N^2) )</td>
<td>( 10 N^2 )</td>
<td>develop upper bounds</td>
</tr>
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<td>Big Omega</td>
<td>( \Theta(N^2) ) and larger</td>
<td>( \Omega(N^2) )</td>
<td>( \frac{1}{2} N^2 ), ( N^5 )</td>
<td>develop lower bounds</td>
</tr>
</tbody>
</table>

Common mistake. Interpreting big-Oh as an approximate model.
This course. Focus on approximate models: use Tilde-notation
Announcements

How did HW3 go?

HW4 is will be released later today. Written assignment on sorting. It will be due before class next Thursday. (Doing the homework will be good preparation for the midterm).

In-class midterm for next Thursday. Closed book. No devices. You should study all of the textbook chapters that have been assigned so far.

We will do our very best to grade the midterm before the drop deadline, so that you know where you stand in the class before you have to make that decision.

My office hours are today immediately after class in Levine 506. I will also be in my office tomorrow from noon-1 even with the university closing.
2.3 QUICKSORT

- quicksort
- selection
- duplicate keys
- system sorts
Two classic sorting algorithms: mergesort and quicksort

Critical components in the world’s computational infrastructure.
- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

Mergesort. [last lecture]

Quick sort. [this lecture]
public static void quicksort(char[] items, int left, int right)
{
    int i, j;
    char x, y;
    i = left; j = right;
    x = items[(left + right) / 2];
    do
    {
        while ((items[i] < x) && (i < right)) i++;
        while ((x < items[j]) && (j > left)) j--;
        if (i <= j)
        {
            y = items[i];
            items[i] = items[j];
            items[j] = y;
            i ++; j --;
        }
    } while (i <= j);
    if (left < j) quicksort(items, left, j);
    if (i < right) quicksort(items, i, right);
}
2.3 QUICKSORT

- quicksort
- selection
- duplicate keys
- system sorts
Quicksort overview

Step 1. Shuffle the array.

Step 2. Partition the array so that, for some \( j \):
   - Entry \( a[j] \) is in place.
   - No larger entry to the left of \( j \).
   - No smaller entry to the right of \( j \).

Step 3. Sort each subarray recursively.

---

**Input**

QuickSortExample

**Shuffle**

KRATELEPUMQCXOS

**Partition**

ECAI EKLPUMQRXOS

**Sort left**

ACEEIKLPUTMQRXOS

**Sort right**

ACEEIKLMOPOQRSTUX

**Result**

ACEEIKLMOPOQRSTUX
Quicksort overview

input

QUICKSORT EXAMPLE
Quicksort overview

Step 1. Shuffle the array.
Quicksort overview

Step 1. Shuffle the array.

shuffle

K R A T E L E P U I M Q C X O S
Quicksort overview

Step 2. Partition the array so that, for some $j$

- Entry $a[j]$ is in place.
- No larger entry to the left of $j$.
- No smaller entry to the right of $j$.

```
| K | R | A | T | E | L | E | P | U | I | M | Q | C | X | O | S |
```
Quicksort overview

Step 2. Partition the array so that, for some $j$

- Entry $a[j]$ is in place.
- No larger entry to the left of $j$.
- No smaller entry to the right of $j$.

---

**partition**

| E | C | A | I | E | K | L | P | U | T | M | Q | R | X | O | S |

not greater

not less
Quicksort overview

Step 3. Sort each subarray recursively.

sort the left subarray

E C A I E K L P U T M Q R X O S
Quicksort overview

Step 3. Sort each subarray recursively.

sort the left subarray

A C E E I K L P U T M Q R X O S

sorted
Quicksort overview

Step 3. Sort each subarray recursively.

sort the right subarray

A C E E I K L P U T M Q R X O S

sorted
Quicksort overview

Step 3. Sort each subarray recursively.

sort the right subarray

```
 |A  |C  |E  |E  |I  |K  |L  |M  |O  |P  |Q  |R  |S  |T  |U  |X |
```

sorted |

sorted
Quicksort overview

sorted array

A C E E I K L M O P Q R S T U X
• Invented quicksort to translate Russian into English.
  • [but couldn't explain his algorithm or implement it!]
• Learned Algol 60 (and recursion).
• Implemented quicksort.
Tony Hoare

- Invented quicksort to translate Russian into English.
- [ but couldn't explain his algorithm or implement it! ]
- Learned Algol 60 (and recursion).
- Implemented quicksort.

“There are two ways of constructing a software design: One way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies. The first method is far more difficult.”

“I call it my billion-dollar mistake. It was the invention of the null reference in 1965… This has led to innumerable errors, vulnerabilities, and system crashes, which have probably caused a billion dollars of pain and damage in the last forty years.”
Bob Sedgewick

- Refined and popularized quicksort.
- Analyzed many versions of quicksort.

Implementing Quicksort Programs

Robert Sedgewick
Brown University

The Analysis of Quicksort Programs*

Robert Sedgewick

Received January 19, 1976

Summary. The Quicksort sorting algorithm and its best variants are presented and analyzed. Results are derived which make it possible to obtain exact formulas describing the total expected running time of particular implementations on real computers of Quicksort and an improvement called the median-of-three modification. Detailed analysis of the effect of an implementation technique called loop unwrapping is presented. The paper is intended not only to present results of direct practical utility, but also to illustrate the intriguing mathematics which arises in the complete analysis of this important algorithm.
Quicksort partitioning demo

Repeat until i and j pointers cross.

- Scan i from left to right so long as (a[i] < a[lo]).
- Scan j from right to left so long as (a[j] > a[lo]).
- Exchange a[i] with a[j].
Quicksort partitioning demo

Repeat until i and j pointers cross.

- Scan i from left to right so long as (a[i] < a[lo]).
- Scan j from right to left so long as (a[j] > a[lo]).
- Exchange a[i] with a[j].

stop i scan because a[i] >= a[lo]
Quicksort partitioning demo

Repeat until i and j pointers cross.

- Scan i from left to right so long as \((a[i] < a[lo])\).
- Scan j from right to left so long as \((a[j] > a[lo])\).
- Exchange \(a[i]\) with \(a[j]\).
Quicksort partitioning demo

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stop j scan and exchange a[i] with a[j]
Repeat until i and j pointers cross.

- Scan i from left to right so long as (a[i] < a[lo]).
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- Exchange \(a[i]\) with \(a[j]\).

```
K C A T E L E P U I M Q R X O S

↑ lo
↑ i
↑ j
```

stop i scan because \(a[i] >= a[lo]\)
Quicksort partitioning demo

Repeat until i and j pointers cross.
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<table>
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<tr>
<th>K</th>
<th>C</th>
<th>A</th>
<th>I</th>
<th>E</th>
<th>L</th>
<th>E</th>
<th>P</th>
<th>U</th>
<th>T</th>
<th>M</th>
<th>Q</th>
<th>R</th>
<th>X</th>
<th>O</th>
<th>S</th>
</tr>
</thead>
</table>

↑  ↓  ↑  ↓  ↑
Repeat until i and j pointers cross.

- Scan i from left to right so long as (a[i] < a[lo]).
- Scan j from right to left so long as (a[j] > a[lo]).
- Exchange a[i] with a[j].
Quicksort partitioning demo

Repeat until i and j pointers cross.
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stop i scan because a[i] >= a[lo]
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Quicksort partitioning demo

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- Exchange \(a[i]\) with \(a[j]\).

stop j scan and exchange \(a[i]\) with \(a[j]\)
Quicksort partitioning demo

Repeat until i and j pointers cross.

- Scan i from left to right so long as (a[i] < a[lo]).
- Scan j from right to left so long as (a[j] > a[lo]).
- Exchange a[i] with a[j].
Quicksort partitioning demo

Repeat until \(i\) and \(j\) pointers cross.
- Scan \(i\) from left to right so long as \((a[i] < a[lo])\).
- Scan \(j\) from right to left so long as \((a[j] > a[lo])\).
- Exchange \(a[i]\) with \(a[j]\).

stop \(i\) scan because \(a[i] \geq a[lo]\)
Quicksort partitioning demo

Repeat until i and j pointers cross.
- Scan i from left to right so long as \( a[i] < a[lo] \).
- Scan j from right to left so long as \( a[j] > a[lo] \).
- Exchange \( a[i] \) with \( a[j] \).

stop j scan because \( a[j] \leq a[lo] \)
Quicksort partitioning demo

Repeat until i and j pointers cross.
- Scan i from left to right so long as (a[i] < a[lo]).
- Scan j from right to left so long as (a[j] > a[lo]).
- Exchange a[i] with a[j].

When pointers cross.
- Exchange a[lo] with a[j].

pointers cross: exchange a[lo] with a[j]
Quicksort partitioning demo

Repeat until i and j pointers cross.
- Scan i from left to right so long as (a[i] < a[lo]).
- Scan j from right to left so long as (a[j] > a[lo]).
- Exchange a[i] with a[j].

When pointers cross.
- Exchange a[lo] with a[j].

partitioned!
private static int partition(Comparable[] a, int lo, int hi) {
    int i = lo, j = hi+1;
    while (true)
    {
        while (less(a[++i], a[lo]))
            if (i == hi) break;
        while (less(a[lo], a[--j]))
            if (j == lo) break;
        if (i >= j) break;
        exch(a, i, j);
    }
    exch(a, lo, j);
    return j;
}
public class Quick
{
    private static int partition(Comparable[] a, int lo, int hi)
    {
        /* see previous slide */
    }

    public static void sort(Comparable[] a)
    {
        StdRandom.shuffle(a);
        sort(a, 0, a.length - 1);
    }

    private static void sort(Comparable[] a, int lo, int hi)
    {
        if (hi <= lo) return;
        int j = partition(a, lo, hi);
        sort(a, lo, j-1);
        sort(a, j+1, hi);
    }
}
Quicksort trace

<table>
<thead>
<tr>
<th>lo</th>
<th>j</th>
<th>hi</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>15</td>
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<td>7</td>
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<td>7</td>
<td>7</td>
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<td>8</td>
<td>8</td>
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<tr>
<td>10</td>
<td>13</td>
<td>15</td>
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<td>10</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

Quicksort trace (array contents after each partition)
Quicksort animation

50 random items

http://www.sorting-algorithms.com/quick-sort
Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is trickier than it might seem.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop scans on keys equal to the partitioning item's key.

Preserving randomness. Shuffling is needed for performance guarantee.

Equivalent alternative. Pick a random partitioning item in each subarray.
Quicksort: empirical analysis (1961)

Running time estimates:
- Algol 60 implementation.
- National-Elliott 405 computer.

Table 1

<table>
<thead>
<tr>
<th>NUMBER OF ITEMS</th>
<th>MERGE SORT</th>
<th>QUICKSORT</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>2 min 8 sec</td>
<td>1 min 21 sec</td>
</tr>
<tr>
<td>1,000</td>
<td>4 min 48 sec</td>
<td>3 min 8 sec</td>
</tr>
<tr>
<td>1,500</td>
<td>8 min 15 sec*</td>
<td>5 min 6 sec</td>
</tr>
<tr>
<td>2,000</td>
<td>11 min 0 sec*</td>
<td>6 min 47 sec</td>
</tr>
</tbody>
</table>

* These figures were computed by formula, since they cannot be achieved on the 405 owing to limited store size.

sorting N 6-word items with 1-word keys
Quicksort: empirical analysis

Running time estimates:
- Home PC executes $10^8$ compares/second.
- Supercomputer executes $10^{12}$ compares/second.

<table>
<thead>
<tr>
<th>Computer</th>
<th>insertion sort ($N^2$)</th>
<th>mergesort ($N \log N$)</th>
<th>quicksort ($N \log N$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>thousand</td>
<td>million</td>
<td>billion</td>
</tr>
<tr>
<td>home</td>
<td>instant</td>
<td>2.8 hours</td>
<td>317 years</td>
</tr>
<tr>
<td>super</td>
<td>instant</td>
<td>1 second</td>
<td>1 week</td>
</tr>
</tbody>
</table>

Lesson 1. Good algorithms are better than supercomputers.
Lesson 2. Great algorithms are better than good ones.
Quicksort: best-case analysis

**Best case.** Number of compares is $\sim N \lg N$.

<table>
<thead>
<tr>
<th>a[]</th>
</tr>
</thead>
<tbody>
<tr>
<td>lo</td>
</tr>
<tr>
<td>j</td>
</tr>
<tr>
<td>hi</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
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<tr>
<td>3</td>
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<td>5</td>
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<td>6</td>
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<tr>
<td>7</td>
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<tr>
<td>8</td>
</tr>
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</table>

**Initial values**

```
H A C B F E G D L I K J N M O
```

**Random shuffle**

```
H A C B F E G D L I K J N M O
```

```
0 7 14 D A C B F E G H L I K J N M O
```

```
0 3 6 B A C D F E G H L I K J N M O
```

```
0 1 2 A B C D F E G H L I K J N M O
```

```
0 0 0 A B C D F E G H L I K J N M O
```

```
2 2 2 A B C D F E G H L I K J N M O
```

```
4 5 6 A B C D E F G H L I K J N M O
```

```
4 4 4 A B C D E F G H L I K J N M O
```

```
6 6 6 A B C D E F G H L I K J N M O
```

```
8 11 14 A B C D E F G H J I K L N M O
```

```
8 9 10 A B C D E F G H I J K L N M O
```

```
8 8 8 A B C D E F G H I J K L N M O
```

```
10 10 10 A B C D E F G H I J K L N M O
```

```
12 13 14 A B C D E F G H I J K L M N O
```

```
12 12 12 A B C D E F G H I J K L M N O
```

```
14 14 14 A B C D E F G H I J K L M N O
```

```
A B C D E F G H I J K L M N O
```
**Quicksort: worst-case analysis**

**Worst case.** Number of compares is $\sim \frac{1}{2} N^2$.

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Quicksort: average-case analysis

**Proposition.** The average number of compares $C_N$ to quicksort an array of $N$ distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

**Pf.** $C_N$ satisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

$$C_N = (N + 1) + \left( \frac{C_0 + C_{N-1}}{N} \right) + \left( \frac{C_1 + C_{N-2}}{N} \right) + \ldots + \left( \frac{C_{N-1} + C_0}{N} \right)$$

- Multiply both sides by $N$ and collect terms:
  $$NC_N = N(N + 1) + 2(C_0 + C_1 + \ldots + C_{N-1})$$

- Subtract from this equation the same equation for $N - 1$:
  $$NC_N - (N - 1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by $N(N + 1)$:
  $$\frac{C_N}{N + 1} = \frac{C_{N-1}}{N} + \frac{2}{N + 1}$$
Quicksort: average-case analysis

- Repeatedly apply previous equation:
  \[
  \frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1} \\
  = \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
  = \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
  = \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \ldots + \frac{2}{N+1}
  \]

- Approximate sum by an integral:
  \[
  C_N = 2(N+1) \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{N+1}\right) \\
  \sim 2(N+1) \int_3^{N+1} \frac{1}{x} \, dx
  \]

- Finally, the desired result:
  \[
  C_N \sim 2(N+1) \ln N \approx 1.39N \lg N
  \]
Quicksort: summary of performance characteristics

Quicksort is a (Las Vegas) randomized algorithm.
  - Guaranteed to be correct.
  - Running time depends on random shuffle.

**Average case.** Expected number of compares is $\sim 1.39 N \lg N$.
  - 39% more compares than mergesort.
  - Faster than mergesort in practice because of less data movement.

**Best case.** Number of compares is $\sim N \lg N$.

**Worst case.** Number of compares is $\sim \frac{1}{2} N^2$.

[ but more likely that lightning bolt strikes computer during execution ]
Quicksort properties

Proposition. Quicksort is an in-place sorting algorithm.

Pf.
- Partitioning: constant extra space.
- Depth of recursion: logarithmic extra space (with high probability).

Proposition. Quicksort is not stable.

Pf. [by counterexample]

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<tr>
<th>i</th>
<th>j</th>
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<td>A1</td>
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<td>A1</td>
<td>B1</td>
<td>C2</td>
<td>C1</td>
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</table>
Quicksort: practical improvements

Insertion sort small subarrays.

- Even Quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for \( \approx 10 \) items.

```java
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo + CUTOFF - 1)
    {
        Insertion.sort(a, lo, hi);
        return;
    }
    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```
Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.

\[
\sim 12/7 \quad N \ln N \text{ compares (14\% fewer)}
\]
\[
\sim 12/35 \quad N \ln N \text{ exchanges (3\% more)}
\]

```java
private static void sort(Comparable[] a, int lo, int hi) {
    if (hi <= lo) return;

    int median = median0f3(a, lo, lo + (hi - lo)/2, hi);
    swap(a, lo, median);

    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```
2.3 Quicksort

- quicksort
- selection
- duplicate keys
- system sorts
Selection

**Goal.** Given an array of $N$ items, find the $k^{th}$ smallest item.

**Ex.** Min ($k = 0$), max ($k = N - 1$), median ($k = N/2$).

**Applications.**

- Order statistics.
- Find the "top $k$.”

**Use theory as a guide.**

- Easy $N \log N$ upper bound. How?
- Easy $N$ upper bound for $k = 1, 2, 3$. How?
- Easy $N$ lower bound. Why?

**Which is true?**

- $N \log N$ lower bound? is selection as hard as sorting?
- $N$ upper bound? is there a linear-time algorithm?
Quick-select

Partition array so that:
- Entry $a[j]$ is in place.
- No larger entry to the left of $j$.
- No smaller entry to the right of $j$.

Repeat in one subarray, depending on $j$; finished when $j$ equals $k$.

```java
public static Comparable select(Comparable[] a, int k) {
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
        int j = partition(a, lo, hi);
        if (j < k) lo = j + 1;
        else if (j > k) hi = j - 1;
        else return a[k];
    }
    return a[k];
}
```
Quick-select demo

Partition array so that:

- Entry \( a[j] \) is in place.
- No larger entry to the left of \( j \).
- No smaller entry to the right of \( j \).

Repeat in one subarray, depending on \( j \); finished when \( j \) equals \( k \).

select element of rank \( k = 5 \)

\[
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
50 & 21 & 28 & 65 & 39 & 59 & 56 & 22 & 95 & 12 & 90 & 53 & 32 & 77 & 33 \\
\end{array}
\]

\( k = 5 \)
Quick-select demo

Partition array so that:

- Entry \( a[j] \) is in place.
- No larger entry to the left of \( j \).
- No smaller entry to the right of \( j \).

Repeat in one subarray, depending on \( j \); finished when \( j \) equals \( k \).

Partition on leftmost entry

\[
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\hline
50 & 21 & 28 & 65 & 39 & 59 & 56 & 22 & 95 & 12 & 90 & 53 & 32 & 77 & 33 \\
\end{array}
\]

\( k = 5 \)
Quick-select demo

Partition array so that:

- Entry $a[j]$ is in place.
- No larger entry to the left of $j$.
- No smaller entry to the right of $j$.

Repeat in one subarray, depending on $j$; finished when $j$ equals $k$.

**Partitioned array**

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$k = 5$
Quick-select demo

Partition array so that:
• Entry $a[j]$ is in place.
• No larger entry to the left of $j$.
• No smaller entry to the right of $j$.

Repeat in one subarray, depending on $j$; finished when $j$ equals $k$.

can safely ignore right subarray

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$k = 5$
Quick-select demo

Partition array so that:

- Entry $a[j]$ is in place.
- No larger entry to the left of $j$.
- No smaller entry to the right of $j$.

**Repeat** in one subarray, depending on $j$; finished when $j$ equals $k$.

### partition on leftmost entry

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</table>

$k = 5$
Quick-select demo

Partition array so that:

- Entry $a[j]$ is in place.
- No larger entry to the left of $j$.
- No smaller entry to the right of $j$.

Repeat in one subarray, depending on $j$; finished when $j$ equals $k$.

### partitioned array

<table>
<thead>
<tr>
<th>0</th>
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</table>

$k = 5$
Quick-select demo

Partition array so that:

- Entry $a[j]$ is in place.
- No larger entry to the left of $j$.
- No smaller entry to the right of $j$.

Repeat in one subarray, depending on $j$; finished when $j$ equals $k$.

can safely ignore left subarray

\[
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
12 & 21 & 22 & 33 & 39 & 32 & 28 & 50 & 95 & 56 & 90 & 53 & 59 & 77 & 65 \\
\end{array}
\]

$k = 5$
Quick-select demo

Partition array so that:

- Entry $a[j]$ is in place.
- No larger entry to the left of $j$.
- No smaller entry to the right of $j$.

Repeat in one subarray, depending on $j$; finished when $j$ equals $k$.

**partition on leftmost entry**

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
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</tbody>
</table>

$k = 5$
Quick-select demo

Partition array so that:

- Entry \( a[j] \) is in place.
- No larger entry to the left of \( j \).
- No smaller entry to the right of \( j \).

Repeat in one subarray, depending on \( j \); finished when \( j \) equals \( k \).

**partitioned array**

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</tr>
</tbody>
</table>

\[ k = 5 \]
Quick-select demo

Partition array so that:

- Entry \( a[j] \) is in place.
- No larger entry to the left of \( j \).
- No smaller entry to the right of \( j \).

Repeat in one subarray, depending on \( j \); finished when \( j \) equals \( k \).

stop: partitioning item is at index \( k \)

\[
\begin{array}{cccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
12 & 21 & 22 & 32 & 28 & 33 & 39 & 50 & 95 & 56 & 90 & 53 & 59 & 77 & 65 \\
\end{array}
\]

\( k = 5 \)
Quick-select: mathematical analysis

Proposition. Quick-select takes linear time on average.

Pf sketch.
- Intuitively, each partitioning step splits array approximately in half:
  \[ N + N/2 + N/4 + \ldots + 1 \sim 2N \text{ compares.} \]
- Formal analysis similar to quicksort analysis yields:
  \[
  C_N = 2N + 2k \ln (N/k) + 2(N-k) \ln (N/(N-k)) \\
  \leq (2 + 2 \ln 2) N
  \]
- Ex: \((2 + 2 \ln 2) N \approx 3.38 N\) compares to find median \((k = N/2)\).
Theoretical context for selection


![Image](image.png)

**Abstract**

The number of comparisons required to select the $i$-th smallest of $n$ numbers is shown to be at most a linear function of $n$ by analysis of a new selection algorithm -- PICK. Specifically, no more than $5.4305n$ comparisons are ever required. This bound is improved for extreme values of $i$.

**Remark.** Constants are high $\Rightarrow$ not used in practice.

**Use theory as a guide.**

- Still worthwhile to seek practical linear-time (worst-case) algorithm.
- Until one is discovered, use quick-select (if you don’t need a full sort).
2.3 Quicksort

- quicksort
- selection
- duplicate keys
- system sorts
Duplicate keys

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.
War story (system sort in C)

A beautiful bug report. [Allan Wilks and Rick Becker, 1991]

We found that qsort is unbearably slow on "organ-pipe" inputs like "01233210":

```c
main (int argc, char**argv) {
    int n = atoi(argv[1]), i, x[100000];
    for (i = 0; i < n; i++)
        x[i] = i;
    for ( ; i < 2*n; i++)
        x[i] = 2*n-i-1;
    qsort(x, 2*n, sizeof(int), intcmp);
}
```

Here are the timings on our machine:

```
$ time a.out 2000
real  5.85s
$ time a.out 4000
real  21.64s
$ time a.out 8000
real  85.11s
```
War story (system sort in C)

**Bug.** A `qsort()` call that should have taken seconds was taking minutes.

At the time, almost all `qsort()` implementations based on those in:
- Version 7 Unix (1979): quadratic time to sort organ-pipe arrays.
- BSD Unix (1983): quadratic time to sort random arrays of 0s and 1s.
Duplicate keys: stop on equal keys

Our partitioning subroutine stops both scans on equal keys.

Q. Why not continue scans on equal keys?
Partitioning an array with all equal keys

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</tbody>
</table>
Duplicate keys: partitioning strategies

**Bad.** Don't stop scans on equal keys.

\[
\sim \frac{1}{2} N^2 \text{ compares when all keys equal }
\]

B A A B A B B C C C A A A A A A A A A A A

**Good.** Stop scans on equal keys.

\[
\sim N \lg N \text{ compares when all keys equal }
\]

B A A B A B C C B C B A A A A A A A A A A A A

**Better.** Put all equal keys in place. How?

\[
\sim N \text{ compares when all keys equal }
\]

A A A B B B B B C C C A A A A A A A A A A A A
3-way partitioning

**Goal.** Partition array into three parts so that:
- Entries between `lt` and `gt` equal to the partition item.
- No larger entries to left of `lt`.
- No smaller entries to right of `gt`.

Dutch national flag problem. [Edsger Dijkstra]
- Conventional wisdom until mid 1990s: not worth doing.
- Now incorporated into C library `qsort()` and Java 6 system sort.
Dijkstra 3-way partitioning demo

- Let $v$ be partitioning item $a[lo]$.
- Scan $i$ from left to right.
  - $(a[i] < v)$: exchange $a[lt]$ with $a[i]$; increment both $lt$ and $i$
  - $(a[i] > v)$: exchange $a[gt]$ with $a[i]$; decrement $gt$
  - $(a[i] == v)$: increment $i$
Dijkstra 3-way partitioning demo

- Let \( v \) be partitioning item \( a[lo] \).
- Scan \( i \) from left to right.
  - \( (a[i] < v) \): exchange \( a[lt] \) with \( a[i] \); increment both \( lt \) and \( i \)
  - \( (a[i] > v) \): exchange \( a[gt] \) with \( a[i] \); decrement \( gt \)
  - \( (a[i] == v) \): increment \( i \)

invariant
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo) return;
    int lt = lo, gt = hi;
    Comparable v = a[lo];
    int i = lo;
    while (i <= gt)
    {
        int cmp = a[i].compareTo(v);
        if (cmp < 0) exch(a, lt++, i++);
        else if (cmp > 0) exch(a, i, gt--);
        else i++;
    }
    sort(a, lo, lt - 1);
    sort(a, gt + 1, hi);
}
3-way quicksort: visual trace

equal to partitioning element
Duplicate keys: lower bound

**Sorting lower bound.** If there are \( n \) distinct keys and the \( i^{th} \) one occurs \( x_i \) times, then any compare-based sorting algorithm must use at least

\[
\log \left( \frac{N!}{x_1! x_2! \cdots x_n!} \right) \sim - \sum_{i=1}^{n} x_i \log \frac{x_i}{N}
\]

\( \sim N \log N \) when all distinct; linear when only a constant number of distinct keys compares in the worst case.

**Proposition.** The expected number of compares to 3-way quicksort an array is **entropy optimal** (proportional to sorting lower bound).

**Pf.** [beyond scope of course]

**Bottom line.** Quicksort with 3-way partitioning reduces running time from linearithmic to linear in broad class of applications.
## Sorting summary

<table>
<thead>
<tr>
<th></th>
<th>inplace?</th>
<th>stable?</th>
<th>best</th>
<th>average</th>
<th>worst</th>
<th>remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>selection</td>
<td>✔️</td>
<td></td>
<td>$\frac{1}{2} N^2$</td>
<td>$\frac{1}{2} N^2$</td>
<td>$\frac{1}{2} N^2$</td>
<td>$N$ exchanges</td>
</tr>
<tr>
<td>insertion</td>
<td>✔️ ✔️</td>
<td>✔️</td>
<td>$N$</td>
<td>$\frac{1}{4} N^2$</td>
<td>$\frac{1}{2} N^2$</td>
<td>use for small $N$ or partially ordered</td>
</tr>
<tr>
<td>shell</td>
<td>✔️</td>
<td></td>
<td>$N \log_3 N$</td>
<td>?</td>
<td>$c N^{3/2}$</td>
<td>tight code; subquadratic</td>
</tr>
<tr>
<td>merge</td>
<td>✔️</td>
<td>✔️</td>
<td>$\frac{1}{2} N \lg N$</td>
<td>$N \lg N$</td>
<td>$N \lg N$</td>
<td>$N \log N$ guarantee; stable</td>
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<tr>
<td>timsort</td>
<td>✔️</td>
<td>✔️</td>
<td>$N$</td>
<td>$N \lg N$</td>
<td>$N \lg N$</td>
<td>improves mergesort when preexisting order</td>
</tr>
<tr>
<td>quick</td>
<td>✔️</td>
<td></td>
<td>$N \lg N$</td>
<td>$2N \ln N$</td>
<td>$\frac{1}{2} N^2$</td>
<td>$N \log N$ probabilistic guarantee; fastest in practice</td>
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<tr>
<td>3-way quick</td>
<td>✔️</td>
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<td>$N$</td>
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<td>$\frac{1}{2} N^2$</td>
<td>improves quicksort when duplicate keys</td>
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<td>✔️</td>
<td>$N$</td>
<td>$N \lg N$</td>
<td>$N \lg N$</td>
<td>holy sorting grail</td>
</tr>
</tbody>
</table>
2.3 Quicksort

- quicksort
- selection
- duplicate keys
- system sorts
Sorting applications

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS feed in reverse chronological order.
- Find the median.
- Identify statistical outliers.
- Binary search in a database.
- Find duplicates in a mailing list.
- Data compression.
- Computer graphics.
- Computational biology.
- Load balancing on a parallel computer.

...
Engineering a system sort (in 1993)

Bentley-McIlroy quicksort.

- Cutoff to insertion sort for small subarrays.
- Partitioning item: median of 3 or Tukey's ninther.
- Partitioning scheme: Bentley-McIlroy 3-way partitioning.

Very widely used. C, C++, Java 6, ....
Replacement of quicksort in java.util.Arrays with new dual-pivot quicksort

Hello All,

I'd like to share with you new Dual-Pivot Quicksort which is faster than the known implementations (theoretically and experimental). I'd like to propose to replace the JDK's Quicksort implementation by new one.

... 

The new Dual-Pivot Quicksort uses *two* pivots elements in this manner:

1. Pick an elements P1, P2, called pivots from the array.
2. Assume that P1 <= P2, otherwise swap it.
3. Reorder the array into three parts: those less than the smaller pivot, those larger than the larger pivot, and in between are those elements between (or equal to) the two pivots.
4. Recursively sort the sub-arrays.

The invariant of the Dual-Pivot Quicksort is:

\[
\text{[ < P1 | P1 <= & <= P2 } \geq P2] 
\]

...
Replacement of quicksort in java.util.Arrays with new dual-pivot quicksort

Date: Thu, 29 Oct 2009 11:19:39 +0000
Subject: Replace quicksort in java.util.Arrays with dual-pivot implementation

Changeset: b05abb410c52
Author: alanb
Date: 2009-10-29 11:18 +0000
URL: http://hg.openjdk.java.net/jdk7/tl/jdk/rev/b05abb410c52

6880672: Replace quicksort in java.util.Arrays with dual-pivot implementation
Reviewed-by: jjb
Contributed-by: vladimir.yaroslavskiy at sun.com, joshua.bloch at google.com, jbentley at avaya.com

! make/java/java/FILES_java.gmk
! src/share/classes/java/util/Arrays.java
+ src/share/classes/java/util/DualPivotQuicksort.java

http://mail.openjdk.java.net/pipermail/compiler-dev/2009-October.txt
Dual-pivot quicksort

Use two partitioning items $p_1$ and $p_2$ and partition into three subarrays:

- Keys less than $p_1$.
- Keys between $p_1$ and $p_2$.
- Keys greater than $p_2$.

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degenerates to Dijkstra's 3-way partitioning

Recursively sort three subarrays.

Note. Skip middle subarray if $p_1 = p_2$. 
Initialization.

- Choose $a[lo]$ and $a[hi]$ as partitioning items.
- Exchange if necessary to ensure $a[lo] \leq a[hi]$.
Main loop. Repeat until $i$ and $gt$ pointers cross.
- If $(a[i] < a[lo])$, exchange $a[i]$ with $a[lt]$ and increment $lt$ and $i$.
- Else if $(a[i] > a[hi])$, exchange $a[i]$ with $a[gt]$ and decrement $gt$.
- Else, increment $i$. 

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Dual-pivot partitioning demo

Finalize.
- Exchange $a[hi]$ with $a[++gt]$.

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3-way partitioned
Dual-pivot quicksort

Use two partitioning items $p_1$ and $p_2$ and partition into three subarrays:

- Keys less than $p_1$.
- Keys between $p_1$ and $p_2$.
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Three-pivot quicksort

Use three partitioning items $p_1$, $p_2$, and $p_3$ and partition into four subarrays:

- Keys less than $p_1$.
- Keys between $p_1$ and $p_2$.
- Keys between $p_2$ and $p_3$.
- Keys greater than $p_3$.

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Multi-Pivot Quicksort: Theory and Experiments

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System sort in Java 7

Arrays.sort().

- Has one method for objects that are Comparable.
- Has an overloaded method for each primitive type.
- Has an overloaded method for use with a Comparator.
- Has overloaded methods for sorting subarrays.

Algorithms.

- Dual-pivot quicksort for primitive types.
- Timsort for reference types.

Q. Why use different algorithms for primitive and reference types?

Bottom line. Use the system sort!