# CIS121 - Fall 2008

**Lab 3** – Monday/Tuesday, September 22/23

$$1 + q + q^2 + q^3 + \ldots + q^{n-1} = \frac{q^n - 1}{q - 1}.$$

## 1

Solve the following recurrence relation
$T(1) = 0.5$ and $T(n) = 0.5n + T(n-1)$
**Solution:** $T(n) = 0.5n + T(n-1)$.
$T(n-1) = 0.5(n-1) + T(n-2)$
$T(n-2) = 0.5(n-2) + T(n-3)$

$T(n) = 0.5n + [0.5(n-1) + T(n-2)]$
$T(n) = 0.5n + 0.5(n-1) + T(n-2)$
$T(n) = 0.5n + 0.5(n-1) + [0.5(n-2) + T(n-3)]$

$T(n) = 0.5(n + (n-1) + (n-2) + \cdots + 1) = 0.5(n(n+1)/2) = n(n+1)/4$

## 2

Solve the following recurrence relation
$T(1) = 3$ and $T(n) = 2T(n/2) + 1$
You can assume that $n = 2^k$ for some integer $k$.
**Solution:** $T(n) = 2T(n/2) + 1$
$T(2^k) = 2T(2^{k-1}) + 1$

Let $S(m) = T(2^m)$

$S(m) = 2S(m-1) + 1$
$S(m-1) = 2S(m-2) + 1$
$S(m-2) = 2S(m-3) + 1$

$S(m) = 2[2S(m-2) + 1] + 1$
$S(m) = 2^2 S(m-2) + 2 + 1$
$S(m) = 2^2[2S(m-3) + 1] + 2 + 1$
$S(m) = 2^3 S(m-3) + 2^2 + 2 + 1$

$S(m) = 2^m \cdot S(0) + \sum_{i=0}^{m-1} 2^i$
$S(m) = 2^m \cdot S(0) + 2^m - 1$
$S(m) = 2^m \cdot T(1) + 2^m - 1$

$S(m) = 4 \cdot 2^m - 1$

$T(n) = T(2^k) = S(k) = 4 \cdot 2^m - 1 = 4 \cdot 2^{\log_2 n} - 1 = 4n - 1$

# 3

Solve the following recurrence relation
$T(1) = 2$ and $T(n) = T(n-1)$
**Solution:** $T(n) = T(n-1)$
$T(n-1) = T(n-2)$

$T(n) = T(n-1) = T(n-2) = \cdots = 2$
$T(n) = 2$

# 4

Solve the following recurrence relation
$T(1) = 1$ and $T(n) = 4T(n-1) + 1$
**Solution:** $T(n) = 4T(n-1) + 1$
$T(n-1) = 4T(n-2) + 1$
$T(n-2) = 4T(n-3) + 1$

$T(n) = 4 \cdot (4T(n-2) + 1) + 1$
$T(n) = 4^2 \cdot (4T(n-3) + 1) + 4 + 1$
$T(n) = 4^3 \cdot (4T(n-4) + 1) + 4^2 + 4 + 1$

$T(n) = 4^{n-1} \cdot T(1) + \sum_{i=0}^{n-2} 4^i$
$T(n) = 4^{n-1} \cdot T(1) + (4^n - 4)/12$

$T(n) = (4^n - 1)/3$

## 5

Solve the following recurrence relation
$T(1) = 1$ and $T(n) = 3T(n/3) + 1$
You can assume that $n = 3^k$ for some integer $k$.
**Solution:** $T(n) = 3T(n/3) + 1$
$T(3^k) = 3T(3^{k-1}) + 1$

Let $S(m) = T(3^m)$

$S(m) = 3S(m-1) + 1$
$S(m-1) = 3S(m-2) + 1$
$S(m-2) = 3S(m-3) + 1$

$S(m) = 3[3S(m-2) + 1] + 1$
$S(m) = 3^2 S(m-2) + 3 + 1$
$S(m) = 3^2[3S(m-3) + 1] + 3 + 1$
$S(m) = 3^3 S(m-3) + 3^2 + 3 + 1$
$S(m) = 3^m \cdot S(0) + \sum_{i=0}^{m-1} 3^i$
$S(m) = 3^m \cdot S(0) + (3^m - 1)/2$
$S(m) = 3^m \cdot T(1) + (3^m - 1)/2$

$S(m) = (3 \cdot 3^m - 1)/2$

$T(n) = T(3^k) = S(k) = (3 \cdot 3^m - 1)/2 = (3 \cdot 3^{\log_3 n} - 1)/2 + 1 = (3n - 1)/2$

## 6

Solve the following recurrence relation
$T(1) = 0$ and $T(n) = 2T(n/2) + n + 1$
**Solution:** Solve by recurrsion tree

$$\begin{array}{cccccccccc}
 & & & & n+1 & & & & = & n+1 \\
 & & n/2+1 & & & & n/2+1 & & = & n+2 \\
 & n/4+1 & & n/4+1 & & n/4+1 & & n/4+1 & = & n+4 \\
\vdots & & & & & & & & \vdots & \\
T(1)+1 & T(1)+1 & & \cdots & & & T(1)+1 & T(1)+1 & = & n
\end{array}$$

$T(n) = n + n + \cdots + n + 2^{\log(n)} - 1 = n\log(n) + n - 1$