Midterm 1 Review WITH SOLUTIONS

Posted Thursday February 22

On Thursday March 1 we will have our first midterm exam during the usual class time. The exam will take place in CHEM 102 and in COLL 200.

The students whose last name begins with a letter in the range N-Z will have to take the exam in COLL 200. (We'll switch alphabetical range for the other midterms.) The rest (A-M) should come take the exam in our lecture room, CHEM 102.

The exam will last for 80 minutes. Please be in CHEM 102 or COLL 200 at 1:30PM so we have time to seat everybody properly.

This here is a midterm review document with readings, a mock (practice) midterm, and more practice problems. You should solve the practice exam while timing yourselves.

Solutions to the practice exam will be posted Sunday February 25, late afternoon.

Val will hold a review session on Tuesday February 27, 6:00-7:30PM in Heilmeier Hall (TOWNE 100).

The TAs will hold a review session on Monday February 26, 6-9PM in Heilmeier Hall (TOWNE 100).

1 Readings

STUDY IN-DEPTH... ...the posted notes for lectures 1-12 (not lecture 13).

STUDY IN-DEPTH... ...the posted guides for recitations 1-7.

STUDY IN-DEPTH... ...the posted solutions to homeworks 1-5. Compare with your own solutions.

STUDY IN-DEPTH... ...the solutions to the mock exam and the additional problems contained in this document, to be posted Sunday February 25, late afternoon. Until then, try very hard to solve these on your own.

2 Mock Exam (80 minutes for 160 points)

1. (35 pts)

For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a very brief explanation of your answer.

(a) \( \binom{100}{51} \) is strictly bigger than \( \binom{100}{49} \).

(b) In Pascal’s Triangle, there exist four binomial coefficients \( c_1, c_2, c_3, c_4 \) such that \( c_1 = c_2 + c_3 + c_4 \).

(c) The boolean expressions \( \neg[(p \Rightarrow q) \lor q] \) and \( p \land \neg q \) are logically equivalent.

(d) Let \( A \) be a finite set. All functions \( f : A \rightarrow A \) are bijections.
(e) Let $A, B$ be finite set. Then, there are exactly as many subsets of $A \times B$ as there are functions with domain $A$ and codomain $2^B$.

(f) The function $f : \mathbb{N} \to \mathbb{N}$, defined by $f(n) = \lfloor n/2 \rfloor$ has an inverse.

(g) The Fibonacci number $F_{100}$ is even.

**Answer**

(a) **FALSE** In fact, we see that they are equal. This stems from the identity

$$\binom{n}{k} = \binom{n}{n-k}$$

presented in lecture.

(b) **TRUE** Consider the following values for $c_1, c_2, c_3, \text{ and } c_4$:

$$c_1 = \binom{3}{1} \quad c_2 = \binom{160}{0} \quad c_3 = \binom{160}{160} \quad c_4 = \binom{0}{0}$$

We thus see that:

$$c_1 = c_2 + c_3 + c_4$$

$$\binom{3}{1} = \binom{160}{0} + \binom{160}{160} + \binom{0}{0}$$

$$3 = 1 + 1 + 1$$

which we know to be true.

**Alternate Solution:**

We also could have solved this problem by applying Pascal’s Identity twice, for example:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k}$$

By choosing appropriate values for $n$ and $k$ (i.e., $n \geq k \geq 1$, we know the above equality holds. Concretely, take $n = 5$ and $k = 3$. We thus have:

$$\binom{5}{3} = \binom{4}{2} + \binom{3}{2} + \binom{3}{3}$$

$$10 = 6 + 3 + 1$$

which we know to be true.

(c) **TRUE** We demonstrate this by showing they have equivalent truth tables, as follows:

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<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \implies q$</th>
<th>$(p \implies q) \lor \neg q$</th>
<th>$\neg [(p \implies q) \lor q]$</th>
<th>$\neg q$</th>
<th>$p \land \neg q$</th>
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<tbody>
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</table>
For a counterexample, consider the set $A = \{1, 2\}$ and the function $f : A \to A$ given by $f(1) = f(2) = 1$. The function is not injective, since $f(1) = 1 = f(2)$. Therefore it cannot be a bijection. Alternatively, we can argue that $f$ is not surjective because there is no $a \in A$ with $f(a) = 2$, and hence, it is not bijective.

Let $|A| = n$ and $|B| = m$. From class, we know that:

$$ |A \times B| = |A| \cdot |B| = nm $$

Furthermore, for a given set $S$, we also know that $|2^S| = 2^{|S|}$. Thus, we conclude that the number of subsets of $A \times B$ is given by:

$$ |2^{A \times B}| = 2^{|A \times B|} = 2^{nm} $$

From class, we also know that the number of functions with domain $X$ and codomain $Y$ is given by $|Y|^{|X|}$. In this problem, we know that $X = A$ and $Y = 2^B$, thus the number of such functions is:

$$ |2^B|^{|A|} = (2^m)^n = 2^{nm} $$

Therefore, we have shown that there are exactly as many subsets of $A \times B$ as there are functions with domain $A$ and codomain $2^B$.

Assume for contradiction that the function has an inverse. If the function has an inverse, then it is bijective. In particular, it must be injective. However, the function is not injective because $f(0) = 0 = f(1)$. Thus, we have reached a contradiction.

Let’s see. $F_0 = 0$ is even, $F_1 = 1$ and $F_2 = 1$ are odd. $F_3 = 2$ is even again, and the pattern repeats. More generally, for any positive integer $n$, $F_{3n-2}$ and $F_{3n-1}$ are odd and $F_{3n}$ is even.

[This can be easily justified by ordinary induction on $n$ but we are asking here only for a brief justification.]

Since $100 = 3 \cdot 34 - 2$ it follows that $F_{100}$ is odd.

2. (15pts)

In how many ways can we distribute 20 indistinguishable apples and 30 indistinguishables oranges to 10 distinguishable children so that every child has at least 1 apple and at least 2 oranges?

**Answer**

We can consider making arrangements by using the Multiplication Rule, specifically, we want to first place the apples then place the oranges. Note that these steps are indeed independent of each other.

**Step 1:** distribute the apples among the children so that each gets at least 1.

**Step 2:** distribute the oranges among the children so that each gets at least 2.

In order to do Step 1, we first give each child one apple to satisfy the constraint. The remaining $20 - 10 = 10$ apples can be distributed any way we want therefore the stars and bars technique applies for this. the 10 apples correspond to 10 stars. These we separate for the 10 distinguishable children using $10 - 1 = 9$ bars. Thus, we have

$$ \binom{19}{9} $$
ways to distribute the apples. (Solutions that use \( \binom{10}{10} \) or \( \binom{19}{10} \) instead of \( \binom{19}{9} \) are also OK.)

Step 2 is solved similarly. After giving each child 2 oranges, we are left with 10 oranges to distribute any way we want, which by the stars and bars technique can be done in

\[
\binom{19}{9}
\]

Finally, applying the Multiplication Rule, we get our final answer of \( \binom{19}{9} \cdot \binom{19}{9} \).

3. (15pts)

Count the number of distinct sequences of bits (0’s, 1s) of length 101 such that:

- there are 3 more 1’s than 0’s in the sequence; and...
- . . . also the middle bit is a 1.

**Answer**

Note that given the first constraint, there must be exactly 52 1’s and 49 0’s in the sequence. We find this by letting \( x \) be the total number of 0’s in the sequence, and we see that

\[
x + (x + 3) = 101
\]

\[
2x = 98
\]

\[
x = 49
\]

To count the total number of sequences, we use multiplication rule.

- **Step 1**: place a 1 bit in the middle position.
- **Step 2**: place the remaining 1’s.
- **Step 3**: place the remaining 0’s.

There’s only one way of placing a 1 bit in the middle position. For Step 2, we are left with 51 1’s to distribute among 101 − 1 = 100 spots (since the middle spot is now occupied). This can be done in \( \binom{100}{51} \) ways. Finally, note that once the 1’s are in their positions, there’s only one way to place the 0’s. Specifically, the 0’s *must* go into the empty positions. Thus our final answer is, by the Multiplication Rule:

\[
1 \cdot \binom{100}{51} \cdot 1 = \binom{100}{51}
\]

4. (15pts)

Prove that for any \( x, y, z \in \mathbb{Z} \) such that \( x + 2y = z \), if \( z - x \) is not divisible by 4 then \( x + y + z \) is odd.

**Answer**

Proof by contradiction. Let \( x, y, z \in \mathbb{Z} \) such that \( x + 2y = z \) and \( z - x \) is *not* divisible by 4. Assume toward a contradiction that \( x + y + z \) is *not* odd.
Hence \( x + y + z \) is even, and therefore \( x + y + z = 2k \), for some \( k \in \mathbb{Z} \). We also have that \( z = x + 2y \), so we may write:

\[
\begin{align*}
x + y + (x + 2y) &= 2k \\
2x + 3y &= 2k \\
3y &= 2(k - x)
\end{align*}
\]

Therefore \( 3y \) must be even. It follows that \( y \) must also be even. Indeed, if \( y \) were odd then \( 3y \) would also be odd. So \( y = 2\ell \) for some \( \ell \in \mathbb{Z} \).

But then we have

\[
z - x = (x + 2y) - x \\
= 2y \\
= 4\ell
\]

Since \( \ell \) is an integer, this means that \( 4 \mid z - x \), and we have reached a contradiction.

**Alternate Solution:** (direct proof).

We rewrite the given equation as

\[
2y = z - x
\]

Since \( y \in \mathbb{Z} \), this tells us that \( z - x \) is even. Additionally, since \( z - x \) is not divisible by 4, we know that \( \frac{z - x}{2} \) must be odd – if it were even, \( z - x \) would be divisible by 4.

We write

\[
z - x = 2(2k + 1), k \in \mathbb{Z}
\]

But then \( 2y = 2(2k + 1) \), so \( y = 2k + 1 \) is odd.

Finally, we know that \( 2x \) is even, since \( x \in \mathbb{Z} \). Putting this together gives us:

\[
x + y + z = (z - x) + 2x + y \\
= 2(2k + 1) + 2x + 2k + 1 \\
= 2(3k + x + 1) + 1
\]

As the sum of products of integers, \( 3k + x + 1 \) must be an integer, so \( x + y + z \) is odd.

5. (15pts)

Let \( A, B \) be any sets such that \( A \cap \{1, 2\} = B \cap \{1, 2\} \). Prove that the sets \( (A \setminus B) \cup (B \setminus A) \) and \( \{1, 2\} \) are disjoint.

(You are NOT allowed to use in the proof any set algebra facts. Your proof should use only the definitions of disjoint, set difference, union, and intersection, as well as logical manipulation of statements.)

**Answer**

If we can show that no element in \((A \setminus B) \cup (B \setminus A)\) can be in \( \{1, 2\} \), then clearly the intersection of these 2 sets is empty and we are done.
We consider an arbitrary element \( x \in ((A \setminus B) \cup (B \setminus A)) \) and show that \( x \notin \{1, 2\} \). There are 2 cases:

Case 1: \( x \in A \setminus B \). Then \( x \in A \land x \notin B \). Assume for the sake of contradiction that \( x \in \{1, 2\} \). Then \( x \in A \cap \{1, 2\} \) as it is in both sets and \( x \notin B \cap \{1, 2\} \) as it is not in \( B \). Therefore, \( A \cap \{1, 2\} \neq B \cap \{1, 2\} \), a contradiction. So, \( x \notin \{1, 2\} \).

Case 2: This analogous to Case 1, by symmetry. (Switch A and B in all of the above statements).

In both cases \( x \notin \{1, 2\} \).

Since no element \( x \in ((A \setminus B) \cup (B \setminus A)) \) can be in \( \{1, 2\} \), these 2 sets are disjoint.

**Alternate Solution:**

We prove the contrapositive. Assume that the sets \((A \setminus B) \cup (B \setminus A)\) and \(\{1, 2\}\) are not disjoint, i.e., there exists some \( x \in ((A \setminus B) \cup (B \setminus A)) \cap \{1, 2\} \)

Therefore

\[
[(x \in A \setminus B) \lor (x \in B \setminus A)] \land (x = 1) \lor (x = 2)
\]

We have 4 cases.

Case 1: \((x \in A \setminus B) \land (x = 1)\) i.e., \(1 \in A \setminus B\). Then \(1 \in A\) and \(1 \notin B\). Since \(1 \in A\) we have \(1 \in A \cap \{1, 2\}\). Since \(1 \notin B\) we have \(1 \notin B \cap \{1, 2\}\). Therefore, \(A \cap \{1, 2\} \neq B \cap \{1, 2\}\).

The other three cases:

Case 2: \((x \in A \setminus B) \land (x = 2)\).
Case 2: \((x \in B \setminus A) \land (x = 1)\).
Case 2: \((x \in B \setminus A) \land (x = 2)\).

are analogous by symmetry. (Switch A and B and/or 1 and 2.)

In all four cases we have shown \(A \cap \{1, 2\} \neq B \cap \{1, 2\}\) and this proves the contrapositive.

6. (15pts)

Let \( n \in \mathbb{N} \) and \( n \geq 3 \). Give a combinatorial proof (no other kinds of proofs will be accepted) for the following identity

\[
\left(\begin{array}{c}n \\
3\end{array}\right) = \left(\begin{array}{c}n \\
1\end{array}\right) + 2 \left(\begin{array}{c}n \\
2\end{array}\right) + \left(\begin{array}{c}n \\
3\end{array}\right)
\]

**Answer**

We ask the following question:

After the success of the special line dance in Krishna’s Zumba class, Krishna decides to re-
ward his instructors with special gold star stickers! Given \( n \) distinguishable instructors and
3 indistinguishable gold star stickers, how many ways can Krishna distribute the stickers
to the instructors, assuming that an instructor can receive any number of stickers?

We see that the LHS counts this question by employing the stars and bars technique from lecture. The stickers can be treated as the indistinguishable stars, while the instructors can be treated as the distinguishable categories, thus yielding \( \binom{n}{3} \) total ways to distribute the stickers to his instructors.
Alternatively, we can count the number of ways Krishna can give out stickers by looking at the number of instructors that receive stickers. We thus consider the following cases:

Case 1: One instructor receives all three stickers. We see that this can be done in \( n = \binom{n}{1} \) ways, as Krishna can simply choose his favorite instructor to get all the stickers!

Case 2: One instructor receives two stickers and another receives one sticker. We see that we can count this case as follows:

1. **Step 1**: Pick two of the instructors to receive stickers.
2. **Step 2**: Pick one of the two instructors chosen above to receive two of the three stickers.
3. **Step 3**: Give the stickers to the instructors.

We see that the number of ways to perform Step 1 is simply \( \binom{n}{2} \). After Krishna has chosen the instructors to receive stickers, there are 2 choices for who receives more stickers, thus yielding 2 ways to perform Step 2. Finally, Step 3 can be done in only 1 way, as the stickers are indistinguishable. Thus, by the Multiplication Rule, we see that there are \( 2 \binom{n}{2} \) ways to give one instructor two stickers and another instructor one sticker.

Case 3: Three instructors receive one sticker each. We see that, in this case, Krishna can simply pick the three instructors to get one sticker, yielding \( \binom{n}{3} \) ways.

Combining all these cases together, we see that, by the Sum Rule, there are

\[
\binom{n}{1} + 2 \binom{n}{2} + \binom{n}{3}
\]

ways for Krishna to distribute the stickers to the instructors, which is exactly the RHS!

Thus, as we have answered our question in two different ways, we know that the two sides are equal to each other.

7. (15pts)

Consider 5 points in the 2D plane points with integer coordinates. These five points determine \( \binom{5}{2} = 10 \) line segments. Prove that the middle of at least one of these 10 segments also has integer coordinates. (If the ends of a line segment have coordinates \((x_1,y_1)\) and \((x_2,y_2)\) then the the middle of that segment has coordinates \(\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\).)

**Answer**

Observe that for two integers \(a\) and \(b\), if \(a + b\) is even, then \(\frac{a+b}{2}\) is an integer. Recall that \(a + b\) is even iff \(a\) and \(b\) have the same parity i.e., they are both odd or both even.

Let us use the pigeonhole principle to prove that there must exist a midpoint with integer coordinates. Note that the number of segments is not actually relevant. Let the pigeons be the 5 points and the pigeonholes be the four possibilities for the parities of Cartesian coordinates: (even, even), (even, odd), (odd, even), (odd, odd).

By PHP, at least two pigeons will be in the same pigeonhole. That is, at least two points (WLOG \((x_1,y_1)\) and \((x_2,y_2)\)) will have \(x_1\) and \(x_2\) with the same parity and \(y_1\) and \(y_2\) with the same parity. Thus, as noted at the beginning of the solution, \(\frac{x_1+x_2}{2}\) is an integer. Similarly, \(\frac{y_1+y_2}{2}\) is an integer. Therefore the middle of the segment determined by \((x_1,y_1)\) and \((x_2,y_2)\) will have integer coordinates.
8. (15pts)

Alice is splitting atoms in Wonderland. She has two elements, madium and hatium. Every second, every madium atom splits into two madium atoms and one hatium atom, and every hatium atom splits into two hatium atoms. Assume that Alice starts in second 1 with one madium atom and one hatium atom. Then, for example, in second 2 she will have 2 madium atoms and 3 hatium atoms. Et caetera.

Guess, as an expression in terms of $n$, the number of hatium atoms that Alice will have in second $n$, and prove your answer.

**Answer**

Let $M_n$ and $H_n$ be the number of atoms that Alice has in second $n$. We know that $M_1 = 1$ and $H_1 = 1$ and that

$$M_{n+1} = 2M_n \quad H_{n+1} = 2H_n + M_n \quad (n \geq 1)$$

First we solve the recurrence for the $M$’s. We can see that $M_n = 2^{n-1}$, because the number of madium atoms doubles each second and $M_1 = 1 = 2^{1-1} = 2^0$. We can further prove this by induction (omitted).

Now, the second recurrence becomes $H_{n+1} = 2H_n + 2^{n-1}$. For this we use the telescopic trick. First we write

$$
H_n &= 2H_{n-1} + 2^{n-2} \\
H_{n-1} &= 2H_{n-2} + 2^{n-3} \\
H_{n-2} &= 2H_{n-3} + 2^{n-4} \\
\cdots & \cdots \cdots \\
H_4 &= 2H_3 + 2^2 \\
H_3 &= 2H_2 + 2^1 \\
H_2 &= 2H_1 + 2^0
$$

If we add them like this there is not enough cancelation. So we multiply both side of the second equation with 2. Then we have to multiply both sides of the third equation with $2^2$. In general, we multiply both sides of the $i$th equation with $2^{i-1}$ obtaining, for $n \geq 2$:

$$
H_n &= 2H_{n-1} + 2^{n-2} \\
2H_{n-1} &= 2^2H_{n-2} + 2^{n-2} \\
2^2H_{n-2} &= 2^3H_{n-3} + 2^{n-2} \\
\cdots & \cdots \cdots \\
2^{n-4}H_4 &= 2^{n-3}H_3 + 2^{n-2} \\
2^{n-3}H_3 &= 2^{n-2}H_2 + 2^{n-2} \\
2^{n-2}H_2 &= 2^{n-1}H_1 + 2^{n-2}
$$

Now we add all the LHSs and all the RHSs and perform the cancelations. We obtain, again for $n \geq 2$

$$H_n = 2^{n-1}H_1 + (n-1)2^{n-2} = 2^{n-1} + (n-1)2^{n-2} = (n+1)2^{n-2}$$
For \( n = 1 \) the same formula gives \((1 + 1)2^{1-2} = 2 \cdot 2^{-1} = 1\) which is also correct. So the answer is \( H_n = (n + 1)2^{n-2} \).

It is also possible to guess this answer by computing from the recurrence \( H_2 = 2H_1 + 1 = 3 \), \( H_3 = 2H_2 + 2 = 8 = 4 \cdot 2^1 \), \( H_4 = 2H_3 + 2^2 = 20 = 5 \cdot 2^2 \), \( H_5 = 2H_4 + 2^3 = 48 = 6 \cdot 2^3 \), and then prove the guess by induction.

9. (10pts)

Let \( X \) be a nonempty finite set. Consider the set \( W = \{(A,B) \mid A, B \in 2^X \land A \subseteq B\} \).

Prove that \( W \) has exactly as many elements as there are functions with domain \( X \) and codomain \( \{1, 2, 3\} \).

**Answer**

We find the sizes of these two sets: First, we consider \( W \). We are counting ordered pairs \((A, B)\) such that \( A \subseteq B \subseteq X \). For this constraint to hold, each element \( x \in X \) must have either \( x \notin B \), \( x \in B \setminus A \) or \( x \in A \). (You may recall this from an earlier homework.) We can construct two such sets \( A, B \) in \(|X| \) steps; in each step, one element \( x \in X \) is assigned one of the 3 states \( x \notin B \), \( x \in B \setminus A \) or \( x \in A \). Thus, this whole process can be done in \( 3^{|X|} \) ways.

For the size of the other set we note that the size of the codomain is 3 therefore (as shown in lecture) there are \( 3^{|X|} \) functions with domain \( X \) and codomain \( \{1, 2, 3\} \). Thus, \( W \) and the set of functions \( f \) described in the problem have equal size.

**NOTE** This also follows from the Bijection Rule. Indeed, in each \((A, B)\) of \( W \), each \( x \in X \) must be mapped to one of 3 states "not in \( B \), "in \( B \setminus A \)" and "in \( A \). This establishes a one-to-one correspondence between the pairs \((A, B)\) in \( W \) and the functions with domain \( X \) and a codomain formed by these three states. As this codomain is of the same size as the set \( \{1, 2, 3\} \), then there are also an equal number of functions with domain \( X \) and codomain \( \{1, 2, 3\} \).

10. (10pts)

Let \( n, p, r \) be natural numbers such that \( 1 \leq p \leq n \) and \( p \leq r \). In how many ways can we distribute \( r \) indistinguishable coins to \( n \) distinguishable children so that exactly \( p \) of the children get exactly 1 coin each? You are not required to give the answer in closed form, leaving it with summations is OK.

**Answer**

We start by choosing \( p \) of the \( n \) children to have exactly 1 coin. We can do this in \( \binom{n}{p} \) ways, and we can distribute 1 coin to each of them in 1 way, as the coins are indistinguishable. So now we have \( n - p \) children and \( r - p \) coins remaining.

All of the children who remain must have received either 0 coins or at least 2 coins. We proceed by considering separately the cases where 0 children receive 0 coins, 1 child does, 2 children do, etc. We examine the case where \( i \) children receive 0 coins. There are \( \binom{n-p}{i} \) ways to choose these \( i \) children and 1 way to give them all 0 coins. The remaining children must have each received at least 2 coins, so we give each of these \( n - p - i \) children 2 coins and perform stars and bars with the remaining coins. There are

\[
r - p - 2(n - p - i) = r - 2n + p + 2i
\]
coins remaining and $n - p - i$ children, so we apply stars and bars:

$$\binom{n - p - i}{r - 2n + p + 2i} = \binom{r - n + i - 1}{n - p - i - 1}$$

This gives us $\binom{r - n + i - 1}{n - p - i - 1}$ ways to apportion these coins. Multiplying these two steps, we get

$$\binom{n - p}{i} \binom{r - n + i - 1}{n - p - i - 1}$$

We sum over all possible values of $i$ from 0 to $n - p$ (note that if some case is impossible because the provided variables are too small or large, one of the two terms will simply be $\binom{a}{b}$ with $a < b$, which evaluates to 0). We also multiply by the combination from the first step to have:

$$\binom{n - p}{i} \sum_{i=0}^{n - p} \binom{n - p}{i} \binom{r - n + i - 1}{n - p - i - 1}$$

**Alternate Solution:**

We first consider choosing a group of $j$ children to receive either 0 or 1 coin. Then we know that the remaining children must each receive at least 2 coins, so we give 2 to each of them. We can then distribute the remaining coins among the children who receive at least 2 coins.

Pick some $p \leq j \leq n$ children to receive at most 1 coin. This can be done in $\binom{n}{j}$ ways. We then choose $p$ of these children to receive 1 coin (the rest get 0 coins). This can be done in $\binom{j}{p}$ ways. Next, give each of the remaining $n - j$ children 2 coins each. Since the coins are indistinguishable, this can be done in 1 way. Finally, distribute the remaining coins among the children.

We know that we have already given out $p + 2(n - j)$ coins, so there are only $r - (p + 2(n - j))$ coins left. We have $n - j$ children to whom we may give these coins. So the number of ways to distribute the remaining coins is

$$\binom{n - j}{r - p - 2n + 2j} = \binom{r - n - p + j - 1}{r - p - 2n + 2j}$$

Putting this together, by applying the multiplication rule, we find that for a fixed value of $j$, there are a total of

$$\binom{n}{j} \binom{j}{p} \binom{r - n - p + j - 1}{r - p - 2n + 2j}$$

ways to distribute the coins.

To account for all possible values of $j$, we note that two distinct values of $j$ cannot give us the same arrangement (since we would have different children receiving 0 coins). Hence, the cases for the different $j$’s are disjoint, so we may apply the Sum Rule. This gives the following number of arrangements:

$$\sum_{j=p}^{n} \binom{n}{j} \binom{j}{p} \binom{r - n - p + j - 1}{r - 2n - p + 2j}$$

**Solution Equivalence:**
We now show that these two solutions are equivalent. In order to do this, we first “shift down” the summation from the alternate solution. We do this by employing a trick where we define a new summation index variable. In order to see how this works, consider the sum:

$$3 \sum_{j=1}^{j=1} (j + 1)$$

We see that this summation is really equal to:

$$3 \sum_{j=1}^{j=1} (j + 1) = 2 + 3 + 4 = 9$$

Alternatively, we could express the same summation in a different manner by letting $i = j + 1$. We thus see that the lower bound on the summation becomes $1 + 1 = 2$, and the upper bound becomes $3 + 1 = 4$. Rewriting our summation, we thus see:

$$3 \sum_{j=1}^{j=1} (j + 1) = 4 \sum_{i=2}^{i=2} (i - 1 + 1) = 2 + 3 + 4 = 9$$

as expected. Applying this trick to our alternate solution, we let $i = j - p$, or equivalently, $j = i + p$. Thus, our new summation is given by:

$$\sum_{i=0}^{n-p} \binom{n}{i+p} \binom{i+p}{p} \left( \frac{r - n - p + i + p - 1}{r - 2n - p + 2(i + p)} \right)$$

We now show that this is equivalent to our original solution. We do this by showing that

$$\binom{n}{i+p} \binom{i+p}{p} = \binom{n}{i+p} \binom{n-p}{i}$$

and that

$$\frac{r - n + i - 1}{r - 2n + p + 2i} = \frac{r - n + i - 1}{n - p - i - 1}$$

For the first equality, we observe that, by definition of the binomial coefficient, we have that:

$$\binom{n}{i+p} \binom{i+p}{p} \frac{n!}{(i+p)!((n-p-i)!} \times \frac{(i+p)!}{p!}$$

$$= \frac{n!}{p!} \times \frac{1}{i!(n-p-i)!} \times \frac{(n-p)!}{(n-p)!}$$

$$= \frac{n!}{p!(n-p)!} \times \frac{1}{i!(n-p-i)!} \times \frac{(n-p)!}{(n-p)!}$$

$$= \frac{n!}{p!(n-p)!} \times \frac{1}{i!(n-p-i)!} \times \frac{(n-p)!}{(n-p)!}$$

Alternatively, we give a combinatorial proof to show that the first equality holds. We pose the following question:
AJ decides to start his own K-pop company, PHP Entertainment. Some \( n \) of the CIS160 TAs decide to quit their jobs to pursue their dreams as K-pop idols. However, AJ can only accept \( i + p \) of them to form the main hit group. Further, AJ, who is well versed in the tactics of K-pop labels, seeks to form a subgroup of the main group, containing only \( p \) idols. How many ways can AJ do this?

We see that the LHS answers this question in the following way:

**Step 1:** Pick the \( i + p \) TAs to form the main group. \( \binom{n}{i + p} \) ways

**Step 2:** Pick \( p \) idols from the main group to make the subgroup. \( \binom{i + p}{p} \) ways

Thus, by the Multiplication Rule, AJ can form his first K-pop group and subgroup in:

\[
\binom{n}{i + p} \binom{i + p}{p}
\]

ways, which is exactly the LHS!

We now show that the RHS answers this question as well, in the following way:

**Step 1:** Pick \( p \) of the TAs to form the subgroup. \( \binom{n}{p} \) ways

**Step 2:** Pick \( i \) of the remaining TAs to fill out the main group. \( \binom{n - p}{i} \) ways

Thus, by the Multiplication Rule, AJ can form his first K-pop group and subgroup in:

\[
\binom{n}{p} \binom{n - p}{i}
\]

ways, which is exactly the RHS, thus showing the two are equivalent!

For the second equality, we note that \( \binom{n}{k} = \binom{n}{n-k} \) (for proof, see 1.(a)). We now realize that:

\[
\binom{r - n + i - 1}{r - 2n + p + 2i} = \binom{r - n + i - 1}{r - n + i - 1 - (r - 2n + p + 2i)} = \binom{r - n + i - 1}{n - p - i - 1}
\]

Hence, we have now shown that:

\[
\sum_{j=p}^{n} \binom{n}{j} \binom{j}{p} \binom{r - n - p + j - 1}{r - 2n - p + 2j} = \sum_{i=0}^{n-p} \binom{n}{i + p} \binom{i + p}{p} \binom{r - n + i - 1}{r - 2n + p + 2i}
\]

\[
= \sum_{i=0}^{n-p} \binom{n}{i} \binom{n - p}{i} \binom{r - n + i - 1}{r - 2n + p + 2i}
\]

\[
= \sum_{i=0}^{n-p} \binom{n}{p} \binom{n - p}{i} \binom{r - n + i - 1}{n - p - i - 1}
\]

\[
= \binom{n}{p} \sum_{i=0}^{n-p} \binom{n - p}{i} \binom{r - n + i - 1}{n - p - i - 1}
\]

(by the first equality)

\[
= \binom{n}{p} \sum_{i=0}^{n-p} \binom{n - p}{i} \binom{r - n + i - 1}{n - p - i - 1}
\]

(by the second equality)

\[
= \binom{n}{p} \sum_{i=0}^{n-p} \binom{n - p}{i} \binom{r - n + i - 1}{n - p - i - 1}
\]

(since \( \binom{n}{p} \) is constant)
So we have shown algebraically that our two solutions are indeed the same! In fact, we already knew this, since we have a combinatorial proof for this fact: namely the problem and its two solutions.

3 Additional Problems

1. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a very brief explanation of your answer.

   (a) Assume that $B$ is a set with 7 elements and that $A$ is a set with 15 elements. Then, for any function $f : A \rightarrow B$ there exist at least 3 distinct elements of $A$ that are mapped by $f$ to the same element of $B$, true or false?

   (b) There are exactly three surjective functions with domain $\{1, 2\}$ and codomain $\{a, b\}$.

   (c) Exactly two of the following three boolean expressions: $p \Rightarrow q$, $p \land \neg q$, and $\neg p \lor q$ are logically equivalent.

   (d) Let $A$ be a finite set. For any function $f : A \rightarrow A$ we have $|\text{Ran}(f)| = |A|$.

   (e) There exist two distinct functions with domain and codomain $\{a, b\}$ that are their own inverses.

   (f) For any two finite sets $A, B$, $|2^{A \times B}| > |2^A \times 2^B|$.

   (g) Recall that for any $n = 0, 1, 2, 3, \ldots$ row $n$ of the Pascal Triangle contains the binomial coefficients of the form $\binom{n}{k}$ for $k = 0, 1, \ldots, n$. $\binom{7}{4}$ can be expressed as a sum of binomial coefficients from row 5.

**Answer**

(a) FALSE.

There are exactly two. One that maps 1 to $a$ and 2 to $b$ and one that maps 1 to $b$ and 2 to $a$.

(b) TRUE.

The first and the third are logically equivalent. The second is logically equivalent to the negation of the first. We see this with a quick table:

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$p \Rightarrow q$</th>
<th>$p \land \neg q$</th>
<th>$\neg p \lor q$</th>
</tr>
</thead>
<tbody>
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</tr>
</tbody>
</table>

(c) FALSE.

Counterexample: Let $A = \{0, 1\}$, and $f(0) = f(1) = 0$.

Then $|A| = 2$ but $|\text{Ran}(f)| = 1$ since $\text{Ran}(f) = \{0\}$.

(d) FALSE.

Take $A = \{a\}$ and $B = \{b\}$. Then $A \times B = \{(a, b)\}$.

So $|A| = |B| = |A \times B| = 1$.

$|2^{A \times B}| = 2^{|A \times B|} = 2$. 

13
\[
|2^A \times 2^B| = |2^A| \times |2^B| = 2^{|A|} \times 2^{|B|} = 2 \times 2 = 4.
\]

2 \neq 4.

(e) TRUE.

By Pascal’s Identity
\[
\binom{7}{4} = \binom{6}{3} + \binom{6}{4} = \left( \binom{5}{2} + \binom{5}{3} \right) + \left( \binom{5}{3} + \binom{5}{4} \right)
\]

2. Recall (and remember!) that the sum of the squares of the first \(n\) positive integers is given by the following formula:
\[
1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = n(n+1)(2n+1)/6.
\]

(a) Using only the formula above, (no credit in part (a) for proof by induction, see part (b)), derive the following formula for the sum of the squares of first \(m\) odd positive integers. Show your work.
\[
1^2 + 3^2 + 5^2 + \cdots + (2m-3)^2 + (2m-1)^2 = \frac{m(4m^2-1)}{3}
\]

(b) Now prove by induction the formula from part (a).

**Answer**

(a) Let \(D(m) = \sum_{k=1}^{m}(2k-1)^2\). To this we add \(E(m) = \sum_{k=1}^{m}(2k)^2\) and we obtain \(T(m) = \sum_{i=1}^{2m}i^2\). By the formula, \(T(m) = 2m(2m+1)(4m+1)/6\). Meanwhile, \(E(m) = \sum_{k=1}^{m}(2k)^2 = 2^2 \sum_{k=1}^{m}k^2 = 2^2 \cdot m(m+1)(2m+1)/6\) (using again the formula). Hence
\[
D(m) = T(m) - E(m) = \frac{2m(2m+1)(4m+1)}{6} - \frac{4m(m+1)(2m+1)}{6}
\]
\[
= \frac{2m(2m+1)(4m+1 - 2(m+1))}{6} = \frac{2m(2m+1)(2m-1)}{6} = \frac{m(4m^2-1)}{3}
\]

(b) (Base Case) \(m = 1\). We have:
\[
\frac{1(4 \cdot 1^2 - 1)}{3} = \frac{1 \cdot 3}{3} = 1 = 1^2 \checkmark
\]

(Induction Step) Let \(k\) be an arbitrary integer. Assume that (IH) \(1^2 + 3^2 + \cdots + (2k-3)^2 + (2k-1)^2 = \frac{k(4k^2 - 1)}{3}\). In other words, the sum of the squares of the first \(k\) odd integers is equal to \(\frac{k(4k^2 - 1)}{3}\).

We wish to show that the claim holds for \(n = k + 1\), i.e. the sum of the squares of the first \(k + 1\)
odd integers is equal to \( \frac{(k+1)(4(k+1)^2-1)}{3} \). We have:

\[
\sum_{i=1}^{k+1} (2i - 1)^2 = \sum_{i=0}^{k} (2i - 1)^2 + (2(k+1) - 1)^2 \\
= \frac{k(4k^2 - 1)}{3} + (2k + 1)^2 \\
= \frac{k(4k^2 - 1)}{3} + (2k + 1)^2 \\
= \frac{k(4k^2 - 1)}{3} + 3(2k + 1)^2 \\
= 4k^3 - k + 3(4k^2 + 4k + 1) \\
= 4k^3 - k + 12k^2 + 12k + 3 \\
= 4k^3 + 12k^2 + 11k + 3 \\
= 4k^3 + 8k^2 + 3k + 4k^2 + 8k + 3 \\
= k(4k^2 + 8k + 3) + (4k^2 + 8k + 3) \\
= \frac{(k+1)(4(k+1)^2-1)}{3}
\]

as desired. Thus, we have shown our claim is true when \( n = k + 1 \), concluding our Induction Step and completing our proof.

3. Let \( A, B, C \) be three (finite) sets. Prove \( A \setminus C \subseteq A \setminus B \) given \( A \cap B \subseteq C \).

(You are NOT allowed to use set algebra facts in the proof. Your proof can only utilize the definitions of subset, set difference, and intersection, as well as logical manipulation of statements.)

**Answer**

Let \( x \in A \setminus C \). Then \( x \in A \) and \( x \notin C \).

WTS (want to show) \( x \in A \setminus B \).

Suppose, toward a contradiction that \( x \notin A \setminus B \). Then \( \neg(x \in A \cap B) \) and by De Morgan and double negation \( x \notin A \cup x \in B \).

**Case 1:** \( x \notin A \). Since \( x \in A \), we have contradiction.

**Case 2:** \( x \in B \). Since \( x \in A \) it follows that \( x \in A \cap B \).

Since \( A \cap B \subseteq C \), \( x \in C \), which contradicts \( x \notin C \).

Contradiction in both cases.

4. Give a boolean expression \( e \) with three variables \( p, q, r \) such that \( e \) has the following properties:

- \( e = T \) when \( p = q = T \) and \( r = F \), AND
• $e = F$ when $p = F$ and $q = r = T$.

Also, construct a truth table for $e$. Make sure if you include all intermediate propositions as a separate column. (Yes, there are many possible answers.)

**Answer**

Here’s how to solve such a problem systematically. Note that $p \land q \land \neg r$ is $T$ exactly when $p = q = T$ and $r = F$. Similarly $\neg p \land q \land r$ is $T$ exactly when $p = F$ and $q = r = T$, therefore $\neg(\neg p \land q \land r)$ is $F$ exactly when $p = F$ and $q = r = T$. If there are more such constraints, you find more such conjunctions or negated conjunctions.

Then can take $e = (p \land q \land \neg r) \lor \neg(\neg p \land q \land r)$.

(Actually, because we only have two constraints, it suffices to take $e = p \land q \land \neg r$, for example.)

We skip the truth table in this solution set.

5. A lottery urn contains $n \geq 2$ distinct balls labeled with the numbers $1, \ldots, n$. You extract two distinct balls from the urn. Suppose they are labeled $i$ and $j$. You compute $i + j$ and write down the answer on a piece of paper. Then you put the two balls back.

You repeat this $m$ times. What is the smallest value of $m$ that ensures (no probabilities in this problem!) that you will end up writing the same number at least twice on the piece of paper. Prove your answer.

**Answer**

The smallest number you can write down is $1 + 2 = 3$. The largest number you can write down is $(n - 1) + n = 2n - 1$.

In fact, every $3 \leq k \leq 2n - 1$ could be one of the numbers written down. Indeed, when $3 \leq k \leq n + 1$ then $k = 1 + (k - 1)$ because $2 \leq k - 1 \leq n$. And when $n + 2 \leq k \leq 2n - 1$ then $k = n + (k - n)$ because $2 \leq k - n \leq n - 1$.

The piece of paper can contain any of the numbers in $[3, (2n - 1)]$ hence $2n - 1 - 3 + 1 = 2n - 3$ distinct numbers (pigeonholes). the pigeons are the $m$ extractions. By PHP, if $m = (2n - 3) + 1 = 2n - 2$ then at least one number will be written down at least twice.

However, PHP does not tell us specifically whether $2n - 2$ is the smallest such $m$. But if $m$ is smaller, namely $m = 2n - 3$, then we get get all distinct numbers with the following $2n - 3$ extractions: $\{1, i\}$ for $i = 2, \ldots, n$ ($n - 1$ extractions) followed by $\{j, n\}$ for $j = 2, \ldots, n - 1$ ($n - 2$ extractions). If $m$ is even smaller we just consider a subset of these extractions.

6. Let

$$R_n = \sum_{k=1}^{2n} (-1)^{k+1} k \quad \text{for } n \geq 1$$

(a) Compute $R_1, R_2, R_3$. Guess a simple way to express $R_n$ in terms of $n$. Prove your guess by induction.

(b) Prove by induction that for all $n \geq 1$ we have

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$
(c) Use the identity in part (b) and other identities that you were supposed to memorize to prove the identity in part (a).

**Answer**

(a) \( R_1 = 1 - 2 = -1. \) \( R_2 = 1 - 2 + 3 - 4 = -2. \) \( R_3 = 1 - 2 + 3 - 4 + 5 - 6 = -3. \)

We guess \( \forall n \geq 1 \ R_n = -n. \) And we prove it by induction. We already have the base case \( R_1 = -1. \)

**Induction Step:** Let \( k \geq 1 \) arbitrary. Assume \( R_k = -k \) (IH).

Then, using IH, \( R_{k+1} = R_k + (2k + 1) - (2k + 2) = -k + 2k + 1 - 2k - 2 = -k - 1 = -(k + 1). \)

Done.

(b) **Base Case:** \( n = 1. \) \( 1 = 1^2 \) Check.

**Induction Step:** Let \( k \geq 1 \) arbitrary, fixed. Assume \( 1 + 3 + \cdots + (2k - 1) = k^2 \) (IH).

Then, using IH \( 1 + 3 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2. \) Done.

(c) We know that

\[
1 + 2 + 3 + 4 + \cdots + (2n) = \frac{2n(2n + 1)}{2}
\]

Let’s write the definition of \( R_n \) in part (a) like this

\[
1 - 2 + 3 - 4 + \cdots - (2n) = R_n
\]

Adding LHS and the RHS of these two equalities and canceling \( 2 - 2 = 0 \) etc., we have

\[
1 + 1 + 3 + 3 + \cdots + (2n - 1) + (2n - 1) = \frac{2n(2n + 1)}{2} + R_n
\]

By part (b)

\[
2 \cdot n^2 = \frac{2n(2n + 1)}{2} + R_n
\]

Now it’s just algebra

\[
R_n = 2n^2 - \frac{2n(2n + 1)}{2} = 2n^2 - 2n^2 - n = -n
\]

7. In how many different ways can we arrange all the letters from the English alphabet (26 characters) in a sequence such that:

- each letter occurs exactly once, AND
- the 5 vowels (a,e,i,o,u) occur in 5 consecutive positions.

**Answer**

We can solve this using the multiplication rule.

*Step 1:* Arrange the consonants. (21! ways)

*Step 2:* Choose a position for the vowels. (22 ways)

*Step 3:* Arrange the vowels. (5! ways)
Thus, our final answer is \[21! \times 22 \times 5! = 22! \times 5!\]

**Alternate Solution:**

Since the vowels appear in consecutive positions think of them as a block. There are 21 consonants plus the vowel block, a total of 22 elements, and in Step 1 they are arranged in a sequence. This can be done in 22! ways. However, we must still arrange the vowels within the block, and again we can arrange them in 5! ways.

Answer: \(22! \times 5!\) (the same)

8. Give a combinatorial proofs (no other kind of proofs will be accepted) for the following identities

(a) \[
\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}
\]

(where \(k \leq r \leq n\))

(b) \[
\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}
\]

(where \(k \leq n\))

**Answer**

(a) Our question is as follows:

Given \(n\) pieces of (distinguishable) sushi, how many ways can we choose \(r\) pieces to eat if we also want to put wasabi on \(k\) of them?

**LHS:** We first choose the \(r\) pieces of sushi to eat \(\binom{n}{r}\), and then we choose \(k\) of them out of \(r\) to put wasabi on \(\binom{r}{k}\).

**RHS:** We first choose the \(k\) pieces of sushi to both eat and put wasabi on. This can be done in \(\binom{n}{k}\) ways. Then we choose the \(r-k\) ones without wasabi to eat from the remaining \(n-k\), which can be done in \(\binom{n-k}{r-k}\) ways.

(b) Our question is as follows:

AJ is writing a new song. He wants to include exactly \(k+1\) (distinguishable) chords at some point in his song out of a possible \(n+1\) chords (labeling them \(\{1, 2, \ldots, n+1\}\)). How many collections of \(k+1\) can he make?

**LHS:** We split it up into cases. The first is that we include chord 1. There are \(\binom{n}{k}\) ways to make such a collection. The next is we don’t include chord 1 but include chord 2. There are \(\binom{n-1}{k}\) such collections. The next is we don’t include chords 1 and 2 but include chord 3. This can be done \(\binom{n-2}{k}\) ways. We can do this up until chord \(n-k\) because then we must include the next \(k\) chords. We can express this as the sum \(\sum_{i=k}^{n} \binom{i}{k}\)

**RHS:** We are asked to pick \(k+1\) chords from a total of \(n+1\) distinguishable chords. There are \(\binom{n+1}{k+1}\) ways to do this.

9. Prove that for any integers \(a, b \in \mathbb{Z}\) we have \(a^2 - 4b \neq 2\).

**Answer**
Assume, toward a contradiction that \(a^2 - 4b = 2\). Then \(a^2 = 4b + 2 = 2(2b + 1)\) and therefore \(a^2\) is even. We have shown in lecture that in that case \(a\) is even.

Let \(a = 2k\) for some integer \(k\).

\[
\begin{align*}
(2k)^2 &= 4b + 2 \\
4k^2 &= 4b + 2 \\
2k^2 &= 2b + 1 \\
2k^2 - 2b &= 1 \\
2(k^2 - b) &= 1
\end{align*}
\]

Therefore, because \(k^2 - b\) must be an integer, we can say that 1 is divisible by 2 and thus is even. This, however, we know is not true and thus we have reached a contradiction.

10. Consider \(n\) (distinguishable) bins labeled \(B_1, \ldots, B_n\) and \(r\) indistinguishable (identical) marbles. We wish to put the \(r\) marbles into the \(n\) bins in such a way that each bin will contain at least one marble and at least three of the bins will contain two or more marbles. Assume \(r \geq n + 3\). In how many different ways can this be done?

**Answer**

First we compute the number of ways to put \(r\) marbles into the \(n\) bins such that each bin will contain at least one marble. It’s

\[
\binom{n}{r - n}
\]

But not all of these ways ensure that at least three of the bins will contain two or more marbles. To correct for this over counting, we will count these bad cases and subtract from the number above.

If it is not the case that at least three of the bins will contain two or more marbles then only exactly zero or exactly one or exactly two bins contain two or more marbles. Note that was these are disjoint cases, we do not have to worry about counting a bad case twice.

*Case 1: Exactly zero bins with two or more marbles.*

This is impossible. We have \(r \geq n + 3\) marbles, and if we distribute one to each, we will be left with \(r - n \geq 3\) marbles. As no more bins have more marbles, we have not distributed all the marbles.

*Case 2: Exactly one bins with two or more marbles.*

This means that all \(r - n\) additional marbles go into one bin. There are \(n\) ways to choose that one bin, so this can be done in \(n\) ways.

*Case 3: Exactly two bins with two or more marbles.*

This means means that all \(r - n\) additional marbles go into two bins. We can do this in three steps. In Step 1 we choose two of the bins, in \(\binom{n}{2}\) ways. In Step 2 we put one more marble in each of the two bins (recall that each bin already had a marble). This guarantees that both bins will have two or more marbles. This is done in one way and we are left with \(r - n - 2\) marbles. In Step 3 we put the remaining \(r - n - 2\) marbles in the two (distinguishable) bins we chose, any way we want. So this can be done in \((r - n - 2) + 1 = r - n - 1\) ways (think \(r - n - 2\) stars and 1 bar). By multiplication
The count in the “exactly two” case is:
\[
\binom{n}{2} (r - n - 1)
\]

Putting this all together, the total number of ways is:
\[
\left(\binom{n}{r - n}\right) - n - \binom{n}{2} (r - n - 1)
\]

11. Bob is recycling a set $B$ of $m \geq 1$ distinguishable bottles $B = \{b_1, \ldots, b_m\}$ in a facility that has a set $D$ of $n \geq 2$ distinguishable drums, $D = \{d_1, \ldots, d_n\}$. When Bob shows up all the drums are empty. Each drum is large enough to hold by itself all of Bob’s $m$ bottles. We call a deposit a way of placing the bottles in the drums, i.e., a function $t : B \rightarrow D$. Each deposit may leave some drums (maybe none) empty. Let $\text{empty}(t)$ be the set consisting of all the drums that are left empty by deposit $t$. (Note that it might be the case that $\text{empty}(t) = \emptyset$, depending on $m, n$ and $t$.)

Assume $m \geq n$ and prove that there exist two different deposits, $t_1$ and $t_2$ such that $\text{empty}(t_1) = \text{empty}(t_2)$.

**Answer**

There are $n^m$ deposits. These are the pigeons. The function that maps pigeons to pigeonholes takes a deposit $t$ to the subset $\text{empty}(t) \subseteq D$. But who are the pigeonholes? Not all the subsets of $D$, just those that equal $\text{empty}(t)$ for some deposit $t$.

How many pigeonholes are there? The only subset that cannot equal $\text{empty}(t)$ for some deposit $t$ is $S$ itself (because we have $m \geq 1$ bottles and where do we put them?).

Let’s prove that for every other subset of $S \subseteq D$ such that $S \neq D$ there exists a deposit $t$ such that $S = \text{empty}(t)$. Indeed, $D \setminus S$ has at least one drum and at most $n$ drums. But $m \geq n$ so we can distribute the $m$ bottles among the drums of $D \setminus S$, at least one bottle in each.

Therefore the number of pigeonholes is $2^n - 1$. To apply PHP we must check that $n^m > 2^n - 1$.

Now, $n^m \geq 2^m \geq 2^n \geq 2^n - 1$, using $m \geq n$ and $n \geq 2$. Done.

(Here are some more calculations in the setting of this problem, just for general practice. Assume $n \geq m$.

For any deposit $t : B \rightarrow D$, the minimum number of drums that can be used is 1 (all bottles in one drum), and the maximum number of drums that can be used is $m$ (one bottle in each of $m$ drums, this is possible because there are enough drums, $n \geq m$). That means that the minimum number of drums that can be left empty is $n - m$ and the maximum number of drums that can be left empty is $n - 1$.

For any $n - m \leq k \leq n - 1$ it is clear that there exists at least one deposit $t$ such that $|\text{empty}(t)| = k$ (just distribute the $m$ bottles among $n - k$ drums).

Therefore for a subset $S = \text{empty}(t)$ for some deposit $t$ iff $n - m \leq |S| \leq n - 1$. The number of these subsets is
\[
\binom{n}{n - m} + \cdots + \binom{n}{n - 1}
\]
12. For each statement below, decide whether it is TRUE or FALSE. In each case attach a very brief explanation of your answer.

(a) \( \binom{100}{51} \) is strictly bigger than \( \binom{100}{49} \).

(b) The word QWERTY has 6! anagrams. (Recall that a word is a valid anagram of itself.)

(c) The contrapositive of \( p \rightarrow q \) is logically equivalent to \( p \land \neg q \), TRUE or FALSE?

(d) For any \( 2 \leq k < n \), if \( A \) has \( n \) elements then the number of subsets of \( A \) of \( k \) elements is \( \binom{n}{k} \), TRUE or FALSE?

(e) When we roll two indistinguishable dice at the same time, we have 15 different outcomes, TRUE or FALSE?

(f) If the set \( A \) has \( n \) elements then there are \( n! \) injective functions with domain \( A \) and codomain \( A \), TRUE or FALSE?

(g) There is no set \( X \) such that \( 2^X = \emptyset \), TRUE or FALSE?

**Answer**

(a) FALSE. They are equal. Choosing 51 people to be on a committee from 100 people can be done in the same number of ways as choosing 49 people to not be on the committee. Alternatively, we see that

\[
\frac{100!}{51!(100-51)!} = \frac{100!}{49!(100-49)!}.
\]

(b) TRUE. There are no multiple occurrences of the same letter so the bag is the same as the set and they have the same number of permutations. Alternatively, you could state

\[
\frac{6!}{1!1!1!1!1!1!} = 6!
\]

(c) FALSE. The contrapositive is \( \neg q \Rightarrow \neg p \) and it is not logically equivalent to \( p \land \neg q \) because, for example, for the truth assignment \( p = T, q = F \) we have \( \neg q \Rightarrow \neg p = F \) but \( p \land \neg q = T \). Alternative explanation: We know from class that \( p \rightarrow q \) is logically equivalent to its contrapositive. However we also know that \( p \land \neg q \) is logically equivalent to the negation of \( p \rightarrow q \). A proposition cannot be logically to its negation.

(d) FALSE. \( \binom{n}{k} \) is the number of permutations (not subsets) of \( k \) elements out of \( n \). Alternative explanation: The number of subsets of size \( k \) of a set of \( n \) elements is different, namely \( \binom{n}{k} \).

(e) FALSE. Because the number of outcomes is \( 21 \neq 15 \).

(You are NOT required to justify the number 21, but here is how this goes: When the dice can be distinguished (blue die and red die, say) there are \( 6 \cdot 6 = 36 \) outcomes. But when they cannot be distinguished (both blue, say) we are doing some double counting. For 6 of these outcomes in which the dice show the same number it does not matter that the dice are indistinguishable. However, the other 30 outcomes, when the dice show different numbers, are counted twice each. So the number of outcomes is \( 6 + (30/2) = 6 + 15 = 21 \).)

(f) TRUE. The injective functions from \( A \) to \( A \) correspond one-to-one to the permutations of the elements of \( A \) and there are \( n! \) such permutations.

(Not required but FYI: The one-to-one correspondence can be described as follows. Let \( A = \{a_1, \ldots, a_n\} \) and let \( f : A \rightarrow A \). When \( f \) is injective, the sequence \( f(a_1), \ldots, f(a_n) \) consists of}
n distinct elements of $A$ and is therefore a permutation of $A$. Conversely, given a permutation $b_1, \ldots, b_n$ of the elements of $A$ then the function defined by $f(a_i) = b_i, \ i = 1, \ldots, n$ must be injective because the $b_i$’s are distinct.)

(g) TRUE. For any $X$, the set $2^X$ contains at least one element, namely $\emptyset$ (the empty subset of $X$).

13. Let $n \geq 2$ and let $a_1a_2 \ldots a_n$ be a sequence of $n$ integers (they do not have to be pairwise distinct). Prove that there exist $p, q \in [0..n]$ such that $\sum_{i=p+1}^{q} a_i$ is divisible by $n$.

**Answer**

For $k = 1, 2, \ldots, n$, define

$$b_k = \sum_{i=1}^{k} a_i$$

We also define $b_0 = 0$.

We apply PHP in the following manner: consider $n$ holes numbered $0, 1, 2, \ldots, n - 1$, and the $n + 1$ pigeons $b_0, b_1, b_2, \ldots, b_n$. Assign pigeon $b_i$ to the hole whose label is the remainder when $b_i$ is divided by $n$.

Since there are $n + 1$ pigeons and $n$ pigeonholes, by PHP, there must be two pigeons $b_p, b_q$ that are in the same hole, meaning that $b_p$ and $b_q$ have the same remainder when divided by $n$. WLOG, suppose $p < q$, so $b_p < b_q$. Then, we can write $b_p = c_1n + k$ and $b_q = c_2n + k$, where $c_1, c_2, k$ are integers, and $0 \leq k \leq n - 1$. Furthermore,

$$b_q - b_p = c_2n + k - (c_1n + k) = (c_2 - c_1)n = c'n$$

where $c' = c_2 - c_1$. Therefore, it must be the case that $b_q - b_p$ is divisible by $n$.

But

$$b_q - b_p = \sum_{i=1}^{q} a_i - \sum_{i=1}^{p} a_i = \sum_{i=p+1}^{q} a_i$$

and note that this holds even when $p = 0$. Thus, we have proven the claim.

14. Assume that $A \subseteq B \subseteq C$. Prove that $(C \setminus B) \cup (B \setminus A) \subseteq C \setminus A$. (You are NOT allowed to use in the proof set algebra facts such as $A \subseteq B \iff A \cap B = A$ or $A \setminus A = \emptyset$. Your proof should use only the definitions of subset, set difference, intersection, and empty set and logical manipulation of statements.)

**Answer**

Let $x \in (C \setminus B) \cup (B \setminus A)$.

W.T.S. (want to show) $x \in C \setminus A$.

**Case 1:** $x \in C \setminus B$

$x \in C$ and $x \not\in B$

Now, W.T.S. $x \not\in A$

Suppose, toward a contradiction, that $x \in A$. 

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Because \( A \subseteq B \) we have \( x \in B \) which contradicts \( x \notin B \) above.

Therefore, \( x \notin A \)

And so \( x \in C \setminus A \)

**Case 2:** \( x \in B \setminus A \)

\( x \in B \) and \( x \notin A \)

Because \( B \subseteq C \) we have \( x \in C \)

Therefore \( x \in C \setminus A \)

In both cases \( x \in C \setminus A \). Done.

15. The Taney Dragons are going to the Little League World Series! In appreciation, each of the 12 distinct team members (players) can pick 2 hats from a supply of red (Philly Phillies), blue (Boston Red Sox), and green (Ploiesti Frackers) hats. For each color, the supply is unlimited. For each of the three questions below (see also next page), give the answer and an explanation of how you derived it. No proofs required.

In how many different ways can the hat picking be done if:

(a) There is no ordering among the two hats that each player picks, and both hats can even be of the same color.

(b) The ordering matters and the two hats have a different color: let’s say each player picks a hat to wear in the morning and then a hat (of a different color) to wear in the afternoon.

(c) What is the count for part (15a) above, if you also know that at least one of the hats that Dragon’s pitcher Mo’ne Davis picks is red.

**Answer**

(a) Each player can pick an unordered set of two hats of different color, and there are 3 such options, or two hats of the same color, and there are 3 such options. By the sum (addition) rule each player has 6 options. Because the supplies are unlimited, a hat picking is just like sequence of length 12 of options. There are \( 6^{12} \) such sequences (equivalently \( 6^{12} \) functions from the set of 12 players to the set of 6 options). Hence \( 6^{12} \) ways.

(b) Each player has 3 options for her/his morning hat and once he/she picks that, is left with 2 options for her/his afternoon hat. Therefore, by the multiplication rule, each player has \( 3 \cdot 2 = 6 \) options. Again we count the number of sequences of options (or functions from the set of players to the set of options) and we get \( 6^{12} \) ways.

(c) We are back to the 6 options described in the solution to part (15a) above. But now these 6 options are only available to Mo’ne’s 11 teammates so they can pick hats in \( 6^{11} \) ways. Mo’ne herself must pick a red Philly hat and for her other hat she can pick any of the colors so she has 3 options. Thus she only has 3 options.

By the multiplication rule the total number of ways for this part is \( 3 \cdot 6^{11} \).
16. Recall from HW3 that the boolean expression $e_2$ is a **logical consequence** of the boolean expression $e_1$ if every truth assignment to the variables that makes $e_1$ true also makes $e_2$ true.

Let $x, y$ be arbitrary boolean variables. Prove, using truth tables, that $x \rightarrow y$ is a logical consequence of $\neg x \land y$.

**Answer**

We compute the truth table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\neg x$</th>
<th>$\neg x \land y$</th>
<th>$x \rightarrow y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
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</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

To prove if this is a logical consequence, we need check to that every assignment of $x, y$ that makes $\neg x \land y$ true also makes $x \rightarrow y$ true. Indeed, there is only one assignment $x = F, y = T$ that makes the first expression true, and the same assignment makes the second expression true.

17. For natural numbers $n \geq 1$ we will use the notation $[1..n] = \{1, \ldots, n\}$. In the following, just give the examples, you do not have to prove that they work.

(a) For arbitrary $n \geq 1$, give an example of a set $Y$ and a function $f : [1..n] \rightarrow Y$ such that $f$ is injective but **not** surjective.

(b) For arbitrary $n \geq 2$, give an example of a set $X$ and a function $g : X \rightarrow [1..n]$ that is **not** injective and moreover $|\text{Ran}(g)| = n - 1$.

**Answer**

(a) Define $Y = [1..n + 1]$ and for each $1 \leq k \leq n$ define $f(k) = k$.

(*Not required but FYI*: The function is clearly injective since every value 1 to $n$ maps to itself, and it is not surjective because there is no $k \in [1..n]$ such that $f(k) = n + 1$.)

(b) **Answer** Define $X = [1..n]$ and for each $1 \leq k \leq n$ define

$$g(k) = \begin{cases} 
  k & \text{if } 1 \leq k \leq n - 1 \\
  n - 1 & \text{if } k = n 
\end{cases}$$

(*Not required but FYI*: The function is not injective because two different values map to $n - 1$, specifically $g(n - 1) = n - 1 = g(n)$. Moreover $\text{Ran}(g) = \{1, \ldots, n - 1\}$ therefore $|\text{Ran}(g)| = n - 1$.)

18. Prove by induction on $n$ that for any $n \in \mathbb{N}$, $n \geq 1$ we have

$$\sum_{i=1}^{n} (-1)^i \cdot i = \begin{cases} 
  \frac{n}{2} & \text{if } n \text{ is even} \\
  \frac{-n+1}{2} & \text{if } n \text{ is odd} 
\end{cases}$$

**Answer**
BC: \((-1)^1 \cdot 1 = -1 = -\frac{1+1}{2}\) and \((-1)^1 \cdot 1 + (-1)^2 \cdot 2 = 1 = \frac{2}{2}\). P(1) and P(2) hold.

IS: We wish to show \(P(k) \rightarrow P(k+1)\). We need to consider two cases: \(k\) is even and \(k\) is odd.

Case 1: \(k\) is even. Examining the LHS:

\[
\sum_{i=1}^{k+1} (-1)^i \cdot i = \sum_{i=1}^{k} (-1)^i \cdot i + (-1)^{k+1}
\]

By the Induction Hypothesis,

\[
= \frac{k}{2} + (-1)^{k+1} \cdot (k + 1)
\]

\[
= \frac{k}{2} - (k + 1)
\]

\[
= \frac{k - 2(k + 1)}{2} = -\frac{k + 2}{2}
\]

Case 2: \(k\) is odd. Examining the LHS:

\[
\sum_{i=1}^{k+1} (-1)^i \cdot i = \sum_{i=1}^{k} (-1)^i \cdot i + (-1)^{k+1}
\]

By the Induction Hypothesis,

\[
= -\frac{k + 1}{2} + (-1)^{k+1} \cdot (k + 1)
\]

\[
= -\frac{k + 1}{2} + (k + 1)
\]

\[
= \frac{-k - 1 + 2(k + 1)}{2} = \frac{k + 1}{2}
\]

\(P(k) \rightarrow P(k+1)\) for both the even and odd cases, so the induction step is complete.

19. Consider the recurrence relation

\[a_0 = 0 \quad a_1 = 1 \quad a_n = 2a_{n-1} - a_{n-2} + 1 \quad \text{(for } n \geq 2)\]

Express \(a_n\) as a polynomial in \(n\). (Hint: use the telescopic trick twice.) Then prove by induction the result you obtained.

**Answer**

If we write the recurrence relation for \(n = 2, \ldots, n\), add the equations and cancel terms using the telescopic trick,

\[
a_2 = 2a_1 - a_0 + 1
\]

\[
a_3 = 2a_2 - a_1 + 1
\]

\[
a_4 = 2a_3 - a_2 + 1
\]

\[
\vdots
\]

\[
a_{n-1} = 2a_{n-2} - a_{n-3} + 1
\]

\[
a_n = 2a_{n-1} - a_{n-2} + 1
\]

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we obtain that
\[ a_n + a_{n-1} = 2a_{n-1} + 2a_1 - a_1 - a_0 + (n - 1) \]
\[ a_n = a_{n-1} + a_1 - a_0 + (n - 1) \]
\[ a_n = a_{n-1} + n \]

Applying the same telescopic trick to this recurrence:

\[ a_1 = a_0 + 1 \]
\[ a_2 = a_1 + 2 \]
\[ a_3 = a_2 + 3 \]
\[ \vdots \]
\[ a_{n-1} = a_{n-2} + (n - 1) \]
\[ a_n = a_{n-1} + n \]

we get that

\[ a_n = a_0 + \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \]

Therefore, \( a_n = \frac{n(n+1)}{2} \).

20. Prove that any positive integer can be expressed as the sum of distinct Fibonacci numbers.

**Answer**

We proceed by strong induction.

**BASE CASE** \( n = 1 \). We have that \( 1 = F_1 \), which is the sum of distinct Fibonacci numbers.

**INDUCTION STEP** Let \( k \) be an arbitrary integer \( \in \mathbb{Z}^+ \). Assume that (IH) \( \forall 1 \leq j \leq k, j \) can be expressed as the sum of distinct Fibonacci numbers.

We wish to show that \( n = k + 1 \) can also be expressed as the sum of distinct Fibonacci numbers.

Let \( F_i \leq k + 1 \) be the largest Fibonacci number less than or equal to \( k + 1 \). We have two cases.

**Case 1:** \( F_i = k + 1 \)

Then we can express \( k + 1 \) as \( F_i \), which is the sum of distinct Fibonacci numbers.

**Case 2:** \( F_i < k + 1 \)

Let \( l = k + 1 - F_i \).

We claim that \( l < F_i \). Assume toward contradiction that \( l \geq F_i \). Then \( k + 1 = F_i + l \geq 2F_i \). But \( F_i \geq F_{i-1} \), so \( k + 1 \geq F_i + F_{i-1} = F_{i+1} \), contradicting our claim that \( F_i \) is the largest Fibonacci number less than or equal to \( k + 1 \). Hence, \( l < F_i \).

Then, by the induction hypothesis, since \( l < F_i \), \( l \) can be expressed as the sum of distinct Fibonacci numbers \( F_{m_1} + F_{m_2} + \ldots + F_{m_n} \), and \( F_{m_p} \neq F_i, \forall 1 \leq p \leq n \). Then \( k + 1 = l + F_i = F_{m_1} + F_{m_2} + \ldots + F_{m_n} + F_i \) is the sum of distinct Fibonacci numbers.

Thus, we have shown our claim is true when \( n = k + 1 \), concluding our Induction Step and completing our proof.
21. Punch happily tells Judy that he proved two new theorems and he shares his proofs with her.

(a) **Punch’s First Theorem**: If \( n \) is odd then \( n^2 - 1 \) is a multiple of 4.

*Punch’s Proof*: “We prove the contrapositive instead. Suppose \( n \) is even, then \( n^2 \) is even, then \( n^2 - 1 \) is odd so it cannot be a multiple of 4. Done.” Upon reading these, Judy remarks that while the theorem is true, the proof is not proving the theorem, but another statement, which is not the contrapositive of of the theorem.

i. What is the contrapositive of the theorem and what statement is Punch actually proving?

ii. Give a correct proof of Punch’s First Theorem.

(b) **Punch’s Second Theorem**: For any finite sets \( A, B \), if \( |A| \) and \( |B| \) are even then \( |A \setminus B| \) is even.

*Punch’s Proof*: “The difference of two even numbers is an even number. Done.”

i. Now, Judy remarks that this other theorem is not even true. Give a counterexample that supports Judy’s contention.

ii. Judy also remarks that Punch’s “proof” relies on a false statement about set cardinalities.

(Since the theorem is not true, there had to be a bug in the proof!) What is that false statement?

**Answer**

(a) i. The contrapositive is:

“*If* \( n^2 - 1 \) *is not a multiple of 4 then* \( n \) *is even (or you can say “is not odd”).”

What Punch actually proved is:

“If \( n \) is even then \( n^2 - 1 \) is not a multiple of 4.”

*(Not required but FYI: This is the converse of the contrapositive and in general it’s not logically equivalent to the theorem.)*

ii. Direct proof: if \( n \) is odd then \( n = 2k + 1 \) for some integer \( k \). Thus,

\[
\begin{align*}
n^2 - 1 &= (2k + 1)^2 - 1 \\
&= 4k^2 + 4k + 1 - 1 \\
&= 4(k^2 + k)
\end{align*}
\]

which is clearly divisible by 4.

*(Not required but FYI: Suppose that we attempt to prove the contrapositive. This leads to the following strange but interesting proof:)*

Suppose \( n^2 - 1 \) is not a multiple of 4 and W.T.S. \( n \) is even.

**Case 1**: \( n^2 - 1 \) is odd. In this case \( n^2 \) is even and therefore \( n \) is even.

**Case 2**: \( n^2 - 1 \) is even but still not a multiple of 4. In this case \( n^2 \) is odd so \( n = 2k + 1 \) for some integer \( k \) and then \( n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k) \) hence a multiple of 4. But our statement says that \( n^2 - 1 \) couldn’t have been a multiple of 4, and thus this case is impossible. Therefore since we showed in this case the premise is false, and false implies anything, the statement still holds true!

\( n \) is even in both cases. Done.

The strange thing is that in the process of doing a proof by contrapositive you essentially discover the direct proof too, see inside Case 2. )
(b) i. The counterexample is \( A = \{1, 2\}, B = \{2, 3\} \) therefore \( A \setminus B = \{1\} \).

ii. Punch assumes that \( |A \setminus B| = |A| - |B| \). Note that it is actually \( |A \setminus B| = |A| - |B| + |A \cap B| \).

22. How many sequences of bits (0’s, 1’s) are there that each sequence has all of the following properties:

- Their length is either 3 or 5 or 7.
- Their middle bit is a 1.
- The number of 0’s they have equals the number of 1’s they have minus one.

**Answer**

We partition the possible sequences into three cases by their lengths.

**Case 1:** Sequences with 3 bits Since the middle bit must be a 1 and the bit must contain a total of two 1’s and one 0, there are only 2 ways to arrange the bits: 011 and 110.

**Case 2:** Sequences with 5 bits Since the middle bit must be a 1, we only need to consider the possible arrangements of the remaining 1’s and 0’s. To do this, we choose the 2 of the 4 non-middle positions to place the 1’s and then place the 0’s in the remaining spots. This can be done in \( \binom{4}{2} \) ways.

**Case 3:** Sequences with 7 bits Again, the middle bit must be a 1, so we only need to consider the possible arrangements of the remaining 1’s and 0’s. To do this, we choose 3 of the 6 non-middle positions to place the 1’s and then place the 0’s in the remaining spots. This can be done in \( \binom{6}{3} \) ways.

Since the cases are disjoint, we can apply the Sum Rule to get

\[
2 + \binom{4}{2} + \binom{6}{3} = 2 + \frac{4!}{2!2!} + \frac{6!}{3!3!} = 28
\]

23. How many sequences of bits (0’s, 1’s) of length 100 can we make such that:

- the number of 0’s in the sequence is equal to the number of 1’s in the sequence; and
- the sequence begins with a 1 and ends with a 1.

**Answer**

Since there are an equal number of 0’s and 1’s, we know that there are 50 0’s and 50 1’s. The sequence begins and ends with 1’s, so we only need to count the number of ways to arrange the middle 98 bits, which are 50 0’s and 48 1’s. This can be done by choosing 50 of 98 positions to place the 0’s and then filling the remaining spots with 1’s. Alternatively, we can choose 48 of 98 positions to place the 1’s and fill the remaining spots with 0’s, which will give us the same answer. Thus, the total number of distinct bit sequences is

\[
\binom{98}{50} = \binom{98}{48}
\]
24. A cookie shop has \(k\) different flavors of cookies. Alex wishes to purchase cookies for his recitation, and he has enough money to buy up to 250 cookies. Assuming that he does not have to spend all of the money that he has, in how many ways can he purchase cookies? (For full credit, your solution should be in closed form, so no open summations!)

**Answer**

The problem essentially asks for the number of ways to distribute 250 cookies into distinguishable categories.

Each of these cookies will be either one of the \(k\) flavors or “not purchased,” so we can imagine \(k + 1\) bins to place the cookies in. Since any two unpurchased cookies or cookies of the same flavor are indistinguishable, we can use the stars and bars method, where the cookies are the stars and the \((k + 1) - 1 = k\) dividers between the \(k + 1\) categories are the bars.

For example, if Alex purchases 1 cookie of each of \(n\) varieties and then 250 – \(n\) potential cookies are left unpurchased (because he’s feeling stingy), then we would represent this with one star followed by one bar \(k\) times, and then 250 – \(k\) stars representing unpurchased cookies.

Thus, using the formula for the stars and bars method, the answer is \(\binom{250 + k}{k}\).

**Alternative (incomplete) solution**: You could choose to sum over all possibilities. For each \(i\) between 0 and 250 equal to number of cookies purchased, we want to count the number of ways we can split \(i\) up into the \(k\) varieties. This can still be done with stars and bars, but we now have \(k\) bins, one for each variety. Summing, we have

\[
\sum_{i=0}^{250} \binom{i + k - 1}{k - 1} = \sum_{j=k-1}^{250+k-1} \binom{j}{k-1}
\]

If you leave it like this, you do not get full credit. But this sum actually equals \(\binom{250 + k}{k}\) (see problem 9(c)). In fact these two alternative solutions provide a combinatorial proof for 9(c).

25. Give a combinatorial proof (no other kind of proofs will be accepted) for the following identity

\[(n)_r = (n - 1)_r + r (n - 1)_{r-1} \quad \text{ (where } 1 \leq r \leq n)\]

**Answer**

We use this counting question: There are \(n\) TAs but only \(r\) (distinguishable) chairs at grading. How many ways can we place the \(n\) TAs into the \(r\) chairs?

The LHS is a fairly direct way to solve this problem - for the first chair, there are \(n\) TAs we can place in it. For the second chair, there are \(n - 1\) TAs we can place in it. This will evaluate to the falling factorial (i.e. the permutation) \((n)_r\) on the LHS.

Now we try to solve this problem in such a way that yields the expression on the RHS. Since we add two expressions, it seems logical that we might have two cases. In case 1, Krishna, as the TA who always arrives first, decides to be self-sacrificing and chooses to stand so the other TAs can sit. In case 2, he decides to be selfish and takes a seat.
**Case 1:** Since we know Krishna is not going to sit, there are only \( n - 1 \) TAs who are going to sit, and there are \( r \) chairs. We apply the definition of a permutation to get \((n - 1)_r\).

**Case 2:**

*Step 1:* Krishna chooses a seat for himself. \((r \text{ ways})\)

*Step 2:* We place \( r - 1 \) of the remaining \( n - 1 \) TAs in the remaining \( r - 1 \) chairs. Applying again the definition of a permutation, there are \((n - 1)_{r - 1}\) ways to do this.

We apply the multiplication rule to get \( r(n - 1)_{r - 1} \) in this case.

We add the two cases to get \((n - 1)_r + r(n - 1)r - 1\), which is exactly the expression on the RHS.

Thus, as we have solved the same problem in two valid ways to get the expressions on the LHS and RHS respectively, we have a combinatorial proof.

26. We distribute indistinguishable ungraded papers to \( n \geq 3 \) distinguishable TAs in the following way.

First we select two “lucky” TAs to have the designation of Head TA. Next we distribute the papers in such a way that each TA gets at least 1 paper to grade, and both of the Head TAs get at least 2 papers. What is the minimum number of papers needed to make this work? Now, assume that we have \( r \) papers, where \( r \) is large enough to make this kind of distribution work, in how many different ways can the papers be distributed?

**Answer**

We need at least \( n - 2 \) coins for each of the \( n - 2 \) “unlucky”, non-Head TAs to have 1 paper. Therefore, the minimum, and then 4 for the 2 Head TAs to each have 2, so the number of papers we need is \( n - 2 + 4 = n + 2 \).

For this interpretation a distributions can be constructed as follows (assuming we have \( r \geq n + 2 \) papers):

*Step 1:* Choose two (out of \( n \)) TAs to be Head TAs.

*Step 2:* Give 2 papers each to Head TAs and 1 paper each to the remaining \( n - 2 \) TAs.

*Step 3:* Distribute the remaining \( r - n - 2 \) papers to all the \( n \) TAs.

This process works because, after we designate the head TAs and give everyone their minimum number of papers, the number of ways to distribute the rest of the papers is the same as the number of total distributions.

Step 1 can be done in \( \binom{n}{2} \) ways, as we are simply choosing 2 TAs from \( n \) without order. Step 2 can be done in 1 way. To do Step 3, we apply stars and bars. There are \( r - n - 2 \) papers, which, as the items being distributed, represent the stars. Then there are \( n \) TAs - \( n \) “categories” into which we sort the items - so we have \( n - 1 \) bars. Thus we need to choose \( n - 1 \) bars from the \( r - n - 2 + (n - 1) = r - 3 \) stars and bars. So we have that Step 3 can be done in \( \binom{r - 3}{n - 1} \) ways. By the multiplication rule the answer is

\[
\binom{n}{2} \binom{r - 3}{n - 1}
\]
27. Give a combinatorial proof for the following identity:

\[
\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}
\]

**Answer**

We pose the following counting question.

Shawn bought \( m \) (distinguishable) math textbooks and \( n \) (distinguishable) CS textbooks as summer reading, but Seth forgot to buy books to read, so Shawn agrees to give him \( r \) of his textbooks. How many different combinations of \( r \) math and CS textbooks can Shawn give Seth?

We need to solve this problem in 2 ways which yield the RHS and LHS respectively.

First, we look at the RHS. Since we are choosing \( r \) books from \( m+n \) total distinguishable books, we simply apply the definition of a combination to have \( \binom{m+n}{r} \).

We now look at the LHS. Since we see a summation, we consider breaking this expression into cases. Of the \( r \) textbooks, Shawn could choose 0 to be math textbooks, 1 to be a math textbook, or 2, 3, etc. (the remainder being made up of CS textbooks). We let this value be \( k \). That is, \( k \) is the number of math textbooks and \( r-k \) is the number of CS textbooks Shawn chooses. So we solve this problem in 2 steps:

* **Step 1:** Shawn chooses \( k \) of the \( m \) math textbooks.
* **Step 2:** Shawn chooses \( r-k \) of the \( n \) CS textbooks.

We again apply the definition of the combination to find that there are \( \binom{m}{k} \) ways to perform Step 1 and \( \binom{n}{r-k} \) ways to perform Step 2. With the multiplication rule, we have \( \binom{m}{k} \binom{n}{r-k} \). Then we sum over all possible values of \( k \), which range over all integer values from 0 to \( r \) (Shawn could pick any value from 0 to \( r \) of the \( r \) textbooks to be math books):

\[
\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}
\]

This is exactly the expression on the RHS. Thus, as we have solved the same problem in two valid ways to get the expressions on the LHS and RHS respectively, we have a combinatorial proof.

28. How many sequences of bits of length 10 are there such that every 0 is followed immediately by a 1? Leave the answer as a sum of binomial coefficients.

**Answer**

Since every 0 must be followed by a 1, we can consider 01 as a single unit, so let us denote 01 by \( a \). Then a sequence of bits in which every 0 is followed immediately by a 1 is actually a sequence of 1’s and \( a \)’s. However, note that this sequence is not of the same length as the original sequence. The original sequence had length 10 but the sequence of 1’s and \( a \)’s will have length \( 10-k \), where \( k \) is the number of \( a \)’s and \( k = 0, 1, 2, 3, 4, 5 \) (since it is the same as the number of 0’s in the original sequence, and we can have up to 5 0’s).
For each \( k = 0, 1, 2, 3, 4, 5 \) the number of sequences of length \( 10 - k \) with \( k \) a’s is \( \binom{10-k}{k} \) since it the number of ways to choose the positions where you put the a’s. Therefore the answer is

\[
\binom{10}{0} + \binom{9}{1} + \binom{8}{2} + \binom{7}{3} + \binom{6}{4} + \binom{5}{5}
\]

29. Let \( X \) be a finite set such that \(|X| \geq 3\). Prove that

\[
\left| \{ (x, A) \mid A \subseteq X \text{ and } x \in A \} \right| > 2^{|X|}
\]

**Answer**

Let \( W = \{ (x, A) \mid A \subseteq X \text{ and } x \in A \} \). We wish to show that \(|W| > 2^{|X|}\).

Consider the function \( \sigma : W \rightarrow 2^X \) defined by \( \sigma(x, A) = A \).

Observe that \( \text{Ran}(\sigma) \) consists of all nonempty subsets of \( X \). Indeed, if \( B \subseteq X \) is nonempty, then there exists \( x \in B \) and therefore \( \sigma(x, B) = B \). On the other hand, there is no \( (x, A) \in W \) such that \( \sigma(x, A) = \emptyset \), because then \( x \in \emptyset \), which is impossible.

Let \( m \) be the number of elements of \( \text{Ran}(\sigma) \). \( 2^X \) has one more element that \( \text{Ran}(\sigma) \), namely \( \emptyset \). Therefore \( 2^{|X|} = m + 1 \).

Note that \( W \) has at least \( m \) elements because each element of \( \text{Ran}(\sigma) \) is mapped to at least one element of \( W \).

Recall that \(|X| \geq 3\). So \( X \) has at least three distinct elements, call them \( x_1, x_2, x_3 \). This gives us three distinct elements of \( W \), namely \( (x_1, X), (x_2, X), (x_3, X) \), all of which are mapped by \( \sigma \) to the same element, \( X \). Therefore, \( W \) has at least two more elements than \( \text{Ran}(\sigma) \).

Thus, \( |W| \geq m + 2 > m + 1 = 2^{|X|} \), as desired.

30. Consider 33 distinct boolean expressions in exactly two variables. Prove that 3 or more of them must be logically equivalent. (Some of you may already know the Pigeonhole Principle (PHP), but we did not cover it yet. Therefore, if you want to use PHP, you must prove it.)

**Answer**

We wish to show that given 33 distinct boolean expressions, 3 or more of them have the same truth table (meaning that each group is logically equivalent).

Observe that a truth table in two variables is a function \( \text{tt} : S \rightarrow \{T, F\} \), where \( S \) is the set of sequences of length 2 formed from \( \{T, F\} \). \(|S| = 2^2 = 4\). Thus, there are \( 2^{|S|} = 2^4 = 16 \) distinct truth tables in two variables.

Now, suppose that we can partition the 33 expression into groups with the same truth table, such that each group has at most 2 expressions.

But we showed before that there are only 16 possible truth tables in two variables, so there are at most 16 such groups. Even if each group has two distinct expressions, this adds up to \( 16 \times 2 = 32 \) expressions which is strictly less than 33. This is a contradiction, so 3 or more of the boolean expressions must be logically equivalent.
31. For each statement below, decide whether it is true or false. In each case attach a very brief explanation of your answer.

(a) For any natural numbers $1 \leq k \leq n$ we have that $(n+1)_k - (n)_k$ is divisible by $k$, true or false?

(b) Consider the proposition $P$ where:

$$P: \exists N > 0, \forall n \geq N, \ 100n \leq n^2/100$$

True or false: $\neg P$ is the following proposition:

$$\forall N > 0, \forall n \geq N, \ 100n > n^2/100$$

(c) Let $X$ be a finite nonempty set. The number of functions with domain $X$ and codomain $\{0, 1\}$ is $2^{|X|}$, true or false?

**Answer**

(a) TRUE.  
**Solution I:**

$$(n+1)_k - (n)_k = (n+1) \times n \times \cdots \times (n-k+2) - n \times \cdots \times (n-k+2) \times (n-k+1)$$

$$= [n \times \cdots \times (n-k+2)] \times [(n+1) - (n-k+1)]$$

$$= [n \times \cdots \times (n-k+2)] \times k$$

so it’s divisible by $k$.

**Solution II:** Consider the first term, $(n+1)_k$. This is equal the product of $k$ consecutive integers, $(n+1) \cdot n \cdots (n-k+2)$. Thus, exactly one of these factors is a multiple of $k$, and so $(n+1)_k$ is divisible by $k$. Therefore, $(n+1)_k = k \times a$ for some integer $a$. Now consider the second term, $(n)_k$. Similarly, it is equal to the product of $k$ consecutive integers, $n \cdot (n-1) \cdots (n-k+1)$. Thus, exactly one of these factors is a multiple of $k$, and so $(n)_k$ is divisible by $k$. Therefore, $(n)_k = k \times b$ for some integer $b$.

Using substitution, we derive $(n+1)_k - (n)_k = k \times a - k \times b = k \times (a-b)$, which is divisible by $k$.

(b) FALSE. If we negate the expression and move the negation as far right as possible, we derive:

$$\neg[\exists N > 0, \forall n \geq N, \ 100n \leq n^2/100]$$

$$\forall N > 0, \neg[\forall n \geq N, \ 100n \leq n^2/100]$$

$$\forall N > 0, \exists n \geq N, \neg[100n \leq n^2/100]$$

$$\forall N > 0, \exists n \geq N, \ 100n > n^2/100$$

which is a different statement.

(c) TRUE. We have seen in class that, for a domain of size $a$ and a codomain of size $b$, there are $b^a$ functions. We also accept a solution by counting. For every element of $X$, we need to choose exactly one element of $\{0, 1\}$ to map it to. Thus, there are $|X|$ decisions each with two choices, and so the number of possibilities is $2 \cdot 2 \cdots 2 = 2^{|X|}$. 

33
32. You are in charge of your country’s adoption program. Suppose that we have \( n \geq 3 \) children needing to be adopted and \( n - 2 \) families what can adopt. Each family must have at least one child and no family can take in 3 or more children. The census board now wants to know how many distinct ordered pairs of *siblings* there are, after you distribute all the kids. How many are there?

**Answer**

**Warning:** just to be sure, this question is NOT asking in how many different ways we can distribute \( n \) children among \( n - 2 \) families such that each family has at least one child, each child belongs to exactly one family (the families are pairwise disjoint), and none of the families has exactly 3 siblings. This would be a different problem with a different answer! We are instead asking how many ordered pairs \((b, s)\) are there where \( b \) and \( s \) are both siblings?

Since each of the \( n - 2 \) families has at least one child, start by allocating one child to each family. You are left with two children, say Edward and Mei, to allocate amongst the families. Because no family can have 3 siblings, the only possibility is for Edward and Mei to be in different families. (By the way, this also implies that \( n = 3 \) is impossible.)

Therefore we have two families, Edward’s and Mei’s, with 2 children each and the rest have just one child. Let’s say Edwards’s sibling is Alphonse and Mei’s is Ling. The ordered pairs of siblings are (Edward, Alphonse), (Alphonse, Edward), (Lin, Mei), and (Mei, Lin). So the answer is 4.

33. For sets \( A, B, C, \) and \( D \), suppose that \( A \setminus B \subseteq C \cap D \) and \( x \in A \). Prove that if \( x \notin D \) then \( x \in B \).

**Answer**

We will prove the claim by proving the contrapositive. Suppose that \( A \setminus B \subseteq C \cap D \) and \( x \in A \) but \( x \notin B \). Since \( x \notin B \) and \( x \in A \), it must be that \( x \in A \setminus B \) and hence \( x \in C \cap D \). Thus \( x \in D \).

34. You are choosing a sequence of five characters for a license plate. Your choices for characters are any letter in PERM and any digit in 1223. Your five-character sequence can contain any of these characters at most the number of times they appear in either PERM or 1223. If there are no other restrictions, how many such sequences are possible?

**Answer**

We want to count the number of distinct ways to order 5 characters from “PERM” and “1223”. “2” is the only character that appears more than once in “PERM” and “1223”. Thus, we case upon the number of “2”s.

*Case 1:* Zero “2”s

In this case, all 5 letters come from \( \{P, E, R, M\} \) or \( \{1, 3\} \). Since there are 6 options, there are \( \binom{6}{5} \) ways to select the letters for the string. There are 5! ways to order the letters once they are selected. By Multiplication Rule, Case 1 has \( \binom{6}{5} \cdot 5! \) solutions.

*Case 2:* Exactly one “2”

In this case, 4 of the letters come from \( \{P, E, R, M\} \) or \( \{1, 3\} \), and 1 of the letters is fixed to be “2”. Since there are 6 options for the unknown characters, there are \( \binom{6}{4} \) ways to select the letters for the string. There are 5! ways to order the letters once they are selected since there are still no repetitions. By Multiplication Rule, Case 2 has \( \binom{6}{4} \cdot 5! \) solutions.
Case 3: Two “2”s
In this case, 3 of the 5 letters come from \{P, E, R, M\} or \{1, 3\}, and 2 of the letters are fixed to be “2”. Since there are 6 options for the unknown characters, there are \(\binom{6}{3}\) ways to select the letters for the string. A simple linear ordering of 5! counts each distinct solution exactly twice since the positions of the “2”s could be reversed, so there are \(\frac{5!}{2}\) ways to order the 5 characters. By Multiplication Rule, Case 3 has \(\binom{6}{3} \cdot \frac{5!}{2}\) solutions.

By Sum Rule, the number of solutions is:

\[
\binom{6}{5} \cdot 5! + \binom{6}{4} \cdot 5! + \binom{6}{3} \cdot \frac{5!}{2} = 3720
\]

35. There are a variety of special hands that one can be dealt in poker. For each of the following types of hands, count the number of hands that have that type.

(a) Four of a kind: The hand contains four cards of the same numerical value (e.g., four jacks) and another card.

(b) Three of a kind: The hand contains three cards of the same numerical value and two other cards with two other numerical values.

(c) Flush: The hand contains five cards all of the same suit.

(d) Full house: The hand contains three cards of one value and two cards of another value.

(e) Straight: The five cards have consecutive numerical values, such as 7-8-9-10-jack. Treat ace as being higher than king but not less than 2. The suits are irrelevant.

(f) Straight flush: The hand is both a straight and a flush.

Answer

(a) Four of a kind:

\[\text{Step 1: Pick a value for the four-of-a-kind. (13 choices)}\]
\[\text{Step 2: Pick 4 cards with that value. (1 choice)}\]
\[\text{Step 3: Pick a 5th card. (48 choices)}\]

By Multiplication Rule, the number of solutions is \(13 \cdot 48 = 13 \cdot 48\).

(b) Three of a kind:

\[\text{Step 1: Pick a value for the three-of-a-kind (13 choices)}\]
\[\text{Step 2: Pick 3 cards for the three-of-a-kind (4 choices)}\]
\[\text{Step 3: Pick 2 values for the other two cards (12 choices)}\]
\[\text{Step 4: Pick the specific cards for the other two cards given the 2 values (4^2 choices)}\]

By Multiplication Rule, the number of solutions is \(13 \cdot 4 \cdot \binom{12}{2} \cdot 4^2 = 13 \cdot 4 \cdot \binom{12}{2} \cdot 4^2\).

(c) Flush:

\[\text{Step 1: Pick a suit (4 choices)}\]
\[\text{Step 2: Pick 5 cards of that suit (13 choices)}\]

By Multiplication Rule, the number of solutions is \(4 \cdot 13\).

(d) Full House:
**Step 1:** Pick a value for the three-of-a-kind \((\binom{13}{1})\) choices.

**Step 2:** Pick 3 cards for the three-of-a-kind \((\binom{4}{3})\) choices.

**Step 3:** Pick 1 values for the other two cards \((\binom{12}{1})\) choices.

**Step 4:** Pick 2 cards for the pair \((\binom{4}{2})\) choices.

By Multiplication Rule, the number of solutions is \(13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6\).

(e) **Straight:**

**Step 1:** Pick the starting value (2 through 10) \((\binom{9}{1})\) choices.

**Step 2:** Pick a card for each value (4^5 choices)

By Multiplication Rule, there are \(9 \cdot 4^5\) solutions.  

(f) **Straight Flush:**

**Step 1:** Pick the suit \((\binom{4}{1})\) choices.

**Step 2:** Pick the starting value (2 through 10) \((\binom{9}{1})\) choices.

By Multiplication Rule, the number of solutions is \(9 \cdot 4\).

36. Let \(A, B\) be arbitrary sets. Prove by contradiction that

\[ A \subseteq B \implies A \setminus (A \cap B) = \emptyset. \]

You are NOT allowed to use in the proof set algebra facts (such as \(A \subseteq B \iff A \cap B = A\) or \(A \setminus A = \emptyset\)). Your proof should use only the definitions of subset, set difference, intersection, and empty set and logical manipulation of statements.

**Answer**

Assume for the sake of contradiction that \(A \subseteq B\) and \(A \setminus (A \cap B) \neq \emptyset\).

Since \(A \setminus (A \cap B)\) is nonempty, there exists some \(x \in A \setminus (A \cap B)\).

\[ x \in A \setminus (A \cap B) \]

\[ (x \in A) \land (x \notin A \cap B) \]  
(\text{By definition of set difference})

\[ (x \in A) \land ((x \in A \cap B)) \]

\[ (x \in A) \land ((x \notin A) \lor (x \notin B)) \]  
(De Morgan’s Law)

We now case on whether \(x \notin A\) or \(x \notin B\).

**Case 1:** \(x \notin A\)

\[ (x \in A) \land (x \notin A) \]

An element \(x\) cannot be in \(A\) and not in \(A\), so we reach a contradiction.

**Case 2:** \((x \in A) \land (x \notin B)\)

We assumed that \(A \subseteq B\) and that \(x \in A\). By definition of subsets, \(x \in B\). However, this is a contradiction by the case condition.

Since both cases reach a contradiction, our original claim must be true: \(A \subseteq B \implies A \setminus (A \cap B) = \emptyset\).

\(^1\)Some variants of poker have a different definition for what a straight is.
37. Prove that if for some integer \( a \), \( a \geq 3 \), then \( a^2 > 2a + 1 \).

**Answer**

Let \( a \in \mathbb{Z} \) s.t. \( a \geq 3 \).

We observe that \( 3a > 2a + 1 \) since \( a > 1 \). So if we can show that \( a^2 \geq 3a \), then we’re done.

Note that \( a^2 = a \times a \), and \( 3a = 3 \times a \). Since we know that \( a \geq 3 \), we can conclude \( a \times a \geq 3 \times a \).

Hence our proof is complete.

38. Give a combinatorial proof of the following identity for \( N, a, b \in \mathbb{N} \):

\[
\binom{N}{a} \binom{N}{b} = \sum_{i=0}^{\min(a,b)} \binom{N}{i} \binom{N-i}{a-i} \binom{N-a}{b-i}
\]

**Answer**

Consider the situation: there are \( N \) people, and Oprah gives \( a \) of them a boat, and \( b \) of them a house (some may receive both). We pose the question: how many ways can Oprah do this? We can choose the people to give a boat to in \( \binom{N}{a} \) ways, and we can choose the people to give a house to in \( \binom{N}{b} \) ways. Thus there are \( \binom{N}{a} \times \binom{N}{b} \) total ways to distribute the boats and houses, which is exactly the left side of the equation.

The RHS starts with iterating on the number of people who are going to get both boats and houses (\( i \) iterates from 0 to \( \min(a,b) \)). We then choose the \( i \) people who receive both boats and houses. Next out of the remaining \( N - i \) we choose \( a - i \) who receive only boats. Note, that so far we have chosen \( i + a - i = a \) people. Finally, out of the remaining \( N - a \) people we choose \( b - i \) who receive only houses.

39. There are 100 guests at a fundraising party, excluding the host. As part of a “fun” party game, the host pairs up the dinner guests into 50 pairs that the host calls “fundraising pairs”. In the game, the individual with the smaller net worth in each pair declares the amount of money that they wish to donate, which the individual with the higher net worth must match in double. For example, if the individual with the smaller net worth in one pair donates $100 dollars, the individual with the larger net worth must donate $200 dollars.

The host says that the aim of the game is to raise a total of 9 million dollars between all of the individuals. Given this set up, how many ways can the game unfold? Assume that the net worth of each of the individuals is unique, that all donations are in whole dollars, and that all of them can donate up to 9 million dollars each.

**Answer**

First, let us calculate how many ways there are to pair up the individuals. We can do this by arranging all 100 individuals in line and interpreting the \( 2k+1 \)th and \( 2k+2 \)th individuals as paired up, where \( 0 \leq k \leq 49 \), and then removing the ordering from these pairs. There are 100! ways of arranging the individuals in a line, which we divide by 50! to remove the ordering of the pairs. We also need to divide by \( 2^{50} \), since there is no ordering within the pairs either.

Second, we seek the number of possible ways that the individuals can donate money. Since the individual with the higher net worth must donate twice what the individual with the lower net
worth donates, each pairs donation must be a multiple of 3. We can therefore find the number of arrangements using the stars and bars method seen in lecture. Each star is a donation of $3 dollars, and there are 3 million stars. There are 49 bars since there are 50 pairs. Therefore, the number of arrangements is \( \binom{3000000+49}{49} \).

Finally, we seek to the number of ways that the two individuals within the pair can donate the money. There is simply 1 way, since the ratio of donation between the two is fixed.

Therefore, the total number of ways that the game can unfold is \( \frac{100!}{50! \cdot 2^{50}} \times \binom{3000049}{49} \).

40. Let \( A \) be a finite set with \( n \geq 1 \) elements and let \( f : 2^{A} \times A \to \mathbb{N}, f(R) = |R| \). How many elements does the set \( \text{Ran}(f) \) have?

**Answer**

The set \( \text{Ran}(f) \) contains \( n^2 + 1 \) elements and here is why.

The domain of the function is all subsets of \( A \times A \) (the power set of \( A \times A \)), and the function \( f \) simply maps a subset to its cardinality. The set \( A \times A \) will have \( n \cdot n = n^2 \) elements, and a subset could contain any number of these pairs (even all of them) or none at all. Hence, the range of the function is \( \{0, 1, \cdots n^2\} \) and its cardinality is \( n^2 + 1 \).

41. Let \( x, y \) be arbitrary boolean variables. Give a truth table for the expression \( x \Rightarrow (\neg y \Rightarrow F) \). Then, find, two other, distinct, boolean expressions that are logically equivalent to the previous expression.

**Answer**

The two boolean expressions are \( x \rightarrow y \) and \( \neg x \lor y \). This is justified by the truth table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( x \Rightarrow (\neg y \Rightarrow F) )</th>
<th>( x \Rightarrow y )</th>
<th>( \neg x \lor y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
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</tr>
</tbody>
</table>

*Additional explanations, not required for your solution:* The statement \( \neg y \rightarrow F \) is logically equivalent to \( y \), so \( x \rightarrow (\neg y \rightarrow F) \) is logically equivalent to \( x \rightarrow y \). The only time \( \neg x \lor y \) will be false is when \( x = T \) and \( y = F \), which is consistent with the truth values for the other two expressions.

Thus \( x \rightarrow (\neg y \rightarrow F) \) is logically equivalent to \( x \rightarrow y \) and \( \neg x \lor y \).

(These are not the only possibilities. In fact there are infinitely many solutions!)

42. Let \( A \) be arbitrary sets. Prove using contradiction that

\[
A \subseteq B \Rightarrow A \setminus (A \cap B) = \emptyset
\]

**Answer**

It is stated that \( A \subseteq B \). Now, assume toward a contradiction that \( A \setminus (A \cap B) \neq \emptyset \). Therefore, \( \exists x \) such that \( x \in A \setminus (A \cap B) \). Thus, \( x \in A \) but \( x \notin (A \cap B) \).
By De Morgan’s Law, \( x \notin (A \cap B) \) is logically equivalent to \( (x \notin A) \vee (x \notin B) \).

**Case 1:** \( x \notin A \). Contradicts \( x \in A \).

**Case 2:** \( x \notin B \). But \( x \in A \) and \( A \subseteq B \) implies \( x \in B \). Contradiction.

Both cases, which are exhaustive, end up with a contradiction.

43. Give an example of nonempty finite sets \( A, B, C \) and an example of a function \( f : A \to B \) such that

- \( C \subseteq A \)
- \( A \times C \in 2^{f(C)} \)

(Just give the example. You do not need to prove that it works.)

Note: For a function \( f : X \to Y \) and a set \( S \), \( f(S) \) denotes the image of a set.

That is, \( f(S) = \{ y \in Y \mid \exists x \in S \text{ s.t. } y = f(x) \} \).

**Answer**

Let \( A = \{1\} \), \( B = \{(1,1)\} \), and \( C = \{1\} \).

Let \( f : A \to B \) be defined as \( f(1) = (1,1) \).

Then \( A \times C = \{(1,1)\} \), and \( f(C) = \{(1,1)\} \). Therefore, \( 2^{f(C)} = \{\emptyset, \{(1,1)\}\} \), and we have that \( A \times C \in 2^{f(C)} \).

44. Prove that for any natural number \( n \geq 1 \) there exist two natural numbers \( m_1 \) and \( m_2 \) such that \( m_1 < n < m_2 \) and \( n = (m_1 + m_2)/2 \).

**Answer**

Let \( m_1 = n - 1 \) and \( m_2 = n + 1 \). Clearly, \( m_1 < n < m_2 \). WTS that:

\[
\begin{align*}
n &= \frac{m_1 + m_2}{2} \\
&= \frac{n - 1 + n + 1}{2} \\
&= \frac{2n}{2} \\
n &= n
\end{align*}
\]

(This is OK even when \( n = 1 \) because zero is a natural number.)

45. For each statement below, decide whether it is true or false. In each case attach a *very brief* explanation of your answer.

(a) The function \( f : \mathbb{N} \to \mathbb{N} \) \( f(x) = 2^x \) is a bijection, true or false?

(b) Let \( S \) be the set of the first 100 natural numbers: \( S = 0, 1, \ldots, 99 \). There are as many subsets of \( S \) of size 50 that contain the number 50 as subsets of \( S \) of size 50 that do not contain the number 50, true or false?

(c) Suppose that \( A, B, C \) are finite nonempty sets with an even number of elements and that \( A \) and \( B \) are disjoint. Then \(|(A \cup B) \times C|\) is divisible by 4, true or false?
(a) FALSE.
   It is not a surjection. For example, take 3 ∈ N - this element in the codomain never gets mapped
to because it is not a power of 2.

(b) TRUE. There are \( \binom{99}{49} \) subsets of size 50 with the number 50 in them because we need only count
how to choose the 49 remaining elements (since 50 is already chosen), from the remaining 99. To
count subsets of size 50 without 50, we need to count how many ways we can select 50 numbers
from the remaining 99, which can be done in \( \binom{99}{50} \) ways. And \( \binom{99}{49} = \binom{99}{50} \) since \( \binom{n}{k} = \binom{n}{n-k} \).

(c) TRUE. Since \( A, B, C \) have even cardinality, there are \( x, y, z \) ∈ \( \mathbb{N}^+ \) such that \( |A| = 2x, |B| = 2y \)
and \( |C| = 2z \). Since \( A \) and \( B \) are disjoint, \( |A \cup B| = 2x + 2y = 2(x + y) \). By the multiplication
rule, \( |(A \cup B) \times C| = 2(x + y) \times 2z = 4z(x + y) \), which is clearly divisible by 4.

46. Let \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) be two functions such that for all \( x \in \mathbb{R} \) we have \( g(f(x)) = 2x + 3 \).
Prove that \( f \) is an injection.

**Answer**

We want to show that \( f \) is an injection. From the lecture notes, we know that one way to prove that
a function \( h \) is an injection is to show that, for any pair of (not-necessarily distinct) elements \( a \) and
\( b \) in the domain of \( h \), if \( h(a) = h(b) \), then the two elements must be the same. Here we see that the
domain of \( f \) is the real numbers, \( \mathbb{R} \), so we seek to show that for any for any \( a, b \in \mathbb{R} \), if \( f(a) = f(b) \),
then \( a = b \).

To this end, let \( a, b \in \mathbb{R} \) be such that \( f(a) = f(b) \). Since we know that \( g(f(x)) = 2x + 3 \), we have:

\[
g(f(a)) = 2a + 3 \quad \quad \quad \quad \quad \quad g(f(b)) = 2b + 3
\]

Since \( f(a) = f(b) \), and \( g \) is a function, we must have that \( g(f(a)) = g(f(b)) \), as otherwise we have
found an element in the domain of \( g \) that maps to two distinct values in the codomain, violating the
fact that \( g \) is a function. Therefore, we set \( g(f(a)) \) equal to \( g(f(b)) \) and solve, from which we find:

\[
g(f(a)) = g(f(b))
\]

\[
2a + 3 = 2b + 3
\]

\[
a = b
\]

From this, we conclude that \( f \) is an injection.

47. Let \( X, Y \) be two nonempty sets and \( f : X \to Y \) a function. Define

\[
g : Y \to 2^X \quad \quad \quad \quad \quad \quad g(y) = \{ x \in X | f(x) = y \}
\]

Let \( z \in Y \) arbitrary. Prove by contrapositive that if \( g(z) = 0 \) then \( z \notin \text{Ran}(f) \)

**Answer**

The contrapositive states that if \( z \in \text{Ran}(f) \) then \( g(z) \neq 0 \).
Let \( z \in \text{Ran}(f) \). By the definition of the range of \( f \), we know that there must be some element, say
\( a \), in the domain of \( f \) such that \( f(a) = z \). Since the domain of \( f \) is \( X \), we thus know that \( \exists a \in X \)
such that \( f(a) = z \). By the definition of \( g \), we know that \( g(z) \) contains all elements in \( X \) that are mapped to \( z \) by \( f \). In other words,
\[
g(z) = \{ x \in X \mid f(x) = z \}
\]
Note that, since we have just shown that \( f(a) = z \), we must have that \( a \in g(z) \). Therefore, we have found at least one element in the set \( g(z) \), which implies \( g(z) \neq \emptyset \), concluding the proof.

48. Let \( X \) be a set and \( A, B, C \) three subsets of \( X \). Define \( \overline{C} = X \setminus C \). Prove that
\[
(A \cup B) \cap \overline{C} \subseteq A \cup (B - A - C)
\]

**Answer**
In order to show that
\[
(A \cup B) \cap \overline{C} \subseteq A \cup (B - A - C)
\]
we will show that an arbitrary element in the LHS is also an element of the RHS. To this end, let \( x \) be an arbitrary element of the LHS. We thus know that:
\[
x \in \text{LHS} \implies x \in (A \cup B) \cap \overline{C}
\]
\[
\implies (x \in (A \cup B)) \land (x \in \overline{C})
\]
\[
\implies (x \in (A \cup B)) \land (x \in X \setminus C)
\]
\[
\implies (x \in (A \cup B)) \land (x \in X \land x \notin C)
\]
From here, we see that we can split into the following two cases. Namely, that \( x \in A \) and \( x \notin A \).

**Case 1:** \( x \in A \). In this case, we know that:
\[
x \in A \implies x \in A \cup ((B - A) - C)
\]
Thus, we have shown \( x \in \text{LHS} \implies x \in \text{RHS} \), completing this case.

**Case 2:** \( x \notin A \). In this case, we have that, using our facts from above:
\[
(x \notin A) \land (x \in (A \cup B)) \implies (x \in A) \land (x \in B)
\]
\[
\implies x \in B - A
\]
From above, we note that \( x \notin C \)
\[
\implies (x \in B - A) \land (x \notin C)
\]
\[
\implies x \in (B - A) - C
\]
\[
\implies x \in A \cup ((B - A) - C)
\]
Thus, we have shown that \( x \in \text{LHS} \implies x \in \text{RHS} \), completing this case.

In both cases, we have shown that
\[
x \in (A \cup B) \cap \overline{C} \implies x \in A \cup ((B - A) - C)
\]
from which we conclude that \( (A \cup B) \cap \overline{C} \subseteq A \cup ((B - A) - C) \), completing the proof.
49. Consider sequences of bits such that $m$ of the bits are 0, where $m \geq 2$, $n$ of the bits are 1, where $n \geq 1$, and start with a 1 and end with two 0’s. How many such sequences are there?

**Answer**

Since the sequence of bits must start with a one and end with 2 zeroes, we can place those three elements in their respective positions. We now need to deal with ordering the remaining $m-2$ zeroes and $n-1$ ones. We note that we can choose $m-2$ of them to be zeroes and leave the other $n-1$ to be ones, which yields the answer: $\binom{m+n-3}{m-2}$

50. For each statement below, decide whether it is true or false. In each case attach a very brief explanation of your answer.

(a) Let $A, B$ be finite sets with $|A| = 2$ and $|B| = 3$. There are more functions $A \rightarrow B$ than functions $B \rightarrow A$, true or false?

(b) Let $X, Y$ be nonempty finite sets such that $|Y| = 1$ and such that there exists an injection $f : X \rightarrow Y$. Then $|X| = 1$, true or false?

(c) There are as many sequences of bits of length 100 that start with a 0 as sequences of bits of length 100 that end with a 1, true or false?

(d) Let $S$ be the set of the first 100 natural numbers: $S = 0, 1, \ldots, 99$. There are as many subsets of $S$ of size 40 that contain the number 40 as subsets of $S$ of size 40 that do not contain the number 40, true or false?

**Answer**

(a) TRUE.

The number of functions $f : X \rightarrow Y$ is $|Y|^{|X|}$, since for each element $x \in X$ we can have $|Y|$ possible values for $f(x)$. Each such assignment results in a function, since we map every element of $X$ to exactly one element of $Y$. There are $|B|^{|A|} = 3^2 = 9$ functions $A \rightarrow B$ and $|A|^{|B|} = 2^3 = 8$ functions $B \rightarrow A$, there are more functions $A \rightarrow B$.

(b) TRUE.

$X$ is nonempty so we must have $|X| \geq 1$. Suppose towards contradiction that $|X| \geq 2$. Then there exist $x_1 \neq x_2 \in X$. Since $Y$ has exactly one element, $y \in Y$ we must have $f(x_1) = y = f(x_2)$. This contradicts the fact that $f$ is injective, as we have two different elements of $X$ with the same image in $Y$. Thus, the only possibility left is $|X| = 1$.

(c) TRUE.

If we fix any specific bit in a 100-bit sequence, we can ignore it and treat the others as a 99-bit sequence, which has $2^{99}$ possible configurations. We find that this is the case because each of these 99 positions has two choices, independent of all other positions, and applying the multiplication rule. Here, we can fix the first to be a 0 and consider only the following 99, or fix the last bit to be a 1 and consider only the first 99, both of which lead to $2^{99}$ possible sequences.
(d) FALSE.

There are \( \binom{99}{39} \) sets that do contain the number 40 and \( \binom{99}{40} \) that don’t contain the number 40. By simplifying these values, we see that

\[
\frac{99!}{60!39!} = \frac{40 \times 99!}{59!40!} < \frac{99!}{59!40!}
\]

So the two quantities are not the same; there are more such subsets of \( S \) that do not contain the number 40 than do.

51. Let \( A, B \) be arbitrary sets. Prove that

\[
A \subseteq B \implies 2^A \subseteq 2^B
\]

**Answer**

Assume that \( A \subseteq B \). Let \( X \in 2^A \). Since \( 2^A \) is the set of subsets of \( A \):

\[
X \in 2^A \implies X \subseteq A \\
\implies X \subseteq B \\
\implies X \in 2^B
\]

(since \( A \subseteq B \))

Hence, since we have shown that any element of \( 2^A \) must be an element of \( 2^B \), we must have \( 2^A \subseteq 2^B \).

52. Let \( A, B, C, D \) be sets. Consider the following predicate (call it \( P(A, B, C, D) \))

if \( A \subseteq C \) and \( B \subseteq D \) then \( A \setminus B \subseteq C \setminus D \)

(a) This predicate is an implication. Write down its contrapositive without using any conjunction (any “and”).

(b) This predicate does not hold for arbitrary \( A, B, C, D \). Give an example of such sets for which the predicate is false.

(c) Now give an example of sets \( A, B, C, D \) for which the predicate is true.

(d) Below are two statements. Circle the one that is true.

\[
\forall A, B, C, D \ P(A, B, C, D) \quad \exists A, B, C, D \ P(A, B, C, D)
\]

**Answer**

(a) if \( A \setminus B \not\subseteq C \setminus D \) then \( A \not\subseteq C \) or \( B \not\subseteq D \)

(b) \( A = \{1\}, B = \emptyset, C = \{1\}, D = \{1\} \)

\[
A \setminus B = \{1\}, C \setminus D = \emptyset \\
A \setminus B \not\subseteq C \setminus D
\]

(Note: There are many possible correct answers for this part.)

(c) \( A = \{1\}, B = \emptyset, C = \{1, 2\}, D = \{2\} \)

\[
A \setminus B = \{1\}, C \setminus D = \{1\} \\
A \setminus B \subseteq C \setminus D
\]

(Note: There are many possible correct answers for this part too.)
(d) The true one is $\exists A, B, C, D \ P(A, B, C, D)$. This is because there is at least one assignment to the variables $A, B, C, D$ under which the predicate holds, which was proved in part (c). The alternative would require that the predicate holds under all assignments to the variables $A, B, C, D$ and this was disproved in part (b).

53. Let $a \in \mathbb{R}$ such that $a \neq 0$. Prove by contradiction that there exist at most two distinct real numbers $x$ such that $ax^2 - 1 = 0$.

**Answer**

*First solution:* Suppose, toward a contradiction, that it is not true that there exist at most two distinct real numbers $x$ such that $ax^2 - 1 = 0$.

Therefore, there exist at least three distinct real numbers $x$ such that $ax^2 - 1 = 0$. W.l.o.g. let’s name them $x_1, x_2, x_3$. We have

$$ax_1^2 - 1 = 0 \quad ax_2^2 - 1 = 0 \quad ax_3^2 - 1 = 0$$

Therefore

$$ax_1^2 - 1 = ax_2^2 - 1 = ax_3^2 - 1 \quad \text{then} \quad ax_1^2 = ax_2^2 = ax_3^2 \quad \text{and, since} \ a \neq 0 \quad x_1^2 = x_2^2 = x_3^2$$

Case 1: at least one of the $x_i$’s is 0. Then all of them are 0 and therefore equal. Contradiction.

Case 2: none of $x_i$’s are 0. Then, two of them must have the same sign, i.e., both negative or both positive. W.l.o.g. let these be $x_1$ and $x_2$. Then, since $x_1^2 = x_2^2$ we must have $x_1 = x_2$. Contradiction.

*Second solution:* Suppose for contradiction that there are at least 3 distinct real number solutions. We now manipulate the equation:

$$ax^2 - 1 = 0$$

$$ax^2 = 1$$

$$x^2 = \frac{1}{a}$$

$$x = \pm \sqrt{\frac{1}{a}}$$

Thus, $x$ equals $\sqrt{\frac{1}{a}}$ or $-\sqrt{\frac{1}{a}}$ (if $a$ is positive these are real numbers. If $a$ is negative these are complex numbers. Even if these two numbers were both real and distinct, there would still be less than 3 distinct real solutions. Hence, a contradiction.

(This was really a direct proof, lightly dressed to look like a proof by contradiction.)

54. In the remote town of Plictisitor a local ordinance prevents inhabitants from having first names, they can only have last names. These last names must start with an upper case letter followed by one to three lower case letters followed by a number between 1 and 22 (to accomodate families, you see). The lower case letters must be distinct among themselves but they can be the same letter as the
upper case at the beginning of the names. Moreover, no two inhabitants can have the same name. The alphabet used in Plictisitor has 31 letters, with lower and upper case for each of them.

What is the maximum population of Plictisitor? (Just give it as an arithmetical expression since you cannot use a calculator during the exam.)

**Answer**

All we know about Plictisitor is how many different names are allowed in this weird town. So we will assume that the maximum population of Plictisitor is the same as the total number of distinct names which we can create given the constraints mentioned in the question.

Let \( A \) be the set containing all names which have only one lower case letter, \( B \) be the set containing all names which have two lower case letters and \( C \) be the set whose members are names that have three lower case letters. It is important to note that three sets are pairwise disjoint. Therefore the total number of distinct possible names is just the sum of the cardinalities of the three sets.

We compute the number of ways of forming names that belong to set \( A \):

Choose the first upper case letter. There are 31 ways of doing this (As there are 31 letters in Plictisitor’s alphabet). Choose the lower case letter. There are again 31 ways of doing this. Finally, choose the number. There are 22 ways of doing this. Therefore the \(|A|\) is \(31 \times 31 \times 22\).

Now we compute the number of ways of forming names that belong to \( B \):

There are 31 ways of choosing the first uppercase letter. There are \(31 \times 30\) ways of choosing the two lower case letters (As they have to be distinct). Finally, there are 22 ways of choosing the number. Therefore, \(|B| = 31 \times 31 \times 30 \times 22\).

Finally, we compute the number of ways of forming names that belong to \( C \):

There are, once again, 31 ways of choosing the first uppercase letter. There \(31 \times 30 \times 29\) of choosing the three lower case letters and finally 22 ways of choosing the number. therefore, the size of \( C \) is \(31 \times 31 \times 30 \times 29 \times 22\).

Overall, the maximum possible population of Plictisitor is equal to the number of possible unique names:

\[|A| + |B| + |C| = 31 \times 31 \times 22 + 31 \times 31 \times 30 \times 22 + 31 \times 31 \times 30 \times 29 \times 22\]

You can leave it like this. Or, if you suffer from MOCD like me you can write \(31 \times 31 \times 22(1 + 30 + 30 \times 29) = 31 \times 31 \times 22(1 + 30 \times 30) = 31^2 \times 22 \times 901\). (Even my MOCD has limits!) Anyway, that’s at least 16 million so Plictisitor has, unfortunately, lots of room to grow.

55. Consider the function \(f : \mathbb{N} \rightarrow \mathbb{N}\) where \(f(x)\) equals the number of sequences of bits with \(x\) 0’s and \(x\) 1’s. Prove that \(f\) is injective.

**Answer**

We shall prove that \(f\) is strictly increasing and infer from that that \(f\) is injective. First we derive an explicit formula for \(f\) via counting principles.

Notice that for a given \(x\) a sequence of bits with \(x\) 0’s and \(x\) 1’s forms an overall sequence of \(2x\) characters. The number of such sequences can be easily counted by noticing that once we choose
the positions for the 0’s, the positions for the 1’s are entirely determined. The number of possible placements for the 0’s is \(\binom{2x}{x}\). Thus we have that: 
\[
f(x) = \frac{(2x)!}{(x!)^2}.
\]

Now, notice that:
\[
f(x + 1) = \frac{(2x + 2)(2x + 1)(2x)!}{(x + 1)^2(x!)^2} = \frac{2(x + 1)(2x + 1)}{(x + 1)^2} f(x) = \frac{4x + 2}{x + 1} f(x)
\]

It is easy to see that \(\frac{4x + 2}{x + 1} > 1, \forall x \in \mathbb{N}\), which means that \(f(x + 1) > f(x), \forall x \in \mathbb{N}\).

We have thus proven that \(f\) is strictly increasing. To prove injectivity, let \(m \neq n \in \mathbb{N}\). Assume, WLOG that \(m > n\) (when \(m < n\) the proof is essentially the same). Since \(f\) is strictly increasing: \(f(m) > f(n)\) which in turn means \(f(m) \neq f(n)\). Thus \(m\) and \(n\) are mapped to distinct values which implies \(f\) is injective.

56. Let \(n \geq 1\) be a natural number and consider the set \(\{1, 2, \ldots, n\}\). We will denote this set by \([1..n]\). (This is not entirely standard notation but it seems quite reasonable.) define \(f : \mathbb{Z}^{[1..n]} \rightarrow [1..n]\) as follows. For each non-empty subset \(S\) of \([1..n]\), define \(f(S)\) as the smallest element of \(S\), and moreover define \(f(\emptyset) = 1\)

(a) How many subsets \(S\) of \(A\) have the property \(S = \{f(S)\}\)?

(b) Prove that \(f\) is surjective.

**Answer**

(a) There are \(n\) subsets \(S\) of \(A\) such that \(S = \{f(S)\}\). These \(n\) subsets are those which have exactly one element. Given a subset \(S\) of a single natural number \(x\), the minimum of that subset will be \(x\). Thus, \(f(S) = x\).

\(S = \{x\}\) and \(\{f(S)\} = \{x\}\). Thus, for a subset of a single natural number, \(S = \{f(S)\}\).

(b) There are \(n\) subsets of \(A\) of size 1. Each of these subsets contain a different single natural number from 1 to \(n\) because each subset must be distinct. Each subset of size 1 has a value \(f(S)\) that is equal to its single element. Thus, given that there are \(n\) distinct possible values of \(f(S)\) for subsets of size 1 and given that these values range from 1 to \(n\), \(f(S)\) must output all value from 1 to \(n\) from all the input subsets of size 1.