On Thursday March 16 we will have our second midterm exam during the usual class time. The exam will take place in COHN G17 and in COLL 200.
The students whose last name begins with a letter in the range A-J will have to take the exam in COLL 200. (We’ll switch alphabetical range again for the third midterm.) The rest (K-Z) should come take the exam in our lecture room, COHN G17.
The exam will last for 60 minutes. Please be in COHN G17 or COLL 200 at 1:30PM so we have time to seat everybody properly.
This here is a midterm review document with readings, a mock (practice) midterm, and more practice problems. You should solve the practice exam while timing yourselves.
Solutions to the practice exam will be posted sometime during the Spring Break.
Val will hold a review session on Tuesday March 14, 7:30-9:00PM in Heilmeier Hall (TOWNE 100).
The TAs will hold a review session (TBA).

1 Readings

STUDY IN-DEPTH... ...the posted notes for lectures 8, 9, 10, 11, 12, 13, 14.

STUDY IN-DEPTH... ...the posted guides for recitations 4, 5, and 6 (when posted).

STUDY IN-DEPTH... ...the posted solutions to homeworks 4, 5, and 6 (when posted). Compare with your own solutions.

STUDY IN-DEPTH... ...the solutions to the mock exam and the additional problems contained in this document, to be posted at the end of the Spring Break. Until then, try very hard to solve these on your own.

2 Memorize!

Find and memorize formulas:

- For the sum of a geometric progression.
- For the sum of the integers in [1..n].
- For the sum of the squares of the integers in [1..n].
3 Mock Exam (60 minutes for 120 points)

1. (25 pts) For each statement below, decide whether it is TRUE or FALSE. In each case attach a very brief explanation of your answer.

(a) For any finite nonempty sets $A, B, C$ such that there exists a surjection with domain $A$ and codomain $B$, and also such that there exists an injection with domain $B$ and codomain $C$, we have $|A| \leq |C|$.

(b) Assume that $B$ is a set with 7 elements and that $A$ is a set with 15 elements. Then, for any function $f : A \to B$ there exist at least 3 distinct elements of $A$ that are mapped by $f$ to the same element of $B$, true or false?

(c) Let $(\Omega, P)$ be a probability space such that $|\Omega| \geq 2$. Assume that there exists $u \in \Omega$ such that $\Pr[u] > 1/2$. Then, there exists $v \in \Omega$ such that $\Pr[v] < 1/2$.

(d) There exist two distinct functions with domain and codomain $\{a, b\}$ that are their own inverses.

(e) For any three events $E, F, G$ in the same probability space, if $E \perp F$ and $F \perp G$ then $E \perp G$.

2. (20pts) Recall the definition of the Fibonacci numbers: $F_0 = 0, F_1 = 1$ and for all $m \geq 2$, $F_m = F_{m-1} + F_{m-2}$.

(a) Apply the telescopic trick to show that $\forall n \in \mathbb{N} \ F_0 + \cdots + F_n = F_{n+2} - 1$.

(b) Now prove by induction that $\forall n \in \mathbb{N} \ F_0 + \cdots + F_n = F_{n+2} - 1$.

3. (15pts) My 6th grade teacher of Russian was unable to pay attention to what we were answering and it appeared to us that he was assigning grades completely randomly. Let’s assume that his grading rubric consisted of tossing a fair coin six times, counting the number $k$ of heads and assigning the grade $4 + k$ (our grades were in the 1-10 range).

(a) What was the probability that I would get a 10?

(b) What was the probability that I would pass (get a grade of 5 or more)?

(c) What was the probability of the following event: “my grade was divisible by 4 or (non-exclusive or!) it was bigger than or equal to Lady Gaga’s shoe size (a 6)”?

4. (20 pts) Recall (and remember!) that the sum of the squares of the first $n$ positive integers is given by the following formula: $1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = n(n+1)(2n+1)/6$.

(a) Using only the formula above, (no credit in part (a) for proof by induction, see part (b)), derive the following formula for the sum of the squares of first $m$ odd positive integers. Show your work.

$$1^2 + 3^2 + 5^2 + \cdots + (2m-3)^2 + (2m-1)^2 = \frac{m(4m^2 - 1)}{3}$$

(b) Now prove by induction the formula from part (a).

5. (15pts) Let $A, B, C$ be three events in the same probability space such that $A \subseteq B$, $A \subseteq C$, $B \perp C$, and $\Pr[A] = 1$. Prove that $\Pr[A \cap B \cap C] = \Pr[A] \Pr[B] \Pr[C]$.
6. (15pts) Let $A, B$ be two sets such that $|A \cup B| = 12$ and $|A \cap B| = 8$. Prove that $96 \leq |A \times B| \leq 100$.

7. (10pts) A lottery urn contains $n \geq 2$ distinct balls labeled with the numbers $1, \ldots, n$. You extract two distinct balls from the urn. Suppose they are labeled $i$ and $j$. You compute $i + j$ and write down the answer on a piece of paper. Then you put the two balls back.

You repeat this $m$ times. What is the smallest value of $m$ that ensures (no probabilities in this problem!) that you will end up writing the same number at least twice on the piece of paper. Prove your answer.

4 Additional Problems

1. For each statement below, decide whether it is true or false. In each case attach a very brief explanation of your answer.

(a) Let $A, B, C$ be three events of non-zero probability in a probability space $(\Omega, P)$. If $A \cap B = B \cap C$, $A \perp B$, and $B \perp C$ then $\Pr[A] = \Pr[C]$.

(b) Let $P(n)$ be a predicate defined on natural numbers. Suppose we proved $P(0)$ and $P(1)$ and also $\forall k \ P(k) \Rightarrow P(k + 2)$. Then $P(a)$ is true, where $a$ is the integer closest to the product of your height, your weight and your shoe size, TRUE or FALSE?

(c) The sum $1 - 2 + 4 - 8 + \cdots + (-1)^n2^n$ is positive when $n$ is odd, TRUE or FALSE?

(d) If a probability space has an event of probability $2/3$ then it must have some outcome of probability at most $1/3$, TRUE or FALSE?

2. Consider the recurrence relation

$$a_{n+1} = a_n + 3 \ (n \geq 0) \quad \text{and} \quad a_0 = 4$$

Prove that $\forall \ n \geq 4 \ a_n \leq 2^n$.

3. Let

$$R_n = \sum_{k=1}^{2n} (-1)^{k+1}k \quad \text{for } n \geq 1$$

(a) Compute $R_1, R_2, R_3$. Guess a simple way to express $R_n$ in terms of $n$. Prove your guess by induction.

(b) Prove by induction that for all $n \geq 1$ we have

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

(c) Use the identity in part (b) and other identities that you were supposed to memorize to prove the identity in part (a).
4. Bob is recycling a set $B$ of $m \geq 1$ distinguishable (he likes variety) bottles $B = \{b_1, \ldots, b_m\}$ in a facility that has a set $D$ of $n \geq 2$ distinguishable drums, $D = \{d_1, \ldots, d_n\}$. When Bob shows up all the drums are empty. Each drum is large enough to hold by itself all of Bob’s $m$ bottles. We call a deposit a way of placing the bottles in the drums, i.e., a function $t : B \rightarrow D$. Each deposit may leave some drums (maybe none) empty. Let $\text{empty}(t)$ be the set consisting of all the drums that are left empty by deposit $t$. (Note that it might be the case that $\text{empty}(t) = \emptyset$, depending on $m, n$ and $t$.)

Assume $m \geq n$ and prove that there exist two different deposits, $t_1$ and $t_2$ such that $\text{empty}(t_1) = \text{empty}(t_2)$.

5. Let $n \geq 2$ and let $a_1a_2\ldots a_n$ be a sequence of $n$ integers (they do not have to be pairwise distinct). Prove that there exist $p, q \in [0..n]$ such that $p < q$ and $\sum_{i=p+1}^{q} a_i$ is divisible by $n$.

6. For each statement below, decide whether it is true or false. In each case attach a very brief explanation of your answer.

   (a) If $n \geq 1$ then $1 + 3 + \cdots + 3^n < (3/2) 3^n$, true or false?

   (b) Let $A, B$ be events in a probability space such that $\Pr[A] = 0$ and $\Pr[B] \neq 0$. Then, $\Pr[A \mid B] = 0$, true or false?

   (c) For any probability space $(\Omega, P)$ and any event $A \subseteq \Omega$ such that $\Pr[A] \neq 0$ we have $\Pr[\Omega \mid A] = \Pr[A \mid \Omega]$, true or false?

7. Prove by induction on $n$ that for any $n \in \mathbb{N}$, $n \geq 1$ we have

   $$\sum_{i=1}^{n} (-1)^i \cdot i = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

8. In an All-Milky Way course the students receive their graded homeworks consisting of $n \geq 2$ problems, where each problem is given a score between 0 and $m \geq 1$. Assume that there are enough students (hence the galaxy-wide offering :) such that each set of possible scores on each problem is represented in the scores received by the students. For any two students $a$ and $b$ define $\text{Same}(a, b)$ to be the set of all homework problems on which $a$ and $b$ got the same scores.

   Use the Pigeonhole Principle to prove that there exist four students, $a, b, c, d$ such that

   - $\text{Same}(a, b) = \text{Same}(c, d)$, and
   - $a \neq b$, and
   - $c \neq d$, and
   - $a \neq c \text{ OR } b \neq d$

9. Consider the recurrence relation

   $$a_0 = 0 \quad a_1 = 1 \quad a_n = 2a_{n-1} - a_{n-2} + 1 \quad (\text{for } n \geq 2)$$

   Express $a_n$ as a polynomial in $n$. (Hint: use the telescopic trick twice.) Then prove by induction the result you obtained.
10. For each statement below, decide whether it is true or false. In each case attach a very brief explanation of your answer.

(a) Let \( E, F \) be two events in a finite probability space. If \(|E| = |F|\) then \( \Pr[E] = \Pr[F] \), true or false?

(b) If \( E, F \) are two events in a finite probability space such that \( \Pr[E \cap F] > 0 \) then \( E \) and \( F \) can be disjoint, true or false?

11. Let \( A, B, C \) be arbitrary finite sets. Let \( m = |A| + |B| + |C| - |A \cup B \cup C| \) and \( n = |A \cap B| + |B \cap C| + |C \cap A| \). Prove that \( m \leq n \).

12. Let \( E, F \) be two events in a finite probability space such that \( \Pr[E \cap F] > 0 \). Prove that \( \Pr[E \setminus F] + \Pr[F \setminus E] < \Pr[E \cup F] \).

13. Prove by ordinary induction on \( n \) that for any \( n \in \mathbb{N}, n \geq 1 \) we have

\[
1 + 2 + \cdots + n - 1 + n + n - 1 + \cdots + 2 + 1 = n^2
\]

14. For each statement below, decide whether it is true or false. In each case attach a very brief explanation of your answer.

(a) Assume that \( A, B \) are finite nonempty sets and \( f : A \to B \) is a function such that there exist at least 3 distinct elements of \( A \) that are mapped by \( f \) to the same element of \( B \). Then \(|A| > 2 \cdot |B|\), true or false?

(b) Let \( A, B \) be events in a finite probability space such that \( \Pr[A] = 1/4 \) and \( \Pr[A \cup B] = 1/2 \). Then, \( 1/4 \leq \Pr[B] \leq 1/2 \), true or false?

15. Let \( A, B, C \) be arbitrary finite sets. Prove that \( |A| + |B| + |C| \geq |A \cap B| + |B \cap C| + |C \cap A| \).

16. Let \( X \) be a finite nonempty set and \( f : X \to X \). Let \( x \in X \) arbitrary and consider the sequence

\[
x, \ f(x), \ f(f(x)), \ \ldots, f(\cdots f(x)\cdots), \ \ldots
\]

Prove that for any \( k \geq 2 \) there must exist \( k \) distinct positions in this sequence in which the same element of \( X \) occurs.

17. Prove by ordinary induction on \( n \) that for any \( n \in \mathbb{N}, n \geq 1 \) we have that for any two sets \( A, B \) with \(|A| = 2\) and \(|B| = n\) the number of functions with domain \( A \) and codomain \( B \) is \( n^2 \). (Indeed we have counted functions before and we know that this is the correct count. Here you must prove it by induction.)

18. Let \( A, B, C \) be finite sets such that \( B \cap C = \emptyset \). Using the principle of inclusion-exclusion prove that

\[
|A \cup B \cup C| + |A| = |A \cup B| + |A \cup C|
\]

19. Prove that any positive integer can be expressed as the sum of distinct Fibonacci numbers.

20. Prove by induction that given an unlimited supply of 6-cent coins, 10-cent coins, and 15-cent coins, one can make any amount of change larger than 29 cents.