On Thursday March 16 we will have our second midterm exam during the usual class time. The exam will take place in COHN G17 and in COLL 200.

The students whose last name begins with a letter in the range \( A-J \) will have to take the exam in COLL 200. (We’ll switch alphabetical range again for the third midterm.) The rest (\( K-Z \)) should come take the exam in our lecture room, COHN G17.

The exam will last for 60 minutes. Please be in COHN G17 or COLL 200 at 1:30PM so we have time to seat everybody properly.

This here is a midterm review document with readings, a mock (practice) midterm, and more practice problems. You should solve the practice exam while timing yourselves.

Solutions to the practice exam will be posted sometime during the Spring Break.

Val will hold a review session on Tuesday March 14, 7:30-9:00PM in Heilmeier Hall (TOWNE 100).

The TAs will hold a review session (TBA).

1 Readings

**STUDY IN-DEPTH...** ...the posted notes for lectures 8, 9, 10, 11, 12, 13, 14.

**STUDY IN-DEPTH...** ...the posted guides for recitations 4, 5, and 6 (when posted).

**STUDY IN-DEPTH...** ...the posted solutions to homeworks 4, 5, and 6 (when posted). Compare with your own solutions.

**STUDY IN-DEPTH...** ...the solutions to the mock exam and the additional problems contained in this document, to be posted at the end of the Spring Break. Until then, try very hard to solve these on your own.

2 Memorize!

Find and memorize formulas:

- For the sum of a geometric progression.
- For the sum of the integers in \([1..n]\).
- For the sum of the squares of the integers in \([1..n]\).
3 Mock Exam (60 minutes for 120 points)

1. (25 pts) For each statement below, decide whether it is TRUE or FALSE. In each case attach a very brief explanation of your answer.

(a) For any finite nonempty sets $A, B, C$ such that there exists a surjection with domain $A$ and codomain $B$, and also such that there exists an injection with domain $B$ and codomain $C$, we have $|A| \leq |C|$.

**Answer** FALSE. Counterexample: take $A = \{1, 2, 3\}, B = \{4\}$ and $C = \{5, 6\}$, the surjection $f : A \to B$ to map $f(1) = f(2) = f(3) = 4$ and the injection $g : B \to C$ to map $g(4) = 5$. Here, however, we see that $|A| > |C|$ which disproves the claim.

(b) Assume that $B$ is a set with 7 elements and that $A$ is a set with 15 elements. Then, for any function $f : A \to B$ there exist at least 3 distinct elements of $A$ that are mapped by $f$ to the same element of $B$, true or false?

**Answer** TRUE. From the Generalized Pigeonhole Principle we know that if $|A| > k \cdot |B|$ then there exist at least $k + 1$ distinct elements of $A$ that are mapped by $f$ to the same element of $B$. Here we take $k = 2$ and we note that $|A| = 15 > 2 \cdot 7 = k \cdot |B|$.

(c) Let $(\Omega, P)$ be a probability space such that $|\Omega| \geq 2$. Assume that there exists $u \in \Omega$ such that $\Pr[u] > 1/2$. Then, there exists $v \in \Omega$ such that $\Pr[v] < 1/2$.

**Answer** TRUE. Consider $E = \Omega \setminus \{u\}$. Since $|\Omega| \geq 2$, $E$ must be non-empty. This means there must exist an outcome $v \in E$. Then $\Pr[E] = 1 - \Pr[u] < 1 - 1/2 = 1/2$ Therefore, all the outcomes in $E$ have probability $< 1/2$. In particular, $\Pr[v] < 1/2$.

(d) There exist two distinct functions with domain and codomain $\{a, b\}$ that are their own inverses.

**Answer** TRUE. Define $f : \{a, b\} \to \{a, b\}$ by $f(a) = a$ and $f(b) = b$ and observe that for any $x \in \{a, b\}$ we have $f(f(x)) = x$. Then define $g : \{a, b\} \to \{a, b\}$ by $g(a) = b$ and $g(b) = a$ and observe that for any $x \in \{a, b\}$ we have $g(g(x)) = x$. Therefore, both of these functions are their own inverses.

(e) For any three events $E, F, G$ in the same probability space. if $E \perp F$ and $F \perp G$ then $E \perp G$.

**Answer** FALSE. Take $E, F$ such that $E \perp F$, $E$ such that $\Pr[E] = 1/2$, and $G = E$. Obviously, since $E \perp F$ and $G = E$, then $G \perp F$. However, let’s look at if $E \perp G$. we can say that this is not the case $(E \not\perp G)$ because $\Pr[E \cap G] = \Pr[E \cap E] = \Pr[E] = 1/2$ while $\Pr[E] \cdot \Pr[G] = (1/2)(1/2) = 1/4$.

2. (20pts) Recall the definition of the Fibonacci numbers: $F_0 = 0, F_1 = 1$ and for all $m \geq 2, F_m = F_{m-1} + F_{m-2}$.

(a) Apply the telescopic trick to show that $\forall n \in \mathbb{N} \ F_0 + \cdots + F_n = F_{n+2} - 1$.

**Answer** First, let’s write $F_m = F_{m-1} + F_{m-2}$ for $m = 2, \ldots, n + 2$:

\[
F_{n+2} = F_{n+1} + F_n \\
F_{n+1} = F_n + F_{n-1} \\
F_n = F_{n-1} + F_{n-2}
\]
\[
F_3 = F_2 + F_1 \\
F_2 = F_1 + F_0
\]

Now, since these are all definitions known to be true, we can add up all the equations and manipulate both sides of that. When we sum up all the equations, we get

\[
\sum_{i=2}^{n+2} F_i = \sum_{i=2}^{n+2} (F_{i-1} + F_{i-2}) \\
\sum_{i=2}^{n+2} F_i = \sum_{i=2}^{n+2} F_{i-1} + \sum_{i=2}^{n+2} F_{i-2} \\
F_{n+2} + \sum_{i=2}^{n+1} F_i = \sum_{i=1}^{n+1} F_i + \sum_{i=0}^{n} F_i \\
F_{n+2} + \sum_{i=2}^{n+1} F_i = F_1 + \sum_{i=2}^{n+2} F_i + \sum_{i=0}^{n} F_i \\
F_{n+2} = 1 + \sum_{i=0}^{n} F_i \\
F_{n+2} - 1 = \sum_{i=0}^{n} F_i
\]

(b) Now prove by induction that \(\forall n \in \mathbb{N} \ F_0 + \cdots + F_n = F_{n+2} - 1\).

**Answer**

**Base Case:** \(n = 0\). Here, we want to show \(F_0 = F_2 - 1\). Since we know both values, we can simply plug in and check. We get that \(1 = 2 - 1 = 1\). Therefore, the base case holds.

**Induction Hypothesis:** Let \(P(n)\) be the claim that \(F_0 + \cdots + F_n = F_{n+2} - 1\). Assume that for some \(k \in \mathbb{N}\), \(P(K)\) holds.

**Induction Step:** We want to show that \(P(k + 1)\) also holds. In other words, we want to show that \(F_0 + \cdots + F_{k+1} = F_{k+3} - 1\). Let’s look at the left side. We can split it up in a pretty cool way to get back to the induction hypothesis. We see that:

\[
F_0 + \cdots + F_{k+1} \\
(F_0 + \cdots + F_k) + F_{k+1} \\
P(k) + F_{k+1} \\
F_{k+2} - 1 + F_{k+1} \\
(F_{k+1} + F_{k+2}) - 1 \\
F_{k+3} - 1
\]
Please note that we used the identity that $F_{k+1} + F_{k+2} = F_{k+3}$ for the last step. We have brought the left hand side to the right hand side using the induction hypothesis and thus the induction step holds.

3. (15 pts) My 6th grade teacher of Russian was unable to pay attention to what we were answering and it appeared to us that he was assigning grades completely randomly. Let’s assume that his grading rubric consisted of tossing a fair coin six times, counting the number $k$ of heads and assigning the grade $4 + k$ (our grades were in the 1-10 range).

(a) What was the probability that I would get a 10?

(b) What was the probability that I would pass (get a grade of 5 or more)?

(c) What was the probability of the following event: “my grade was divisible by 4 or (non-exclusive or!) it was bigger than or equal to Lady Gaga’s shoe size (a 6)”?

**Answer**

We work with a uniform probability space $\Omega$ with $2^6$ outcomes. Each outcome is a sequence of length 6 of $H$’s and $T$’s and each outcome has probability $1/2^6$.

(a) To get a 10 we must have $k = 6$ therefore the event $E$ of interest consists of the one outcome with exactly 6 heads.

$$\Pr[E] = \frac{|E|}{|\Omega|} = \frac{1}{2^6}$$

(b) To pass we must have $k \geq 1$ therefore the event $F$ of interest consists of all outcomes with at least one head. Its complement $\overline{E}$ consists of just one outcome, the sequence with 6 tails. Therefore

$$\Pr[F] = 1 - \Pr[\overline{E}] = 1 - \frac{|E|}{|\Omega|} = 1 - \frac{1}{2^6}$$

(c) The grade can be (4 or 8) or (6 or 7 or 8 or 9 or 10) therefore $k = 0$ or $k \geq 2$. The event $G$ of interest consists of sequences with no heads or with two or more heads. Its complement, $\overline{G}$ consists of sequences with exactly one head. The one head can be in any of the 6 flips so there are 6 such sequences. Therefore

$$\Pr[G] = 1 - \Pr[\overline{G}] = 1 - \frac{|G|}{|\Omega|} = 1 - \frac{6}{2^6}$$

4. (20 pts) Recall (and remember!) that the sum of the squares of the first $n$ positive integers is given by the following formula: $1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = n(n+1)(2n+1)/6$.

(a) Using only the formula above, (no credit in part (a) for proof by induction, see part (b)), derive the following formula for the sum of the squares of first $m$ odd positive integers. Show your work.

$$1^2 + 3^2 + 5^2 + \cdots + (2m-3)^2 + (2m-1)^2 = \frac{m(4m^2-1)}{3}$$

**Answer**

Let $D(m) = \sum_{k=1}^{m} (2k-1)^2$. To this we add $E(m) = \sum_{k=1}^{m} (2k)^2$ and we obtain $T(m) = \sum_{i=1}^{2m} i^2$. By the formula, $T(m) = 2m(2m+1)(4m+1)/6$. Meanwhile, $E(m) = \sum_{k=1}^{m} (2k)^2 = 2^2 \sum_{k=1}^{m} k^2 = 2^2 \cdot m(m+1)(2m+1)/6$ (using again the formula). Hence

$$D(m) = T(m) - E(m) = \frac{2m(2m+1)(4m+1)}{6} - \frac{4m(m+1)(2m+1)}{6}$$
\[
\frac{2m(2m+1)(4m+1-2(m+1))}{6} = \frac{2m(2m+1)(2m-1)}{6} = \frac{m(4m^2-1)}{3}
\]

(b) Now prove by induction the formula from part (a).

**Answer** Base case \(m = 1\). Induction step requires some algebra.

5. (15pts) Let \(A, B, C\) be three events in the same probability space such that \(A \subseteq B\), \(A \subseteq C\), \(B \perp C\), and \(\Pr[A] = 1\). Prove that \(\Pr[A \cap B \cap C] = \Pr[A] \Pr[B] \Pr[C]\).

**Answer** Since \(A \subseteq B\) and \(A \subseteq C\), we know that \(A \subseteq B \cap C\) (one way to reason about this is to observe that \(A \cap C \subseteq B \cap C\), and plug in \(A = A \cap C\)). Therefore, \(A \cap B \cap C = A\).

Moreover, by monotonicity of probability, \(A \subseteq B\) implies \(\Pr[A] \leq \Pr[B]\). Since \(1 = \Pr[A] \leq \Pr[B] \leq 1\) we have \(1 \leq \Pr[B] \leq 1\), which means that we must have \(\Pr[B] = 1\). Similarly, we can show that \(\Pr[C] = 1\).

Therefore, \(\Pr[A \cap B \cap C] = \Pr[A] = 1 = 1 \cdot 1 = \Pr[A] \Pr[B] \Pr[C]\).

6. (15pts) Let \(A, B\) be two sets such that \(|A \cup B| = 12\) and \(|A \cap B| = 8\). Prove that \(96 \leq |A \times B| \leq 100\).

**Answer** Let \(x = |A|\) and \(y = |B|\). By the Principle of Inclusion-Exclusion, we have

\[12 = |A \cup B| = x + y - |A \cap B| = x + y - 8\]

Hence, \(x + y = 20\) and \(y = 20 - x\).

We know that \(|A \times B| = |A| \times |B| = xy = x(20 - x)\). So we need to show that \(96 \leq x(20 - x) \leq 100\).

Observe that we can rearrange the inequality

\[
x(20 - x) \leq 100 \\
20x - x^2 \leq 100 \\
0 \leq x^2 - 20x + 100 \\
0 \leq (x - 10)^2
\]

which is always true (any real number squared is always nonnegative).

Similarly, we can take

\[
96 \leq x(20 - x) \\
96 \leq 20x - x^2 \\
x^2 - 20x + 96 \leq 0 \\
(x - 8)(x - 12) \leq 0
\]

But \(A \cap B \subseteq A \subseteq A \cup B\) so \(|A \cap B| \leq |A| \leq |A \cup B|\). Hence, \(8 \leq x \leq 12\). It follows that \((x - 8)(x - 12) \leq 0\) (since we multiply a nonnegative number by a nonpositive one). Thus, we know that both inequalities hold.

Alternatively, we could solve this by using cases based on the size of \(A\) and \(B\). We know that \(8 \leq |A| \leq 12\), and that \(|A \cap B| = 8\). This means that \(0 \leq |A \setminus B| \leq 4\). Additionally, \(|A \setminus B| + |B \setminus A| = 4\) (we
can apply the Sum Rule, as we have disjoint sets). Then we know that the ordered pair corresponding to the sizes of $A$ and $B$ must be among the following possibilities:

$$\{(8, 12), (9, 11), (10, 10), (11, 9), (12, 8)\}$$

Testing each of these shows that the inequalities hold in all cases.

7. (10pts) A lottery urn contains $n \geq 2$ distinct balls labeled with the numbers $1, \ldots, n$. You extract two distinct balls from the urn. Suppose they are labeled $i$ and $j$. You compute $i + j$ and write down the answer on a piece of paper. Then you put the two balls back.

You repeat this $m$ times. What is the smallest value of $m$ that ensures (no probabilities in this problem!) that you will end up writing the same number at least twice on the piece of paper. Prove your answer.

**Answer** We first find a value of $m$ that ensures that there will be the same number written twice on the piece of paper using the Pigeonhole Principle. To this end, we let the holes be the possible numbers written on the piece of paper, and let the pigeons be the $m$ extractions from the urn.

We now find the number of possible holes. Note that the smallest number you can write down is $1 + 2 = 3$, and the largest number you can write down is $(n - 1) + n = 2n - 1$. There are $(2n - 1) - 3 + 1 = 2n - 3$ numbers in this range.

However, in order to help us find the smallest value of $m$ possible, we need to make sure that all of these holes are valid. We show that every $k \in [3, 2n - 1]$ could be one of the numbers written down. Indeed, when $3 \leq k \leq n + 1$ then we know that the extraction $\{1, k - 1\}$ will yield $k$. This is a possible extraction since $2 \leq k - 1 \leq n$. Similarly, when $n + 2 \leq k \leq 2n - 1$, then the extraction $\{n, k - n\}$ will yield $k$. This is also a possible extraction because $2 \leq k - n \leq n - 1$.

Thus, we know that all $2n - 3$ holes are valid. By the Pigeonhole Principle, if $m = 2n - 2$, then at least one number will be written down at least twice.

However, the Pigeonhole Principle does not tell us specifically whether $2n - 2$ is the smallest such $m$ that ensures that a number will be written twice on the piece of paper. On the other hand, if $m$ is smaller, then we can generate a sequence of extractions where the paper contains no repeated numbers.

We construct such a sequence when we have $m = 2n - 3$ extractions. We can get all distinct numbers on the paper with the following sequence of $2n - 3$ extractions: $\{1, i\}$ for $i = 2, \ldots, n$, for a total of $n - 1$ extractions, followed by $\{j, n\}$ for $j = 2, \ldots, n - 1$, for another $n - 2$ extractions. It is clear to see that each such extraction defined above yields a unique number. Note that if $m$ is even smaller we just consider a subset of these extractions.

### 4 Additional Problems

1. For each statement below, decide whether it is true or false. In each case attach a very brief explanation of your answer.

   (a) Let $A, B, C$ be three events of non-zero probability in a probability space $(\Omega, P)$. If $A \cap B = B \cap C$, $A \perp B$, and $B \perp C$ then $\Pr[A] = \Pr[C]$.
Answer TRUE. We first observe that, as $A$ and $B$ are independent, we have that

$$\Pr[A] \cdot \Pr[B] = \Pr[A \cap B]$$

However, we also have that $A \cap B = B \cap C$, meaning $\Pr[A \cap B] = \Pr[B \cap C]$. Furthermore, since $B$ and $C$ are independent, we have that:

$$\Pr[B] \cdot \Pr[C] = \Pr[B \cap C]$$

Combining these facts, we have that:

$$\Pr[A] \cdot \Pr[B] = \Pr[B] \cdot \Pr[C]$$

Finally, since $\Pr[B] \neq 0$, we know that $\Pr[A] = \Pr[C]$, completing the proof.

(b) Let $P(n)$ be a predicate defined on natural numbers. Suppose we proved $P(0)$ and $P(1)$ and also $\forall k \ P(k) \Rightarrow P(k + 2)$. Then $P(a)$ is true, where $a$ is the integer closest to the product of your height, your weight and your shoe size, TRUE or FALSE?

Answer TRUE. What we proved amounts to two proofs by induction, the first involving a Base Case of 0, and the second involving a Base Case of 1. Note that the former proves the claim for all even natural numbers, i.e., the first induction proof shows $\forall e \in \mathbb{N}, P(2e)$ holds. Similarly, the second induction proof proves the claim for all odd natural numbers, i.e., $\forall o \in \mathbb{N}, P(2o + 1)$ holds.

Since we know that all natural numbers are either even or odd, we know that together, these two proofs imply $\forall n \in \mathbb{N}, P(n)$ holds. In particular, since $a$ is a positive integer, we know that $a \in \mathbb{N}$, so $P(a)$ holds.

(c) The sum $1 - 2 + 4 - 8 + \cdots + (-1)^n2^n$ is positive when $n$ is odd, TRUE or FALSE?

Answer FALSE. A simple counterexample is when $n = 1$, as we then have the sum:

$$1 - 2 = -1$$

which is clearly negative.

(d) If a probability space has an event of probability $2/3$ then it must have some outcome of probability at most $1/3$, TRUE or FALSE?

Answer TRUE. Let $A$ be the event such that $\Pr[A] = \frac{2}{3}$. Then, we know that the complement of $A$, $A$, has probability

$$\Pr[A] = \frac{1}{3}$$

Since $\Pr[A] \neq 0$, $A \neq \emptyset$, meaning $A$ contains at least one outcome, call it $w$. Then, we know that:

$$\Pr[w] \leq \Pr[A] = \frac{1}{3}$$

Thus, $w$ is an outcome with probability at most $\frac{1}{3}$. 

7
2. Consider the recurrence relation

\[ a_{n+1} = a_n + 3 \quad (n \geq 0) \quad \text{and} \quad a_0 = 4 \]

Prove that \( \forall \ n \geq 4 \ a_n \leq 2^n \).

**Answer** We use the recurrence relation to manually calculate \( a_1, a_2 \) and \( a_3 \) (which will come in handy when we calculate \( a_4 \) for our Base Case). \( a_1 = 4 + 3 = 7. \ a_2 = 7 + 3 = 10. \ a_3 = 10 + 3 = 13. \)

Now we prove by weak induction that \( \forall \ n \geq 4 \ a_n \leq 2^n \).

**Base Case** \( n = 4 \). Then \( a_4 = a_3 + 3 = 13 + 3 = 16 \leq 2^4 \).

**Induction Hypothesis** Assume for some integer \( k \geq 4 \) that \( a_k \leq 2^k \).

**Induction Step** We wish to prove that \( a_{k+1} \leq 2^{k+1} \)

\[
\begin{align*}
a_{k+1} &= a_k + 3 \\
a_{k+1} &\leq 2^k + 3 \quad \text{(by IH)} \\
a_{k+1} &\leq 2^k + 2^k \quad \text{(since} \ k \geq 4) \\
a_{k+1} &\leq 2(2^k) \leq 2^{k+1} \quad \text{and we are done.}
\end{align*}
\]

3. Let

\[ R_n = \sum_{k=1}^{2n} (-1)^{k+1} k \quad \text{for} \ n \geq 1 \]

(a) Compute \( R_1, R_2, R_3 \). Guess a simple way to express \( R_n \) in terms of \( n \). Prove your guess by induction.

(b) Prove by induction that for all \( n \geq 1 \) we have

\[ 1 + 3 + 5 + \cdots + (2n-1) = n^2 \]

(c) Use the identity in part (b) and other identities from class to prove the identity in part (a) without the need of induction.

**Answer**

(a) \( R_1 = 1 - 2 = -1. \ R_2 = 1 - 2 + 3 - 4 = -2. \ R_3 = 1 - 2 + 3 - 4 + 5 - 6 = -3. \)

We guess \( \forall n \geq 1 \ R_n = -n \), which we will now prove by weak induction.

**Base Case** \( n = 1 \) We’ve already shown this above.

**Induction Hypothesis**: Assume for some arbitrary integer \( k \geq 1 \) that \( R_k = -k \)

**Induction Step**: Using IH, \( R_{k+1} = R_k + (2k+1) - (2k + 2) = -k + 2k + 1 - 2k - 2 = -k - 1 = -(k + 1) \). Done.

(b) **Base Case**: \( n = 1. \ 1 = 1^2 \) Check.

**Induction Hypothesis**: Assume for some arbitrary integer \( k \geq 1 \) that \( 1 + 3 + \cdots + (2k-1) = k^2 \)

**Induction Step**: Using IH, \( 1 + 3 + \cdots + (2k-1) + (2(k + 1) - 1) = k^2 + (2k + 1) = (k + 1)^2 \).

Done.
(c) We know that
\[ 1 + 2 + 3 + 4 + \cdots + (2n) = \frac{2n(2n+1)}{2} \]

Let’s write the definition of \( R_n \) in part (a) like this
\[ 1 - 2 + 3 - 4 + \cdots - (2n) = R_n \]

Adding LHS and the RHS of these two equalities and canceling \( 2 - 2 = 0 \) etc., we have
\[ 1 + 1 + 3 + 3 + \cdots + (2n - 1) + (2n - 1) = \frac{2n(2n+1)}{2} + R_n \]

By part (b)
\[ 2 \cdot n^2 = \frac{2n(2n+1)}{2} + R_n \]

Now it’s just algebra
\[ R_n = 2n^2 - \frac{2n(2n+1)}{2} = 2n^2 - 2n^2 - n = -n \]

4. Bob is recycling a set \( B \) of \( m \geq 1 \) distinguishable (he likes variety) bottles \( B = \{b_1, \ldots, b_m\} \) in a facility that has a set \( D \) of \( n \geq 2 \) distinguishable drums, \( D = \{d_1, \ldots, d_n\} \). When Bob shows up all the drums are empty. Each drum is large enough to hold by itself all of Bob’s \( m \) bottles. We call a deposit a way of placing the bottles in the drums, i.e., a function \( t : B \to D \). Each deposit may leave some drums (maybe none) empty. Let empty\((t)\) be the set consisting of all the drums that are left empty by deposit \( t \). (Note that it might be the case that empty\((t) = \emptyset\), depending on \( m, n \) and \( t \).)

Assume \( m \geq n \) and prove that there exist two different deposits, \( t_1 \) and \( t_2 \) such that empty\((t_1) = \) empty\((t_2)\).

Answer We could provide a specific example of two deposits \( a \) and \( b \), such that empty\((a) = \) empty\((b)\) (there are many such examples). For example, let \( a \) be the deposit that places \( b_1 \) into \( d_1 \) and all the remaining bottles into \( d_2 \). Let \( b \) be the deposit that places \( b_2 \) into \( d_1 \) and all the remaining bottles into \( d_2 \). Clearly, \( a \) and \( b \) are different and satisfy the condition empty\((a) = \) empty\((b)\). The claim follows.

Alternatively, we can apply PHP to prove the claim without providing a specific example. There are \( n^m \) deposits. These are the pigeons. The function that maps pigeons to pigeonholes takes a deposit \( t \) to the subset empty\((t) \subseteq D \). But what are the pigeonholes? Not all the subsets of \( D \), but rather just those that equal empty\((t)\) for some deposit \( t \).

How many pigeonholes are there? The only subset that cannot equal empty\((t)\) for some deposit \( t \) is \( D \) itself (because we have \( m \geq 1 \) bottles and we must put them somewhere).

Let’s prove that for every subset of \( S \subseteq D \) such that \( S \neq D \), there exists a deposit \( t \) such that \( S = \) empty\((t)\). Indeed, \( D \setminus S \) has at least one drum and at most \( n \) drums. But \( m \geq n \) so we can distribute the \( m \) bottles among the drums of \( D \setminus S \), at least one bottle in each.

Therefore the number of pigeonholes is \( 2^n - 1 \). To apply PHP we must check that \( n^m > 2^n - 1 \).

Now, \( n^m \geq 2^m \geq 2^n > 2^n - 1 \), using \( m \geq n \) and \( n \geq 2 \). The claim follows.
Here are some more calculations in the setting of this problem, just for general practice. Assume \( n \geq m \).

For any deposit \( t : B \to D \), the minimum number of drums that can be used is 1 (all bottles in one drum), and the maximum number of drums that can be used is \( m \) (one bottle in each of \( m \) drums, this is possible because there are enough drums, \( n \geq m \)). That means that the minimum number of drums that can be left empty is \( n - m \) and the maximum number of drums that can be left empty is \( n - 1 \).

For any \( n - m \leq k \leq n - 1 \) it is clear that there exists at least one deposit \( t \) such that \( |\text{empty}(t)| = k \) (just distribute the \( m \) bottles among \( n - k \) drums).

Therefore, a subset \( S = \text{empty}(t) \) for some deposit \( t \) iff \( n - m \leq |S| \leq n - 1 \). The number of these subsets is

\[
\binom{n}{n-m} + \cdots + \binom{n}{n-1}.
\]

5. Let \( n \geq 2 \) and let \( a_1a_2\ldots a_n \) be a sequence of \( n \) integers (they do not have to be pairwise distinct).

Prove that there exist \( p, q \in [0..n] \) such that \( p < q \) and \( \sum_{i=p+1}^{q} a_i \) is divisible by \( n \).

**Answer** For each \( k = 1, \ldots, n \) define \( b_k = \sum_{i=1}^{k} a_i \). In addition, we define \( b_0 = 0 \).

We apply PHP as follows.

There are \( n + 1 \) pigeons: the numbers \( 0, 1, 2, \ldots, n \).

There are \( n \) pigeonholes: the numbers \( 0, 1, \ldots, n - 1 \).

We assign the pigeon \( k \) to the pigeonhole that corresponds to the remainder of the division of \( b_k \) by \( n \). By PHP, there must exist two distinct pigeons \( p, q \) assigned to the same pigeonhole. WLOG, suppose \( p < q \). Then, \( b_q - b_p \) is divisible by \( n \).

But, \( b_q - b_p = \sum_{i=p+1}^{q} a_i \) (note that this holds even when \( p = 0 \)). This proves the claim.

6. For each statement below, decide whether it is true or false. In each case attach a very brief explanation of your answer.

(a) If \( n \geq 1 \) then \( 1 + 3 + \cdots + 3^n < (3/2) 3^n \), true or false?

**Answer** TRUE

Note that the LHS is the sum \( a \) of a geometric progression. Recall the formula \( S \) for the summation of a geometric sequence.

\[
S = 1 + x + x^2 + \cdots + x^n
\]

\[
x \cdot S = x + x^2 + \cdots + x^n + x^{n+1}
\]

\[
(1-x)S = 1 - x^{n+1}
\]

\[
S = \frac{1 - x^{n+1}}{1-x}
\]
The LHS then evaluates to:

\[
LHS = 1 + 3 + \ldots + 3^n
\]

\[
= \frac{1 - 3^{n+1}}{1 - 3}
\]

\[
= \frac{3^{n+1} - 1}{2}
\]

\[
< \frac{3^{n+1}}{2}
\]

\[
= \frac{3}{2} \cdot 3^n
\]

\[= RHS\]

We have thus showed that \(LHS < RHS\).

(b) Let \(A, B\) be events in a probability space such that \(\Pr[A] = 0\) and \(\Pr[B] \neq 0\). Then, \(\Pr[A \mid B] = 0\), true or false?

**Answer** TRUE

Since \(A \cap B \subset A\), we know by monotonicity of probability that \(0 \leq \Pr[A \cap B] \leq \Pr[A] = 0\) so \(\Pr[A \cap B] = 0\).

Therefore \(\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} = 0\).

(c) For *any* probability space \((\Omega, P)\) and *any* event \(A \subseteq \Omega\) such that \(\Pr[A] \neq 0\) we have \(\Pr[\Omega \mid A] = \Pr[A \mid \Omega]\), true or false?

**Answer** FALSE

We proceed with a disproof by counterexample.

Let \((\Omega, P)\) be the probability space of one flip of a fair coin. Further, let \(A\) be the event that the coin shows heads.

\[
\Pr[\Omega \mid A] = \frac{\Pr[\Omega \cap A]}{\Pr[A]}
\]

\[
= \frac{\Pr[A]}{\Pr[A]} \quad \text{(Because } A \subseteq \Omega\text{)}
\]

\[= 1\]

\[
\Pr[A \mid \Omega] = \frac{\Pr[A \cap \Omega]}{\Pr[\Omega]}
\]

\[
= \frac{\Pr[A]}{\Pr[\Omega]}
\]

\[= \frac{1}{2} \neq 1\]

Since \(\Pr[\Omega \mid A] \neq \Pr[A \mid \Omega]\), the counterexample disproves the claim.
7. Prove by induction on \(n\) that for any \(n \in \mathbb{N}, n \geq 1\) we have

\[
\sum_{i=1}^{n} (-1)^i \cdot i = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
-\frac{n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

**Answer**

Let \(P(n)\) for some \(n \in \mathbb{N}\) be the claim:

\[
\sum_{i=1}^{n} (-1)^i \cdot i = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
-\frac{n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

**(BASE CASE)** \(n = 1, 2\)

\((-1)^1 \cdot 1 = -1 = -\frac{1+1}{2}\) and \((-1)^1 \cdot 1 + (-1)^2 \cdot 2 = 1 = \frac{2}{2}.\) \(P(1)\) and \(P(2)\) hold.

**(INDUCTION STEP)**

(IH): Assume \(P(k)\) for an arbitrary \(k \in \mathbb{N}\).

WTS \(P(k) \rightarrow P(k+1)\). We case on the parity of \(k\).

*Case 1*: \(k\) is even.

Observe the LHS.

\[
\sum_{i=1}^{k+1} (-1)^i \cdot i = \sum_{i=1}^{k} (-1)^i \cdot i + (-1)^{k+1}
= \frac{k}{2} + (-1)^{k+1} \cdot (k + 1) 
= \frac{k}{2} - (k + 1)
= \frac{k - 2(k + 1)}{2} = -\frac{k + 2}{2}
\]

*Case 2*: \(k\) is odd.

Observe the LHS.

\[
\sum_{i=1}^{k+1} (-1)^i \cdot i = \sum_{i=1}^{k} (-1)^i \cdot i + (-1)^{k+1}
= -\frac{k + 1}{2} + (-1)^{k+1} \cdot (k + 1) 
= -\frac{k + 1}{2} + (k + 1)
= -\frac{-k - 1 + 2(k + 1)}{2} = \frac{k + 1}{2}
\]

\(P(k) \rightarrow P(k+1)\) for both the even and odd cases, so the induction step is complete.

8. In an All-Milky Way course the students receive their graded homeworks consisting of \(n \geq 2\) problems, where each problem is given a score between 0 and \(m \geq 1\). Assume that there are enough students
Use the Pigeonhole Principle to prove that there exist four students, $a, b, c, d$ such that

- $\text{Same}(a, b) = \text{Same}(c, d)$, and
- $a \neq b$, and
- $c \neq d$, and
- $a \neq c$ OR $b \neq d$

**Answer** The four students form two ordered pairs, each of two distinct students, $(a, b)$ and $(c, d)$. Thus it comes down to proving that there exist two distinct such ordered pairs such that $\text{Same}(a, b) = \text{Same}(c, d)$.

How many ordered pairs of distinct students are there? Since we can assume that there are enough students so that every possible scoring occurs, this number is at least as big as the number of ordered pairs of distinct scorings. For simplicity we denote scorings also with $a, b, c, d$.

It helps to realize that a scoring is a function with domain $[1..n]$ and codomain $[0..m]$. So there are $(m + 1)^n$ scorings.

For each such scoring there are $(m + 1)^n - 1$ that differ from it. So there are $(m + 1)^n((m + 1)^n - 1)$ ordered pairs $(a, b)$ of distinct scorings. Now, with each ordered pair $(a, b)$ of scorings we associate the set Same$(a, b)$ of all the problems to which the scorings $a$ and $b$ assigned the same score. There are $2^n$ such sets.

To apply the pigeonhole principle let the pairs of distinct scorings be the pigeons, the sets of problems be the pigeonholes and Same be the function that assigns pigeons to pigeonholes.

Our proof will be done if we can show that there are strictly more pigeons than pigeonholes. Indeed, since $m \geq 1$ and $2^n1 > 2^21 > 1$ we have $(m + 1)^n((m + 1)^n1) \geq 2^n(2^n1) > 2^n$.

9. Consider the recurrence relation

$$a_0 = 0 \quad a_1 = 1 \quad a_n = 2a_{n-1} - a_{n-2} + 1 \quad (\text{for } n \geq 2)$$

Express $a_n$ as a polynomial in $n$. (Hint: use the telescopic trick twice.) Then prove by induction the result you obtained.

**Answer** Write the recurrence relation for $n = 2, \ldots, n$. Add the LHSides and the RHSides and perform the cancellations. You get $a_{n-1} + a_n = 2a_{n-1} + 2a_1 - a_1 - a_0 + (n - 1)$ hence $a_n = a_{n-1} + n$. Now perform the telescopic trick again.

It turns out that $a_n = n(n + 1)/2$.

10. For each statement below, decide whether it is true or false. In each case attach a very brief explanation of your answer.

(a) Let $E, F$ be two events in a finite probability space. If $|E| = |F|$ then $\Pr[E] = \Pr[F]$, true or false?
We can find a counterexample by defining a non-uniform probability space and letting $E$ and $F$ be sets of single outcomes with different probabilities. For example, consider the roll of two indistinguishable dice. Let $E$ be the event that the roll results in two 6’s and let $F$ be the event that the roll results in one 5 and one 6.

$|E| = |F| = 1$, but $\Pr[E] = \frac{1}{36}$ and $\Pr[F] = \frac{1}{18}$.

(b) If $E, F$ are two events in a finite probability space such that $\Pr[E \cap F] > 0$ then $E$ and $F$ can be disjoint, true or false?

**Answer** FALSE

If $\Pr[E \cap F] > 0$, there exists some outcome $\omega \in E \cap F$ that occurs with a positive probability, so $E \cap F \neq \emptyset$.

11. Let $A, B, C$ be arbitrary finite sets. Let $m = |A| + |B| + |C| - |A \cup B \cup C|$ and $n = |A \cap B| + |B \cap C| + |A \cap C|$. Prove that $m \leq n$.

**Answer** By the Principle of Inclusion-Exclusion, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Rearranging this equation gives us

$$|A| + |B| + |C| - |A \cup B \cup C| = |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

$$m = n - |A \cap B \cap C|$$

$$m \leq n$$

since $|A \cap B \cap C| \geq 0$.

12. Let $E, F$ be two events in a finite probability space such that $\Pr[E \cap F] > 0$. Prove that $\Pr[E \setminus F] + \Pr[F \setminus E] < \Pr[E \cup F]$.

**Answer** By the potato diagram, visualizing the sets $E$ and $F$, $(E \setminus F) \cup (E \cap F) \cup (F \setminus E) = E \cup F$. By the Sum Rule of Probabilities, since the LHS sets are pairwise disjoint: $\Pr[E \setminus F] + \Pr[E \cap F] + \Pr[F \setminus E] = \Pr[E \cup F]$. Therefore $\Pr[E \setminus F] + \Pr[F \setminus E] = \Pr[E \cap F] - \Pr[E \cap F] < \Pr[E \cup F]$.

13. Prove by ordinary induction on $n$ that for any $n \in \mathbb{N}$, $n \geq 1$ we have

$$1 + 2 + \cdots + n - 1 + n + n - 1 + \cdots + 2 + 1 = n^2$$

**Answer** BC: $1 = 1^2$. $P(1)$ holds.

IS: Assume that for an arbitrary natural number $k$, such that $k \geq 1$, $P(k)$ holds (Induction Hypothesis). We wish to show $P(k) \rightarrow P(k + 1)$. Examining the expanded LHS:

$$1 + 2 + \cdots + k + (k + 1) + k + \cdots + 2 + 1 =$$

$$[(k + 1) + k] + 1 + 2 + \cdots + (k - 1) + k + (k - 1) + \cdots + 2 + 1$$

Using the Induction Hypothesis,

$$[(k + 1) + k] + k^2 = (k + 1)^2$$

Clearly, $P(k + 1)$ holds, and we have our inductive proof.
14. For each statement below, decide whether it is true or false. In each case attach a very brief explanation of your answer.

(a) Assume that $A, B$ are finite nonempty sets and $f : A \to B$ is a function such that there exist at least 3 distinct elements of $A$ such that $f$ maps them to the same element of $B$. Then $|A| > 2 \cdot |B|$, true or false?

Answer FALSE. We provide a counterexample: $A = B = \{1, 2, 3\}$ and $f(1) = f(2) = f(3) = 1$. Clearly, $|A| \neq 2 \cdot |B|$.

(b) Let $A, B$ be events in a finite probability space such that $\Pr[A] = \frac{1}{4}$ and $\Pr[A \cup B] = \frac{1}{2}$. Then, $\frac{1}{4} \leq \Pr[B] \leq \frac{1}{2}$, true or false?

Answer TRUE. By the Principle of Inclusion-Exclusion,

$$
\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]
$$

Substituting, we have

$$
\frac{1}{2} = \frac{1}{4} + \Pr[B] - \Pr[A \cap B]
$$

Hence $\Pr[B] = \frac{1}{4} + \Pr[A \cap B]$. Since probabilities must be non-negative, $\Pr[A \cap B] \geq 0$; thus

$$
\Pr[B] \geq \frac{1}{4}
$$

Further, since $B \subseteq A \cup B$, we know that

$$
\Pr[B] \leq \Pr[A \cup B] = \frac{1}{2}
$$

Thus, $\frac{1}{4} \leq \Pr[B] \leq \frac{1}{2}$.

15. Let $A, B, C$ be arbitrary finite sets. Prove that $|A| + |B| + |C| \geq |A \cap B| + |B \cap C| + |C \cap A|$.

Answer By Inclusion-Exclusion, we have

$$
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|
$$

Rearranging,

$$
|A| + |B| + |C| = |A \cap B| + |B \cap C| + |C \cap A| + |A \cup B \cup C| - |A \cap B \cap C|
$$

If we can show that $|A \cup B \cup C| - |A \cap B \cap C| \geq 0$, then our WTS will follow.

$|A \cup B \cup C| - |A \cap B \cap C| \geq 0$ if and only if $|A \cup B \cup C| \geq |A \cap B \cap C|$. Assume for the sake of contradiction that $|A \cup B \cup C| < |A \cap B \cap C|$. Then, there exists some $x \in A \cap B \cap C, x \notin A \cup B \cup C$. However, $x \in A \land x \in B \land x \in C$, so $x \in A \cup B \cup C$, a contradiction; therefore, $|A \cup B \cup C| - |A \cap B \cap C| \geq 0$, so it follows that

$$
|A| + |B| + |C| \geq |A \cap B| + |B \cap C| + |C \cap A|
$$

16. Let $X$ be a finite nonempty set and $f : X \to X$. Let $x \in X$ arbitrary and consider the sequence

$$
x, f(x), f(f(x)), \ldots, f(\cdots f(x)\cdots), \ldots
$$
Prove that there exists some element of $X$ that occurs in $k$ distinct positions for any $k \geq 2$.

**Answer**  
For each natural number $m$ we can define a function $g_m$, which takes in one of the first $m$ terms in the sequence and returns its value. Because each term in the sequence will return an element in $X$, the codomain of $g_m$ is $X$ and, therefore, has cardinality $|X|$. The domain of $g_m$ is the set of the first $m$ terms of the sequence and, therefore, has cardinality $m$.

Now set $m = k |X|$. By GPHP there must be at least $k$ elements in the domain that map to one element of the codomain. This means that there are at least $k$ distinct positions of the first $m$ in the sequence in which the same element of $X$ occurs. This implies that there are at least $k$ positions in the entire sequence in which the same element of $X$ occurs.

(BTW, an alternative argument uses infinite sets and the following principle, related to PHP: if you put infinitely many pigeons in finitely many pigeonholes, at least one of the pigeonholes must contain infinitely many pigeons. You should think about why this implies the result above.)

17. Prove by ordinary induction on $n$ that for any $n \in \mathbb{N}$, $n \geq 1$ we have that for any two sets $A, B$ with $|A| = 2$ and $|B| = n$ the number of functions with domain $A$ and codomain $B$ is $n^2$. (Indeed we have counted functions before and we know that this is the correct count. Here you must prove it by induction.)

**Answer**  
W.l.o.g. we can fix the set $A = \{a, b\}$.

**Base Case**  
$n = 1$. Let $B = \{c\}$. We note that if $f$ is a function from $A$ to $B$ then $f(a) = f(b) = c$. This determines $f$ uniquely and therefore the number of such functions is $1 = 1^2$. Check.

**Induction Step**  
Let $k \geq 1$ be arbitrary. Assume (IH) that for any set $B$ such that $|B| = k$ the number of functions from $A$ to $B$ is $k^2$.

WTS that for any set $C$ such that $|C| = k + 1$ the number of functions from $A$ to $C$ is $(k + 1)^2$.

Fix an arbitrary $c \in C$. This partitions the counting of the functions from $A$ to $C$ into four disjoint cases: of four sets:

- The functions $f$ such that $f(a) = f(b) = c$. There is exactly one of these.
- The functions $f$ such that $f(a) = c$ but $f(b) \neq c$. There are $k$ ways to map $b$ to an element different from $c$ so there are $k$ of these.
- The functions $f$ such that $f(a) \neq c$ but $f(b) = c$. As in the previous case there are $k$ of these.
- The functions $f$ such that $f(a) \neq c$ and $f(b) \neq c$. The set of such functions is in one-to-one correspondence with the set of functions from $A$ to $B = C \setminus \{c\}$. Since $|B| = k$ the IH applies so we have $k^2$ such functions.

By the sum rule, the number of functions from $A$ to $C$ is $1 + k + k^2 = 1 = 2k + k^2 = (k + 1)^2$. Done.

18. Let $A, B, C$ be finite sets such that $B \cap C = \emptyset$. Using the principle of inclusion-exclusion prove that

$$|A \cup B \cup C| + |A| = |A \cup B| + |A \cup C|$$
Because \( A \cap B \cap C \subseteq B \cap C \) we have also \( A \cap B \cap C = \emptyset \). We can now apply the Principle of Inclusion-Exclusion:

\[
|A \cup B \cup C| + |A| = (|A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|)) + |A|
\]

\[
= |A| + |B| + |C| - |A \cap B| - |A \cap C| + |A|
\]

\[
= (|A| + |B| - |A \cap B|) + (|A| + |C| - |A \cap C|)
\]

19. Prove that any positive integer can be expressed as the sum of distinct Fibonacci numbers.

**Answer** We proceed by strong induction.

**Base Case:** For \( n = 1 \), we have that \( 1 = F_1 \). This is a sum of distinct Fibonacci numbers (because it is only a single number).

**Induction Step:** Let \( k \in \mathbb{Z}^+ \), arbitrary. Assume (Induction Hypothesis) that \( \forall 1 \leq j \leq k, j \) is expressible as a sum of distinct Fibonacci numbers.

We want to show that \( k + 1 \) is also the sum of distinct Fibonacci numbers.

Let \( F_i \leq k + 1 \) be the largest Fibonacci number less than or equal to \( k + 1 \).

**Case 1:** \( F_i = k + 1 \)

In this case, \( k + 1 \) can simply be expressed as \( F_i \), a distinct Fibonacci number.

**Case 2:** \( F_i < k + 1 \)

Let \( l = k + 1 - F_i \). For this, \( l < F_i \) as we will prove. For the sake of contradiction, suppose that \( l \geq F_i \). This would imply that \( k + 1 = F_i + l \geq 2F_i \). If this is true, then \( k + 1 \geq F_i + F_{i-1} = F_{i+1} \).

We have reached as contradiction as this would imply that \( F_{i+1} \) is the largest Fibonacci number less than or equal to \( k + 1 \), not \( F_i \) as was our original condition. Therefore, \( l < F_i \).

Now, we have \( k + 1 = F_i + l \). By our induction hypothesis, we know that \( l \) can be expressed as the sum of distinct Fibonacci numbers \( F_{m_1} + F_{m_2} + \ldots + F_{m_n} \), and \( F_{m_p} \neq F_i, \forall 1 \leq p \leq n \). Thus, \( k + 1 = F_{m_1} + F_{m_2} + \ldots + F_{m_n} + F_i \) is the sum of distinct Fibonacci numbers. We have proven that any positive integer can be expressed as the sum of distinct Fibonacci numbers.

20. Prove by induction that given an unlimited supply of 6-cent coins, 10-cent coins, and 15-cent coins, one can make any amount of change larger than 29 cents.

**Answer**

We will induct on \( n \), the amount of change in cents we can make with an unlimited supply of 6-cent, 10-cent, and 15-cent coins.

**(Base Case)** \( n = 30 \). To make 30 cents, we can simply use 5 6-cent coins (or 3 10-cent coins or 2 15-cent coins).

**(Induction Step)** Let \( k \geq 30 \).

Assume that for \( k \) cents, we have some distribution of coins that adds up to \( k \) cents (this is our IH). Mathematically, we can say that

\[
\exists x, y, z \in \mathbb{N} \text{ s.t. } 6x + 10y + 15z = n
\]
We want to show that we can construct a distribution of coins that sum to \( k + 1 \) cents. Note that we can remove a 15-cent coin and add a 6-cent coin and a 10-cent coin to have a net gain of 1 cent, resulting in a distribution with sum of \( k + 1 \) cents. However, we might not always have at least one 15-cent coin. To fix this problem and the more general problem that we don’t know if any of \( x, y, \) or \( z \) is greater than 0 to begin with, we can devise methods of increasing the sum by one in which we only remove coins of one type. We can use the following:

**Removing only 15-cent coins:** If we remove one 15-cent coin, add one 6-cent coin, and add one 10-cent coin, we have increased the sum by one.

**Removing only 6-cent coins:** If we remove four 6-cent coins, add one 10-cent coin and add one 15-cent coin, we have added one to the sum.

**Removing only 10-cent coins:** If we remove two 10 cent coins, add one 6-cent coin and add one 15-cent coin, we have added one to the sum.

Now we need to show that we can always use one of these methods. The only case in which we could not apply any method is if we had 0 15-cent coins, at most 3 6-cent coins and at most 1 10-cent coin. In this case, however, we’d only have 28 cents and since we are only proving the statement for \( n \geq 30 \), this is impossible, and thus we *must* be able to apply one of these methods.