Homework 5 (100 pts)

Posted Thursday, February 16
Due Thursday, February 23, 1PM, uploaded to Canvas

Logistics  Your homeworaks must be typeset in Latex and turned in as a PDF file. Handwritten
homeworks, or homeworks that are prepared with other tools (e.g., MS Word), will not be accepted. Please use our template and make sure that you include: your name, your recitation number, and the names of your collaborators You must upload to Canvas the pdf file with your submission. Canvas assignments with a file upload option will be created for each homework. The deadline (date and time) is above.

Collaboration Policy  Please check Canvas (under Files) for our Collaboration Policy.

1. [20 pts] Let $A, B, C$ be three finite nonempty sets and let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions. Define a function $h : A \rightarrow C$ such that $h = g \circ f$. ($g \circ f$ is the composition of $f$ and $g$ and means $\forall x \in A, h(x) = g(f(x))$).

Consider the following statements:

- $p$: $f$ is injective
- $q$: $g$ is surjective
- $r$: $h$ is injective
- $s$: $h$ is surjective
- $t$: $h$ is bijective

Prove or disprove each of the following:

(a) $r \Rightarrow p$
(b) $s \Rightarrow q$
(c) $p \land q \Leftrightarrow t$
(d) $t \Rightarrow p \land q$

2. [10 pts] AJ and the Pigeons are working on their first big hit: Written in the Stars and Bars. However, while writing the song, they realize that none of them are good enough rappers to do the song justice. When Will B, an aspiring rapper, hears about this, he decides this could be his big breakout opportunity. He picks up a copy of the application and excitedly rips it open, but to his dismay, instead of actual questions about music, like one would expect, Will instead has to inexplicably solve a CIS160 question. Can you help Will get his one big shot at the limelight?
Here’s the question: Let $|s|$ be the length of a sequence $s$. For $k \in \mathbb{N}$ let $B_k$ be the set of all sequences of bits of length $\leq k$. Note that this will always contain the empty sequence because its length is 0. For arbitrary $m, n \geq 1$ define a function $h : B_m \times B_n \rightarrow [0..m + n]$ by $h(s, t) = |s| + |t|$. Prove that $h$ is surjective.

3. [10 pts] After the booming popularity of Krishna’s first few Zumba classes at Pigeon’s Inductive Energy, the studio sees a decline in enrollment because many students pulled muscles breakdancing. Krishna is crushed. In an effort to win back his customers, Krishna decides to hold a zumbathon for CIS160 students. There are 220 students in CIS160, and each student is enrolled in at least one of the following dance classes: Line Dancing (LD), Salsa (S), and Break Dancing (BD). We know the following enrollments: LD: 100, S: 150, BD: 120. We also know that 50 students are in both LD and S, 60 students are in both S and BD, and 10 students are in all three courses.

(a) How many students take both Line Dancing and Break Dancing (and may or may not take Salsa)?

(b) How many students take only Salsa?

4. [15 pts] Let $A, B, C$ be three finite sets.

(a) Prove that

$$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| \leq |A \cup B \cup C| \leq |A| + |B| + |C|$$

(b) Prove also that if $|A \cup B \cup C| = |A| + |B| + |C|$ then $A, B, C$ are pairwise disjoint (note that this is the converse of the sum rule).

5. [15 pts] After getting so good at Master Archer that the game no longer challenges him, Matt decides to look for a new Facebook Messenger game to beat his friends at. Eventually he finally finds the perfect game: Words with Friends - Special CIS160 Edition. In this game, players are asked to make words of length exactly $2k$. Certain bonuses are given out depending on the word. The most lucrative bonus occurs when your word contains at least $k$ of the same letter in a row.

How many possible words are there that can get this bonus ($2k$ letters long, with at least $k$ of the same letter in a row)? For simplicity’s sake you can assume that any permutation of letters is a valid word (i.e. that even a word like ”abgd” is okay). Also assume that the alphabet has only 26 letters (so uppercase and lowercase don’t matter), and that the players have an unlimited supply of each letter.

Note: This answer does not need to be in closed form.

6. [15 pts] Count Olaf is keeping pigeons, plotting to disguise them as partridges and sell them to the local fowl distributor as soon as they are fat enough. He buys $p$ pigeons and puts them in $h$ holes. Wretched cheapskate that he is, he doesn’t buy enough holes for his pigeons ($h < p/10$)
so his pigeons fall ill and shed their feathers (each bird sheds 100 feathers). So now Olaf is left with no pigeons and a bunch of feathers. What can he do?

To try and recoup his investment, Olaf hatches a clever scheme: he’ll force the Baudelaire orphans to make pillows out of the feathers! In a stroke of brilliance he names his startup “Fowltry Sleepface, Inc.” But Olaf knows the value of “appellation d’origine contrôlée” (a fancy word meaning the value of the origins of his products). His business plan is as follows:

1) Violet Baudelaire will collect the feathers from the holes.
2) Klaus Baudelaire will gather 250 feathers and turn it into a pillow. Each feather must come from the same hole (i.e. a pillow can’t be made combining feathers from pigeons that lived in different holes).
3) Sunny Baudelaire will use her teeth to emboss each pillow with the name of the hole the feathers originally came from.
4) The dashing and clever Count Olaf himself will ensure the orphans make the most pillows possible given the above parameters.

Given that each pillow sells for $100, what’s the maximum amount of money Count Olaf could possibly make? What is the minimum?

7. **[15 pts]** After the long and arduous task of deciding which islands to visit on their treasure hunt, the CIS160 TAs are ready to embark on their journey for gold. Before they begin, Krishna suggests that the TAs split up into three groups so that they can visit the islands in a more timely manner. Krishna appoints Joslyn, Aashil, and Jeffrey to lead three squads, which will between them cover all of the islands but not visit any island twice. According to the new treasure map Joslyn discovered buried in the 5th floor bump space, there are $n$ islands, where $n$ is a positive integer, such that:

- Each island contains some number of gold doubloons* between 1 and $2n$ (inclusive).
- No two islands contain the same number of gold doubloons.

Unfortunately, Aashil and Jeffrey are fighting over who got more roses on Valentine’s day, and Joslyn does not want the treasure hunt to anger them further if one gets more doubloons than the other. To appease them, Joslyn suggests that they divide the islands such that Aashil and Jeffrey get the same number of doubloons.

(a) Prove that there exists an $n$ such that no matter what the exact number of doubloons each island has, the 3 squads can divide the islands between them such that Jeffrey’s squad and Aashil’s squad find the same non-zero number of doubloons.

(b) Find the smallest $n$ (and prove it’s the smallest) for which this condition holds.

* doubloons – “the official booty of pirates,” not to be confused with the cheesy snack.