1. [20 pts] You go to a casino and you decide to play the new game, “Hot Dog Sandwich.” The rules are as follows.

You must pay $10 to play the game. The game consists of two rounds: a coin flip (one side of the coin is labeled with a 1 and the other side is labeled with a 2, and the coin is fair) and a 100-sided dice roll (the sides are labeled from 1 to 100, and the dice is fair). During the first round, you can choose to either flip the coin or roll the dice. If you get a 1, you win $1 and you move to the next round; if you get any other number, you must pay the casino $15, and you lose - you don’t get to move on to the next round. In the second round, you must use the other object. For instance, if you rolled the dice the first round then now you must flip the coin, and vice versa. During the second round, if you get a 1, you win $5010. If you get any other number, you lose and must pay another $15.

You are told the rules and now must weigh your options. You have three options:

(a) Decide not to play the game.
(b) Play the game, and flip a coin for the first round.
(c) Play the game, and roll the dice for the first round.

Find the expected value of each of these options. Which option has the highest expected value?

2. [20 pts] Shawn just recently turned 21. In order to celebrate, Shawn and Waley go to the only place that would make sense for a 21st birthday – Bharath’s Fancy Pigeon Farm (BFPF). Today at the farm, they happen to be playing a special game.

Bharath has 50 distinct holes set up around his farm. The player is given one pigeon. In each round, the player will place a pigeon in one of the 50 holes and bet some amount of money. Bharath will choose a hole uniformly at random, and if the player’s pigeon is in that hole, the player will be paid 47 times their initial bet. If the player’s pigeon is not in the hole Bharath chose, the player will lose his initial bet.
Since it is his 21st birthday, Shawn decides that for each round, he is going to place his pigeon in the 21st hole and bet $1.

Waley dislikes this game. Waley bets Shawn $25 that after 48 rounds of the game, Shawn will have lost more money than he gained. In other words, Shawn will pay Waley $25 if Shawn is behind after 48 rounds; otherwise, Waley will pay Shawn $25.

(a) Calculate Shawn’s expected gain for the 48 rounds only, without Waley’s $25 bet. (If he is expected to lose money, his expected gain will be a negative number).

(b) Calculate the probability that Shawn is behind at the end of 48 rounds.

(c) Calculate Shawn’s expected gain in his bet against Waley.

(d) Calculate Shawn’s overall expected gain from the game and the bet with Waley. Does Waley’s bet dissuade Shawn from playing?

3. [10 pts] After deciding that he needs more money, Krishna installs the Ultimate-Kandy-Machine in Kandyland. This machine is a glorified slot machine that just returns to you a random (non-negative integer) number of Kandys in any particular game. In fact, the machine can return an arbitrarily large number of Kandys in each game as well (wow!).

However, before opening it to the public, Krishna needs to calculate the expected number of Kandys returned per game. Arnab tells him that an easy way to calculate the expected value is as follows: for all integers $i$ from 0 to $\infty$, calculate the probability that the number of Kandys returned is greater than $i$. Then take the sum of all of those numbers. Show that Arnab is correct.

4. [20 pts] Krishna has solicited feedback from Kandy Emporium visitors and discovered that people are lazy and hate walking from ride to ride. Thus, in lieu of colored segments, Krishna installs colored conveyor belts between every pair of rides. However, the conveyor belt only goes in one direction. That is, for every pair of rides $a$ and $b$, there is a colored conveyor belt that goes either from $a$ to $b$ or from $b$ to $a$, but not both. Wenting decides to explore, but she only has a few hours to make the most of her time at the Kandy Emporium. Prove that there is some series of conveyor belts that Wenting can take such that she visits every ride exactly once.