Homework 1w

This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in []. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - no collaboration is allowed.

Note that Questions 4-7 are intended more as brainteasers to get you into the problem solving mindset of CIS 160, and do not directly relate to material in lecture. Such questions generally only appear in the first homework – in future, homework questions will relate more closely with the material seen in lecture and recitations.

1. [12 pts] Let $p$, $q$, and $r$ be the following propositions.

   $p$: Tottenham Hotspur win their first game of the season.

   $q$: Tottenham Hotspur win the Premier League.

   $r$: Tottenham Hotspur sign a new defender.

Express the following propositions using $p$, $q$, $r$ and logical operators. Justification is not needed for this question only.

   (a) Tottenham Hotspur win the Premier League and they sign a new defender.

   (b) Tottenham Hotspur sign a new defender if and only if they win their first game of the season.

   (c) If Tottenham Hotspur don’t win their first game of the season or they don’t sign a new defender then they won’t win the Premier League.

   (d) Tottenham Hotspur don’t sign a new defender but they win their first game of the season anyways.

   (e) Tottenham Hotspur winning their first game of the season is sufficient for them winning the Premier League and signing a new defender.

   (f) Tottenham Hotspur signing a new defender is necessary for them to win the Premier League. Tottenham Hotspur signing a new defender is also necessary for them to win their first game of the season.

2. [10 pts] Decide if the following proposition forms are a tautology using a truth table. Make sure you state your final answer and show all intermediate logical expressions – for example, in showing the truth table for $(p \lor q) \land p$, you should show $p \lor q$ as a separate column on its own.
(a) \( [(p \rightarrow q) \land \neg(p \lor q)] \rightarrow (p \lor q) \)
(b) \( [(p \rightarrow q) \rightarrow (q \rightarrow p)] \rightarrow [q \rightarrow (p \rightarrow q)] \)

3. [10 pts] Prove or disprove the following.

(a) For every prime \( p \), either \( p^2 + 2 \) is a prime, or \( p + 3 \) is a prime.
(b) \( \forall x, y \in \mathbb{R}^+, \text{ s.t. } y \leq x, \lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor \).
(c) \( \forall x \in \mathbb{N}, x^3 - 2x^2 + 3x + 3 \) is odd.
(d) \( \forall a, b, x, y, d \in \mathbb{Z}, (d|a \land d|b) \rightarrow d|(ax + by) \).
(e) \( \forall x, y, z \in \mathbb{Z}, \) if \( x - y \) is odd and \( y - z \) is even, then \( x - z \) is odd.

4. [10 pts] There are 5 couples at a fraternity date-night, for a total of 10 people. During the party, each of the couples break off to mingle, leading to a great deal of fistbumping and polite small talk. Towards the end of the party, Somil asks each of the party guests (including his date) how many people they had fistbumped with – surprisingly, each person that Somil asks replies with a different number! Given this and given that no one fistbumped their partner (that would be strange, since they know each other already!), can you work out how many people Somil’s partner bumped fists with? Rigorously justify your answer.

5. [9 pts] Jessica and Andrew walk into a room and stumble upon a table with a row of 160 coins, of various values, on top. They decide to play a game as follows: Jessica picks a coin from one of the ends and puts it in her pocket; then Andrew chooses a coin from one of the (remaining) ends, and the alternation continues until Andrew picks the last coin. Prove there is a strategy Jessica can use to guarantee that she will earn at least as much money as Andrew.

6. [9 pts] A duck walks up to a lemonade stand and says to the man running the stand, ”Hey... got any grapes?” The man says ”Yes! We have a lot of grapes, but you’ll have to play a game to earn them!” He goes on to describe the game.

He has a bag of 9023 purple grapes and 9024 green grapes. He will repeatedly remove 2 grapes from the bag. If the two grapes are of the same color then we put 1 new green grape in the bag without replacing the original 2. Otherwise, we put one new purple grape in the bag (also without replacing the original 2).

He will repeat this until there is only one grape left. If the duck can guess the color of the last grape in the bag, then the duck can buy grapes from him. Can you help the hungry duck and figure out the color of the last grape in the bag? Explain your answer.
7. [10 pts] Grace, Seth, and Vinai are stuck in a conundrum. They are trapped by a serial brainteaser, Arnab. Arnab traps them in a room, in which lies a box with 5 light bulbs. The light bulbs all look identical, however 2 of the bulbs light up white and 3 of the bulbs light up red. Arnab instructs them to each take a light bulb. They then take turns putting on a blindfold and turning on their light so the other two players can see the color of their light. After this is done, everyone knows the color of everyone else’s light bulb but do not know the color of their own light bulb (since they were blindfolded when it was lit). Arnab tells them that if any of them are able to deduce the color of their light bulb without the others telling them then they well be able to move on from the brainteasers.

Seth starts thinking about the possibilities but after a while, he concludes out loud "I don’t know the color of my light bulb". After hearing this, Vinai decides to take a stab at it, however he too can’t deduce the color of his light bulb and says the same thing. At this point, is Grace able to know what color her light bulb is and save the three of them from the brainteasers and Arnab the serial brainteaser? Rigorously justify your answer.