1. [9 pts] Seth and Shawn are arguing over who eats more Dim Sum. To mediate, Jeffrey takes them both to Dim Sum House and orders soup dumplings. Jeffrey tells Seth and Shawn that they cannot eat the soup dumplings until they make up. Seth, who is still slightly angry at Shawn, says that he will not make up with Shawn until Shawn proves that $\sqrt{6}$ is irrational. Show Shawn the proof so that he can reconcile with Seth and get to eat his soup dumplings.

2. [13 pts] Vinai has a weird obsession with twins, both identical and fraternal. He decides to buy 25 tickets to go to the twin convention in Philadelphia this weekend with his fellow TAs. Unfortunately for Vinai, everyone is busy so they can’t go. Not wanting to waste all the tickets, Vinai goes to the convention by himself. Since he is not a twin, Vinai isn’t allowed to participate in the activities himself and instead he watches as all the twins go about their fun filled day.

He observes $n \geq 2$ sets of twins, all playing musical chairs around a round table with $2n$ chairs. Not having anything else to do, Vinai tries to count the number of ways the twins could sit around the table. However, Vinai isn’t that good at mental math and needs your help figuring out the number of possible seating arrangements.

Two seating arrangements are considered the same if, for each person, the person to his or her left is the same in both arrangements.

(a) How many ways can we seat the $n$ sets of twins if the first $k$ sets insist that they be seated together, i.e., for each of the first $k$ sets of twins, each twin should be seated next to their twin? Your answer must be closed-form. (For example, $\sum_{i=1}^{n} (2i - 1)$ and $1 + 3 + \ldots + 2n - 1$ are not closed forms, but the equivalent quantity $n^2$ is.)

(b) How many ways can we seat the $2n$ people such that no one sits next to their twin? You may leave this answer in non-closed-form (for example, you may have summations from 1 to $n$) for this specific question.

Hint: Use the result that you obtain in part (a).

3. [10 pts] Let $r_1$ and $r_2$ be the roots of $f(x) = ax^2 + bx + c$, where $a, b, \text{ and } c$ are all integers.
Prove that if at least one of \( r_1 \) and \( r_2 \) is rational, then at least one of \( a, b, \) and \( c \) is even.

4. **[10 pts]** The Cleveland Cavaliers are objectively better than the Golden State Warriors. Rohan, unfortunately, has a difficult time understanding this fact. Bharath, being the patient TA that he is, decides to help Rohan realize the wrongness of his ways. After multiple hours, Rohan still refuses to agree. He argues that there exist prime numbers \( J, Z, \) and \( S \) such that \( J^2 + Z^2 = S^2 \), and thus, the Golden State Warriors are the best team in the NBA. Prove that Rohan is wrong on both accounts. That is, prove that for any prime numbers \( J, Z, \) and \( S \), \( J^2 + Z^2 \neq S^2 \).

5. **[10 pts]** Rohan and Arnab love doing arts and crafts together and decide to spend their weekend making dream catchers. However, they are not able to make perfectly circular dream catchers so instead, they settle for convex-shaped dream catchers with \( n \) sides each (convex-shaped means that all diagonals lie entirely within the dream catcher). They then take some freshly spun yarn and knit it between every pair of non-adjacent corners. After all the yarn has been added and every pair of such vertices is connected, they wonder how many interior points intersect. Can you help them figure out how many interior points the yarn pieces intersect at? Assume that no three pieces of yarn intersect at the same point.

(Note: Your answer must be expressed in closed form. For example, \( \sum_{i=1}^{n} (2i - 1) \) and \( 1 + 3 + \ldots + 2n - 1 \) are not closed forms, but the equivalent quantity \( n^2 \) is.)

6. **[6 pts]** Let \( x_1, x_2, \ldots, x_{2017} \) be a permutation of numbers from 1 to 2017. Show that the product
\[
(x_1 - 1)(x_2 - 2) \cdots (x_{2016} - 2016)(x_{2017} - 2017)
\]
must be an even number.

7. **[12 pts]** Krishna’s Kandy Emporium is almost ready for its grand opening! Krishna is having a pre-celebration and allowing all CIS160 students and TA’s to play his game as a test-run. However, Wenting thinks that the game will take too long, and she is anxious to get home to watch *Finding Dory*, her favorite movie. Wenting suggests that the students and TA’s pair up into teams of two. If there are 238 students and TA’s, in how many ways can they form teams of two?

(Note: Your answer must be expressed in closed form. For example, \( \sum_{i=1}^{n} i \) and \( 1 + 2 + \ldots + n \) are not closed forms, but the equivalent quantity \( n(n + 1)/2 \) is.)