This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - no collaboration is allowed.

1. [12 pts] Vinai successfully completes his tour of the world’s twin conventions. However, on his way back home he feels a certain emptiness, but he can’t seem to put his finger on what’s keeping him down. While on the plane, Vinai notices someone that looks eerily like himself. Upon further investigation, Vinai discovers that he has just found his long lost twin, Ianiv!

And with that, Vinai’s empty feeling is gone. Nonetheless, this empty feeling is replaced with a new, fiery one: Vinai wants to know that he is the superior twin. In order to prove this, Vinai challenges Ianiv to the only respectable gauge of talent in the world - thumb war. The competition is a best of 21 - in other words, the two must play up to 21 matches, and the first player to win 11 matches is declared the winner. Assuming that Vinai and Ianiv are evenly matched (they are twins after all!), so that they both have the same probability of winning any game, what is the probability that the twin who wins the first game also wins the series?

2. [11 pts] Andrew, Krishna and Yonah decided to take up Sumo Wrestling as a full time career. They quickly ascend to the top of the leaderboards and find themselves at the World Wide International Sumo Wrestling Championship Tournament. After defeating the past 16 World Champions, they remain the final three wrestlers. Since Andrew was able to body slam everyone of his opponents, he enters the final round as the top contender for the Final Championship.

As a result, he gets a nice little extra bonus in this final round of the World Wide Championship. The first two matches will be between Yonah and Krishna. If one of them is able to defeat the other twice in a row, he will move on to face Andrew. That Sumo Wrestler then has two chances to wrestle Andrew and must win both matches to be crowned the World Wide International Champion. If these conditions are not met, Andrew is immediately given the World Wide International Sumo Wrestling Championship Prize Award and no further matches are played.

Based on their rankings from the past 2 days, we have the following statistics for the match ups:

- The probability that Krishna will beat Yonah in any particular match is 0.6.
- The probability that Andrew will beat Krishna in any particular match is 0.5.
- The probability that Andrew will beat Yonah in any particular match is 0.7.
Assume all matches are independent and that ties are not possible in Sumo Wrestling.

(a) Given that no matches have been played yet, determine the probabilities that:

i. Andrew will be crowned champion without playing a single match.

ii. Krishna will play Andrew.

iii. Andrew will win the International Championship this year.

(b) Given that either Krishna or Yonah wins both matches against the other, determine the conditional probabilities that

i. it is Krishna that is wrestling Andrew.

ii. Andrew wins the championship.

(c) Given that Andrew only played one match in total, what is the conditional probability that Krishna beat Yonah in both matches?

3. [13 pts] There are \(2^{n+1}\) marbles in a bag, where each marble is labeled with an arbitrary natural number (different marbles can have the same label). Prove that for all \(n \in \mathbb{N}\), it is possible to choose \(2^n\) marbles from this bag such that the sum of the labels of the \(2^n\) marbles is divisible by \(2^n\).

4. [10 pts] Somil is given a graph \(G = (V, E)\) and wants to play with the vertices. He realizes that no matter how he splits his vertices into two non-empty sets, \(X, Y\), there exists an edge \((a, b)\) such that \(a \in X\) and \(b \in Y\). Prove that this occurs if and only if the graph \(G\) was a connected graph to begin with.

5. [14 pts] Somil’s fistbumping addiction continues, and he decides to attend the annual fistbumping convention.

After the convention, Somil watches the security footage and records every pair of visitors who fistbumped. He then realizes that he has formed a simple, undirected graph!

He calls the resulting graph \(G\), and to make sure that everyone introduces themselves at the next convention, sets out to create \(\overline{G}\), the graph representing pairs of visitors who haven’t fistbumped yet. More formally, consider the initial graph with the set of visitors \(V\) and the set of fistbumps \(F\) such that \(G = (V, F)\). Define \(\overline{G} = (V, \overline{F})\), where

\[
\overline{F} = \{\{x, y\} \mid x \neq y, \{x, y\} \notin F\}
\]
Somil is concerned that the fistbumping community is breaking into factions and posses, so he decides to investigate it with graph theory.

Define a cut vertex to be a vertex such that the subgraph obtained by deleting it and all of its edges has more connected components than the original graph.

Prove that if a visitor’s vertex \( v \) is a cut vertex in \( G \), then \( v \) is not a cut vertex in \( \overline{G} \).

6. [10 pts] Annie wants to celebrate Christmas a little bit early, but doesn’t want to buy an actual tree (they shed, so it’d make her room dirty). She decides to draw a tree instead.

She notices that in her tree, \( T \), which has at least two vertices, each vertex that is adjacent to a leaf has degree of at least 3. Prove that in \( T \), there exists two distinct leaves, \( x \) and \( y \), such that \( N(x) = N(y) \).