Example. Prove that a matching $M$ in $G$ is maximum iff $G$ contains no $M$-augmenting path.

Solution. We will prove the necessary condition by proving its contrapositive, i.e., we will prove that if $G$ contains an $M$-augmenting path then $M$ is not a maximum matching. Suppose that $G$ contains a $M$-augmenting path $v_0v_1v_2\ldots v_{2m+1}$ (Note that an $M$-augmenting path must be of odd length). Define $M' \subseteq E$ by

$$M' = M \setminus \{(v_1,v_2), (v_3,v_4), \ldots, (v_{2m-1},v_{2m})\} \cup \{(v_0,v_1), (v_2,v_3), \ldots, (v_{2m},v_{2m+1})\}$$

Then $M'$ is a matching in $G$ and $|M'| = |M| + 1$. Thus $M$ is not a maximum matching.

We will prove the converse by proving the contraposition. Assume that $M$ is not a maximum matching. Let $M'$ be a maximum matching in $G$. Then $|M'| > |M|$. Set $H = G[M \oplus M']$. Figure 1 illustrates this operation. Observe that every vertex in $H$ has either degree one or degree two in $H$, since it can be incident with at most one edge of $M$ and one edge of $M'$. Thus each component of $H$ is either an even cycle with edges alternating in $M$ and $M'$ or else a path with edges alternating in $M$ and $M'$. Since $|M'| > |M|$, $H$ contains more edges of $M'$ than of $M$, and $H$ must contain a component which is a path, $P$, that starts and ends with edges in $M'$. Since the start vertex and end vertex of $P$ are $M'$-saturated in $H$ they must be $M$-unsaturated in $G$. Thus, $P$ is an $M$-augmenting path in $G$. This completes the proof.

Figure 1: (a) a graph $G$ with a matching $M$ represented by the bold edges, (b) the dashed edges represent a matching $M'$ in $G$, (c) $G[M \oplus M']$
Matching in Bipartite Graphs

An independent set of a graph is a set of pair-wise non-adjacent vertices. A bipartite graph, \((U, V, E)\), is a graph whose vertex set is \(U \cup V\) and for each edge \(e = (u, v) \in E\), \(u \in U\) and \(v \in V\). In other words, \(U\) and \(V\) are independent sets and each edge in \(E\) connects a vertex in \(U\) to a vertex in \(V\).

Now consider the following scenario. There is a set of girls and a set of boys. Each girl likes some boys and dislikes others. What conditions would guarantee that each girl is paired-up with a boy that she likes and that no two girls are paired-up with the same boy.

We can model this situation using a bipartite graph, \((X, Y, E)\), where each vertex in \(X\) represents a girl, each vertex in \(Y\) represents a boy and edge \((g, b) \in E\) means that girl \(g\) likes boy \(b\). We are interested in the conditions that would guarantee a matching that saturates every vertex in \(X\).

Hall’s theorem gives the necessary and sufficient conditions for the existence of such matchings in bipartite graphs.

Example. [Hall’s Theorem] Let \(G = (X, Y, E)\) be a bipartite graph. For any set \(S\) of vertices, let \(N_G(S)\) be the set of vertices adjacent to vertices in \(S\). Prove that \(G\) contains a matching that saturates every vertex in \(X\) iff \(|N_G(S)| \geq |S|, \forall S \subseteq X\). The condition “For all \(S \subseteq X, |N(S)| \geq |S|\)” is called Hall’s condition.

Solution. We prove that Hall’s condition is necessary as follows. Suppose \(G\) contains a matching \(M\) that saturates every vertex in \(X\). Let \(S\) be a subset of \(X\). Since each vertex in \(S\) is matched under \(M\) to a distinct vertex in \(N_G(S)\), \(|N_G(S)| \geq |S|\).

We will now prove the sufficiency of Hall’s condition, i.e., if \(|N_G(S)| \geq |S|, \forall S \subseteq X\) then \(G\) contains a matching that saturates every vertex in \(X\). We prove this by induction on the size of \(X\).

Base Case: \(|X| = 1\). If the only vertex in \(X\) is connected to at least one vertex in \(Y\) then clearly a matching exists.

Induction Hypothesis: Assume that Hall’s condition is sufficient when \(|X| = j\), for all \(j\) such that \(1 \leq j \leq k\).

Induction Step: We want to prove that the sufficiency of Hall’s condition when \(|X| = k + 1\). Let \(G = (X, Y, E)\) be a graph with \(k + 1\) vertices in \(X\) such that \(\forall S \subseteq X, |N_G(S)| \geq |S|\).

We consider the following two cases.

Case I: For every non-empty proper subset \(W \subset X\), \(|N_G(W)| > |W|\). In this case, we pair-up an arbitrary vertex \(x \in X\) with one of its neighbors, say \(y \in Y\). Now consider the subgraph \(G' = (X', Y', E')\), where \(X' = X \setminus \{x\}\), \(Y' = Y \setminus \{y\}\), and \(E' = E \setminus \{(x, y)\}\).

After the removal of \(y\), the neighborhood of any subset, \(S' \subseteq X'\) in \(G'\) is at most one less than its neighborhood in \(G\). But since \(|N_G(S')| > |S'|\), after removal of \(y\), it must be that
Prove that a graph with maximum degree $k$. Thus, Hall’s condition holds for $G'$. By induction hypothesis, $G'$ contains a matching $M'$ that saturates every vertex in $X'$. Hence, $M' \cup \{(x, y)\}$ is a matching that saturates every vertex in $X$.

**Case II:** For some non-empty proper subset $W \subset X$, $|N(W)| = |W|$. For all $S' \subseteq W$, we have $N_G(S') \subseteq N_G(W)$. Hence, Hall’s condition holds for the subgraph induced by $W \cup N(W)$. By induction hypothesis, there is a matching $M_1$ that matches every vertex in $W$ to a vertex in $N_G(W)$. Note that $M_1$ is a perfect matching. Consider the subgraph $G' = (X', Y', E')$, where $X' = X \setminus W$, $Y' = Y \setminus N(W)$, and $E'$ consists of all edges between $X'$ and $Y'$. If we can prove that Hall’s condition holds for $G'$ then by induction hypothesis, $G'$ has a matching $M_2$ that saturates every vertex in $X'$. Then, $M_1 \cup M_2$ is clearly a matching in $G$ that saturates every vertex in $X$. It now remains to prove that $\forall T \subseteq X', |N_{G'}(T)| \geq |T|$. Note that $N_G(W \cup T) = N_G(W) \cup N_{G'}(T)$, $|N_G(W)| = |W|$, $W$ and $T$ are disjoint, and $N_G(W)$ and $N_G(T)$ are disjoint. Then,

\[
|N_G(W \cup T)| \geq |W \cup T| \quad \text{(follows because } \forall S \subseteq X, |N_G(S)| \geq |S|) \\
|N_G(W)| + |N_{G'}(T)| \geq |W| + |T| \\
|W| + |N_{G'}(T)| \geq |W| + |T| \\
|N_{G'}(T)| \geq |T|
\]

This proves the sufficiency of Hall’s condition.

**Graph Coloring**

Consider the following scenario. There are $n$ courses for which final exams need to be scheduled. Each exam needs a two hour slot. Since each student may be in more than one course, the exams need to be scheduled such that two courses that have common students don’t have their final exams at the same time. The objective is to find minimum number of time slots that would be required to schedule all the exams.

A graph is $k$-colorable if each vertex can be colored using one of the $k$ colors so that adjacent vertices are colored using different colors. The chromatic number of a graph $G$, $\chi(G)$, is the smallest value of $k$ for which $G$ is $k$-colorable. Note that every bipartite graph is 2-colorable.

The problem of scheduling exams can be modeled as a graph coloring problem. Construct a graph in which there is a vertex for each course and two vertices $u$ and $v$ are connected by an edge if there is a student who is taking both the courses corresponding to $u$ and $v$. The chromatic number of the graph will provide the required solution to the problem.

Finding the chromatic number of a graph “quickly” is a very hard problem. Even finding a reasonable approximate solution is very hard!!

**Example.** Prove that a graph with maximum degree $k$ is $(k + 1)$-colorable.

**Solution.** Let $P(n)$ be the property that a graph with $n$ vertices and maximum degree at most $k$ is $(k + 1)$-colorable. We will now prove the claim by doing induction on $n$. 
Base Case: $P(1)$ is clearly true as a graph with just one vertex has maximum degree zero and can be colored using one color.

Induction Hypothesis: Assume that $P(h)$ is true for some $h \geq 1$.

Induction Step: We want to prove that $P(h + 1)$ is true. Let $G$ be a graph with maximum degree at most $k$ and having $h + 1$ vertices. Let $G'$ be the graph obtained from $G$ by removing a vertex $v$ along with the edges incident on $v$. $G'$ has $h$ vertices and has a maximum degree at most $k$. By induction hypothesis, $G'$ is $(k + 1)$-colorable. Now insert $v$ along with its incident edges. Since we have a palette of $k + 1$ colors and $\text{deg}(v) \leq k$, we can always color $v$ using a color that is not used by any of its neighbors. Thus, $P(h + 1)$ is true. This completes the proof.