COUNTING   We continue studying how to use combinations and what are their properties.

Example 5.1 How many 8-letter strings can be constructed by using the 26 letters of the alphabet if each string contains 3, 4, or 5 vowels? There is no restriction on the number of occurrences of a letter in the string.

Solution: Let $E$ be the set of 8-letter strings that contain at least 3 vowels. Let $E_i$ be the set of 8-letter strings containing exactly $i > 0$ vowels (so $i \in \{1, 2, 3, 4, 5\}$)

An element of $E_i$, i.e., a 8-letter string with exactly $i$ vowels, can be constructed using the following steps.

Step 1. Choose $i$ locations for vowels out of the available 8 locations.

Step 2. Choose the vowels for each of the $i$ locations.

Step 3. Choose the consonants for each of the remaining $8 - i$ locations.

Step 1 can be performed in $\binom{8}{i}$ ways. Step 2 can be performed in $5^i$ ways. Step 3 can be performed in $21^{8-i}$ ways. By the multiplication rule, the number of 8-letter strings with exactly $i$ vowels is given by

$$|E_i| = \binom{8}{i} 5^i 21^{8-i}$$

Since the sets $E_3, E_4,$ and $E_5$ partition (divide in pairwise disjoint subsets) the set $E$, by the sum rule we get

$$|E| = \sum_{i=3}^{5} |E_i| = \sum_{i=3}^{5} \binom{8}{i} 5^i 21^{8-i}$$

Addendum. The following question was raised in class in one of the previous offerings of this course. What if we want to count all 8-letter strings with distinct letters that have 3, 4, or 5 vowels? In this case, the above procedure still applies. However, the number of ways of doing each step changes. Step 1 can be performed in $\binom{8}{i}$ ways. Step 2 can be performed in $(5)_i$ ways. Step 3 can be performed in $(21)_{8-i}$ ways. By the multiplication rule, the number of 8-letter strings with distinct letters that have exactly $i$ vowels is given by

$$\binom{8}{i} (5)_i (21)_{8-i}$$
The total number of 8-letter strings with distinct letters that have 3, 4, or 5 vowels is
\[
\sum_{i=3}^{5} \binom{8}{i} (5)^i (21)^{8-i}
\]
It is worthwhile revisiting Example 2.5 from Lecture 2 and comparing it with what we did just now.

Example 5.2 Suppose you are in Manhattan at the NY Public Library (5th Ave and 42nd St) and you wish to walk to Columbus Circle (8th Ave and 59th St) taking a shortest path (but you cannot take Broadway for some reason :). How many blocks do you need to walk? How many different ways are there to walk a shortest path?

Solution: A shortest path consists of making only two kinds of decisions: go-West one block (at most \(8-5 = 3\) times), or, go-North one block (at most \(59-42 = 17\) times), for a total of \(3+17 = 20\) blocks.

How many different paths of length 20 blocks are there from the Library to the Circle?

Any such path is a sequence of 20 decisions of one of two kinds: go-West 3 times, or go-North 17 times. And any such sequence is a valid shortest path.

Such a sequence has 20 positions and choosing the 3 positions in which to make go-West decisions can be done in \(\binom{20}{3}\) ways. If we choose instead the positions in which to make go-North decisions we get \(\binom{20}{17}\) different ways. But these numbers are equal (why?).

More generally, consider the grid defined by the points of integer coordinates in the plane. Assuming that we can move only along lines of integer abscissa or integer ordinate how many different shortest paths are there from the origin to the point \((m, n)\) where \(m, n \in \mathbb{N}\)? The answer is
\[
\binom{m+n}{m} = \binom{m+n}{n}
\]
By the way, you can check that these two are equal by using the expression with factorial, or, better, wait for a “combinatorial” proof coming up below.

Stars and bars (sticks and crosses) Next we explain, via an example, a versatile method for counting certain kind of collections.

Example 5.3 How many different ways are there to buy a dozen donuts when 5 given glazes are available? Assume (1) that there is an unlimited supply of donuts of each glaze, (2) that donuts of the same glaze are indistinguishable, and (3) that the placement of the donuts in, say, a box, does not distinguishes your purchases.
Solution: We apply stars and bars as follows.

Arrange 12 stars in a row. They represent hypothetical donuts before glazes are assigned, let’s call them proto-donuts :) Obviously, all 12 proto-donuts are indistinguishable so their ordering is irrelevant. Now place 4 bars between some of the stars. Note that this separates the proto-donuts into 5 contiguous parts. Assign the 1st glaze to the proto-donuts in the leftmost part, 2nd glaze to the next part, etc. For example:

```
***|**|*|****|**
A B C D E
```

A=chocolate, B=maple, C=dulce-de-leche, D=amaretto-cherry, E=hazelnut

Note that it is possible that you don’t buy donuts of a specific glaze at all. This means that we have to allow for bars at the beginning or end and for adjacent bars:

```
***||******|***| ||*********
A B C D E AB C D E A BCD E
```

In the first example above there are no maple-glazed donuts, and no hazelnut-glazed donuts, etc.

Note also that because the proto-donuts are indistinguishable, the ordering of A,B,C,D,E does not matter, only how many proto-donuts are in each contiguous part matters. For example the following have the same glaze distribution hence they produce the same purchase.

```
***|**|*|****|**
A B C D E
AO B C D E
```

Thus, to avoid overcounting, we fix an ordering of A,B,C,D,E and then we count.

So how do we count the different ways of constructing a sequence of stars and bars that corresponds to a this fixed ordering of glazes? The 12 stars and the 4 bars form a sequence with $12 + 4 = 16$ positions. Out of these, we choose 4 positions where we put the bars, or, equivalently, 12 positions where we put the stars. The answer is $\binom{16}{4} = \binom{16}{12}$.

Let’s review what we have done so far about counting different kinds of collections. We have seen that the number of distinct sequences of length $r$ made of elements from a given set of $n$ elements is

$$n^r$$

This is the same as the number of different ways to construct from elements of a set with $n$ elements a collection of size $r$ that is both ordered and allows repetitions.

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1I am using “part” rather than (sub)sequence because their ordering is irrelevant.
We have also considered counting the number of ways to construct from elements of a given set with $n$ elements a collection of size $r$ that is ordered but does not allow repetitions. We called these *permutations of $r$ out of $n$* and we denoted their number by

$$
(n)_r = \frac{n!}{(n-r)!}
$$

Finally, we have considered counting the number of ways to construct from elements of a given set with $n$ elements a collection of size $r$ that is unordered and does not allow repetitions. That is the same as counting the subsets of size $r$ of the set with $n$ elements. We called these *combinations of $r$ out of $n$* and we denoted their number by

$$
\binom{n}{r} = \frac{n!}{r!(n-r)!}
$$

It is now natural to consider counting the number of ways to construct from elements of a given set with $n$ elements a collection of size $r$ that is unordered but does allow repetitions. Such a collection is called a *bag* or a *multiset* (because it is set-like, but with repetitions). We shall call these *combinations of $r$ out of $n$ with repetition* and denote their number by $\binom{n}{r}$ (read $n$ “multichoose” $r$).

**Notation for bags** There is no standard notation. Let $A = \{a, b, c, d\}$ be the given set. Some people use the normal set notation (braces) for bags (multisets) and just repeat elements. For example, the bag $\{a, b, a, c, a, b\}$ is the same as the bag $\{a, a, a, b, b, c\}$. This can be dangerous for one who confuses them with sets in which the copies of the same element are somehow distinguishable. To avoid this, some people use double braces: $\{\{a, b, a, c, a, b\}\}$ (same as $\{\{a, a, a, b, b, c\}\}$).

An interesting alternative is to use braces, but, for each distinct element, use a natural number that gives the number of copies (repetitions). For instance the bag above would be $\{3 \cdot a, 2 \cdot b, 1 \cdot c\}$ or even $\{3 \cdot a, 2 \cdot b, 1 \cdot c, 0 \cdot d\}$! (we shall see that this can be thought of a certain kind of function.) Don’t use this notation when the elements of $A$ are numbers, say, $A = \{1, 2, 3, 4\}$, it’s confusing!

**Counting bags** Let $A = \{a_1, \ldots, a_n\}$ be the given set. Its elements are (clearly) distinguishable. It turns out that we can use exactly the stars and bars technique that we used to count donut purchases to count the number of bags of size $r$ constructed from elements of $A$ and thus obtain a formula for $\binom{n}{r}$.\footnote{A notation used by some textbooks, including Scheinerman.}

Think of constructing bag of size $r$ as separating $r$ stars into parts corresponding to copies of the same element of $A$. Since $|A| = n$ we use it with $n-1$ bars. This can be done by choosing $n-1$ positions for the bars (equivalently, choosing $r$ positions for the stars) in a sequence of length $n + r - 1$. Therefore

$$
\binom{n}{r} = \binom{n + r - 1}{r} = \binom{n + r - 1}{n - 1}
$$

2A notation used by some textbooks, including Scheinerman.
Warning  Note that is the number of bags of size $r$ (so repetitions are counted in the $r$) made from elements of a set of size $n$ (so no repetitions among the $n$).

Marbles in bins, coins to children  $\binom{n}{r}$ is also

- the number of ways of putting $r$ indistinguishable marbles in $n$ distinguishable bins; or,
- the number of ways of distributing $r$ indistinguishable coins\footnote{Many years ago this was taught with pennies, then with nickels, eventually with quarters, keeping up with inflation, you know. At some point we gave up and started using coins. I expect that at some point in the future we’ll use bitcoins and beam them directly into the children’s electronic personal assistant.} to $n$ distinguishable children; or,
- the number of distinct natural number solutions to the Diophantine equation

$$x_1 + x_2 + \cdots + x_n = r$$

Example 5.4  In how many ways can we distribute 11 indistinguishable coins to 3 distinguishable children?

Solution:  $\binom{3}{11} = \frac{\binom{3+11-1}{3-1}}{\binom{13}{2}}$.

Example 5.5  A (somewhat fairer) distribution of 11 indistinguishable coins to 3 distinguishable children assumes that each child gets at least 3 coins. In how many different ways can this be done?

Solution:  We begin by giving 3 coins each to the 3 children. This leaves us $11 - 3 \times 3 = 2$ coins to distribute to the 3 children according to the general scheme. So the number of ways is $\binom{3}{2} = \frac{\binom{3+2-1}{2}}{\binom{4}{2}} = \binom{4}{2} = 6$.

Definition 5.6  Let $m, n$ be two integers such that $m \leq n$. The integer interval $[m..n]$ is defined as the set of integers $\geq m$ and $\leq n$. That is

$$[m..n] = \{x \in \mathbb{Z} \mid m \leq x \leq n\}$$

Note that both $m$ and $n$ are elements in $[m..n]$. Note also that there are $n - m + 1$ elements in $[m..n]$. That is

$$\mid [m..n] \mid = n - m + 1$$

Do not confuse integer intervals with the usual intervals of real numbers. For instance $[0..1] = \{0, 1\}$ has two elements while $[0, 1]$ is infinite.
Example 5.7  What is the number of non-decreasing sequences of length 12 whose elements are taken from [10..30]? What is the number of increasing (we often say strictly increasing so there is no confusion) such sequences?

Solution:  [10..30] has 30 − 20 + 1 = 21 elements. Non-decreasing means that elements can be repeated, as long as copies are adjacent in the sequence. Here is a procedure for constructing a non-decreasing sequence of length 12 using integers in [10..30].

Step 1: choose an unordered collection of 12 out of the 21 numbers, with repetition.
Step 2: order it.

Step 1 can be done in \( \binom{21}{12} \) ways. Step 2 can be done in exactly one way. The answer is \( \binom{21}{12} \).

On the other hand, to count the number of strictly increasing sequences of length 12 made of numbers from [10..30] we choose a combination of 12 out of 21 (without repetition!) and then order it. The answer is \( \binom{21}{12} \).

Binomial coefficients  Mathematicians typically use this name instead of “(number of) combinations”. We are about to see why. A binomial is a sum of two terms, such as \( a + b \). The binomial theorem gives an expression for \( (a + b)^n \) where \( a \) and \( b \) are real numbers and \( n \) is a natural number.

Theorem 5.8 (Binomial Theorem) For any real numbers \( a \) and \( b \) and natural number \( n \)
\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k
\]

Proof: The formula is easily checked for \( n = 0 \). For \( n \geq 1 \), let’s think what happens when we calculate the expansion
\[
(a + b)^n = (a + b) \times (a + b) \times \cdots \times (a + b)
\]

We obtain a sum of terms (monomials), each of the form \( a^{n-k} b^k \), where \( k \) is 0 or 1, etc., . . . , or \( n \).

Most of these terms occur more than once (which ones occur exactly once?). How many times does \( a^{n-k} b^k \) occur in the expansion? This is the same number of times as there are orderings of \( n - k \) a’s and \( k \) b’s. This is equal to \( \binom{n}{k} \). Thus the coefficient of like terms of the form \( a^{n-k} b^k \) is \( \binom{n}{k} \). This proves the theorem.

Pascal’s Triangle  We know that \( (a + b)^0 = 1 \), \( (a + b)^1 = a + b \), \( (a + b)^2 = a^2 + 2ab + b^2 \), \( (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \), etc.

\[ ^4 \text{How many terms? Use the multiplication rule to figure out that there are } 2^n \text{ terms!} \]
Blaise Pascal observed some interesting relationships between the binomial coefficients when arranged in rows as follows:

Specifically, observe that each binomial coefficient is the sum of the two just above it. Formally:

**Theorem 5.9 (Pascal’s Identity)** If $n$ and $k$ are positive integers with $n \geq k$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

**Proof:** This can be easily verified using the factorial formula for combinations. However, the following proof is more insightful.

Let $S = \{x_1, x_2, \ldots, x_n\}$ be the set of $n$ elements. We count the number of subsets of size $k$ in two ways. One way gives the combinations of $k$ our of $n$, i.e., the LHS of Pascal’s Identity. In the second way, we observe that $k$-element subsets of $S$ can be partitioned into those that contain $x_n$ and those that don’t. For the former type of subset the other $k - 1$ elements come from $S \setminus \{x_n\}$. There are $\binom{n-1}{k-1}$ ways of choosing these subsets. For the latter type of subset all of the $k$ elements must be chosen from $S \setminus \{x_n\}$. There are $\binom{n-1}{k}$ ways of doing this. Thus, by the sum rule the number of $k$-element subsets of $S$ is given by the RHS of Pascal’s Identity. Proofs like the one we just saw are called *combinatorial*.  

**Combinatorial proofs** We now prove a couple of identities using the following technique. To prove an identity we will pose a counting question. We will then answer the question in two ways, one answer will correspond to LHS and the other would correspond to the RHS of the identity. We have seen an example of this technique in the proof of Pascal’s Identity. This technique is often called a *combinatorial proof*.

**Example 5.10** Prove that

$$\binom{n}{r} = \binom{n}{n-r}$$
Solution: We give a combinatorial proof. Let $X$ be a set of size $n$. We count the number of subsets of size $r$ of $X$ in two ways. We already did one such count when we defined combinations and we know the answer is \( \binom{n}{r} \).

Alternatively we count the number of ways we can choose elements that are not in a subset of size $r$. That is, count the number of subsets of size $n - r$ which is, of course, \( \binom{n}{n-r} \).

The subsets of size $r$ are in one-to-one correspondence with their complements (see Example 3.12 in Lecture 3). These complements are exactly the subsets of size $n - r$. Therefore there must be as many subsets of size $n - r$ as there are subsets of size $r$. This proves the identity.

(The argument that uses, informally, the one-to-one correspondence is an example of something called the Bijection Rule. We will learn about it when we study functions.)

Example 5.11 Prove that

\[
\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0
\]

Solution: One way to solve this problem is by substituting $a = 1$ and $b = -1$ in the Binomial Theorem, yielding

\[
0^n = 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.
\]

However, a combinatorial proof will give us more insight into what the expression means. We want to prove that

\[
\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots
\]

Consider a set $X = \{x_1, x_2, x_3, \ldots, x_n\}$. We want to show that the total number of subsets of $X$ that have even size equals the total number of subsets of $X$ that have odd size. We will now show that both these quantities equal $2^{n-1}$ from which the claim follows.

An even-sized subset of $X$ can be constructed as follows.

Step 1. Decide whether $x_1$ belongs to the subset or not.
Step 2. Decide whether $x_2$ belongs to the subset or not.

\[\vdots\]

Step $n$. Decide whether $x_n$ belongs to the subset or not.

In the first $n - 1$ steps one can make either one of the 2 choices, in or out. But in step $n$ only one choice is possible! This is because if we have chosen an even number of elements from $X \setminus \{x_n\}$ to put in the subset then we must leave out $x_n$ otherwise we must include $x_n$ in the subset. Hence using the multiplication rule the total number of even-sized subsets of $X$ equals $2^{n-1}$. 

8
Another way of thinking about this is to count in two steps: in the first step choose a subset of \( \{x_1, \ldots, x_{n-1}\} \); in the second step add or not \( x_n \) to the subset chosen in the first step, making sure the result has even size (don’t forget that 0 is even!).

To compute the number of odd-sized subsets we could proceed similarly. Or, we could count complementarily: since we know that the total number of subsets of \( X \) is \( 2^n \), the total number of odd-sized subsets of \( X \) is \( 2^n - 2^{n-1} = 2^{n-1}(2 - 1) = 2^{n-1} \).

Example 5.12 Prove that \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \).

Solution: We pose the following counting question.

Given a set \( S \) of \( n \) distinct elements how many subsets are there of the set \( S \)?

From earlier lectures, we know that the answer is \( 2^n \). This gives us the RHS.

Another way to compute the answer to the question is as follows. The powerset \( 2^S \) containing all possible subsets can be partitioned into \( S_0, S_1, \ldots, S_n \), where \( S_i, 0 \leq i \leq n \), is the set of all subsets of \( S \) that have cardinality \( i \). Thus

\[
|2^S| = |S_0| + |S_1| + \ldots + |S_n| = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = \text{LHS}
\]

This proves the claim.

(How do you derive the same identity from the Binomial Theorem?)

Example 5.13 Give a combinatorial proof to show that

\[
\sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k} = \binom{n+m}{r}
\]

Solution: We pose the following counting question.

There are \( n \) men and \( m \) women. How many ways are there to form a committee of \( r \) people from this group of people?
By definition, there are \( \binom{n+m}{r} \) distinct committees of \( r \) people. This gives us the RHS.

The set \( S \) of all possible committees of \( r \) people can be partitioned into subsets \( S_0, S_1, S_2, \ldots, S_r \), where \( S_k \) is the set of committees in which there are exactly \( k \) men and the rest \( r - k \) are women. Note that \(|S_k| = \binom{n}{k} \binom{m}{r-k}\). Thus we have

\[
|S| = \sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k}
\]

which gives us the left hand side of the expression. \( \blacksquare \)