PROOFS: STRONG INDUCTION

Example 12.1 Any integer \( n \geq 2 \) can be written as the product of one or more (not necessarily distinct) prime numbers.

Solution: Let’s try to prove this by induction.

(Base Case) \( n = 2 \). Check.

(Induction Step) Let \( k \geq 1 \) be an arbitrary natural number. Assume (IH) that \( k \) can be written as the product of one or more (not necessarily distinct) prime numbers.

Now consider \( k + 1 \). If \( k + 1 \) is prime we are done (without even using the IH!).

If \( k + 1 \) is composite then by Proposition 1.3 (Lecture 1) we have \( k + 1 = r \cdot s \) where \( r, s \) are both integers \( > 1 \). This implies that \( r, s \) are both \( \leq (k + 1)/2 \).

So now we would like to use the IH on \( r \) and \( s \). Unfortunately, none of them equals \( k \)! On the other hand since \( r, s \leq k \) the induction “process” must have gone through them already!

This is the idea of the Strong Induction Principle which we explain next,

Proof Pattern STRONG INDUCTION

(Base Case) Prove \( P(n_0) \).

(Induction Step) Let \( k \in \mathbb{N} \) s.t. \( k \geq n_0 \). Prove \( P(n_0) \land \cdots \land P(k) \Rightarrow P(k + 1) \).

Then you can conclude that for all natural numbers \( n \geq n_0 \) we have \( P(n) \).

In spite of its name Strong Induction is logically equivalent to Ordinary Induction (and hence, further equivalent to the Well-Ordering Principle). This is not hard to prove, you should try it. This equivalence is the main reason why I am against calling Ordinary Induction “weak”.

1The IH for strong induction is indeed stronger that the IH for ordinary induction: \( P(n_0) \land \cdots \land P(k) \) implies \( P(k) \), hence the name. Just because the IH is stronger does not mean that strong induction allows us to prove statements that cannot be shown using ordinary induction. In fact, it is not hard to show that any proof by strong induction has be transformed into a proof by ordinary induction (the idea is to use a different predicate \( P(n) \)). Using strong induction when appropriate makes some proofs easier and clearer.
Now let’s go back to the proof earlier and do it with strong induction. The base case is the same. The IH is now: every integer between 2 and \( k \) (inclusive) can be written as the product of one or more (not necessarily distinct) prime numbers.

If \( k + 1 \) is prime we are done, like before, If \( k + 1 \) is composite then \( k + 1 = r \cdot s \) where \( r, s \) are both integers \( > 1 \) and therefore \( \leq k \). So the IH applies to \( r \) and \( s \):

\[
r = p_1 \cdot \ldots \cdot p_m \quad \text{and} \quad s = q_1 \cdot \ldots \cdot q_n
\]

where \( p_1, \ldots, p_m, q_1, \ldots, q_n \) are primes. Then \( k + 1 = r \cdot s = p_1 \cdot \ldots \cdot p_m \cdot q_1 \cdot \ldots \cdot q_n \). and we are done. ■

**Example 12.2** If a polygon with four or more sides is triangulated then at least two of the triangles formed are exterior.

![Diagram of a polygon with triangulation]

**Solution:** By strong induction on the number \( n \) of vertices of the polygon.

**(BASE CASE)** \( n = 4 \). To triangulate a quadrilateral we draw one diagonal. Both resulting triangles are exterior.

**(INDUCTION STEP)** Let \( k \geq 4 \).

Assume (IH) that for any triangulated polygon with a number of sides between 4 and \( k \) at least two of the formed triangles are exterior.

Now consider a triangulated polygon \( P \) with \( k + 1 \) sides. Let \( d \) be one of the diagonals used in the triangulation. It will divide \( P \) into two other triangulated polygons, \( A \) and \( B \). See picture below.

![Diagram of a polygon with triangulation and diagonal]

Now a crucial observation is that both \( A \) and \( B \) have at most \( k \) sides. Indeed, \( d \) is a diagonal so \( A, B \) have at least three sides, two of which are not \( d \) itself. If one of them had \( k + 1 \) or more sides then \( P \) would have \( k + 2 \) or more sides, contradiction.
Lemma 12.3  The triangulation of $A$ has at least one triangle that is exterior for the triangulation of $P$.

Proof: If $A$ is itself a triangle we are done. Otherwise, $A$ has between 4 and $k$ sides and the IH applies so the triangulation of $A$ has at least two triangles which are exterior for $A$. At most one of these two triangles includes $d$. Therefore, the other one must be exterior for $P$ as well. ■

The lemma applies to $B$ as well so, in total, we have at least two exterior triangles for $P$. ■

Example 12.4  What is the largest number of pieces (not slices!) of pizza that can be made with $n$ distinct straight cuts?

Solution: A bit of thinking makes you realize that the pizza is a red herring one (not as good as anchovies!) because it does not matter where exactly is the crust of the pizza. In fact this problem is known as Steiner’s Plane Cutting Problem.

Some experimentation leads us to the following:

The number of pizza pieces is *NOT* maximized unless (1) every cut crosses every other cut, and (2) no three cuts cross at the same point.

Indeed it is easy to see that if the cuts do *not* satisfy (1) or (2) then you can move lines “a little” and increase the number of pieces.

What is not clear a priori is that any two sets of cuts satisfying (1) and (2) actually produce the same number of pieces (which then must be the largest number of pieces). This will follow from the following observations.

There is exactly one set of 0 cuts, it satisfies (1) and (2) vacuously, and it produces 1 piece.

Let $S$ be any set of $n \geq 1$ cuts satisfying (1) and (2). Delete one of the cuts, call it $c$ in $S$ and obtain a set $S'$ of $n - 1$ cuts. Note that $S'$ also satisfies (1) and (2).

And note that if $p(S)$ and $p(S')$ are the number of pieces yielded by $S$, respectively $S'$, then $p(S) = p(S') + n$. That’s because the $n - 1$ lines in $S'$ together with the pizza crust determine $n$ segments on $c$ and each of these segments is a side for a new piece that the addition of $c$ produces.
Using \( p(S) = p(S') + n \) we can prove by induction the following:

**Lemma 12.5** For any \( n \in \mathbb{N} \), for any two sets of \( n \) cuts \( S_1 \) and \( S_2 \) both satisfying (1) and (2), the number of pieces that \( S_1 \) and \( S_2 \) produce is the same.

We skip the easy proof.

Let \( C(n) \) be the number of pieces produced by a set of \( n \) cuts satisfying (1) and (2). By the lemma, it does not matter which such set of cuts we consider because they all produce the same (maximum) number of pieces. From the considerations above we conclude that

\[
C(0) = 1 \quad \quad \quad C(n) = C(n - 1) + n
\]

This is a recurrence relation. Recurrences are extremely useful in the analysis of algorithm (e.g., the divide-and-conquer kind). How do you solve them, i.e., for above obtain an expression for \( C(n) \)? One way is to guess the answer and the prove it by induction. Another way is to analyze the “recursion tree” constructed from the recurrence. For example, here is the recursion tree for pizza recurrence above (for \( n = 6 \):)

```
C(6)
 / \ 
C(5) 6
 / \ 
C(4) 5
 / \ 
C(3) 4
 / \ 
C(2) 3
 / \ 
C(1) 2
 / \ 
C(0) 1
```

Sometimes we can use a “telescopic” trick to solve such recurrences. We write them one under the
other for \(n, n-1, n-2, \ldots, 2, 1:\)

\[
\begin{align*}
C(n) &= C(n-1) + n \\
C(n-1) &= C(n-2) + n - 1 \\
C(n-2) &= C(n-3) + n - 2 \\
&\vdots \\
C(2) &= C(1) + 2 \\
C(1) &= C(0) + 1 
\end{align*}
\]

Then we add all the LHS and all the RHS. After we (telescopically) cancel the terms that appear on both sides we obtain

\[
C(n) = C(0) + 1 + 2 + \cdots + n = 1 + \frac{n(n+1)}{2} = \frac{n^2 + n + 2}{2}
\]

**Another way of counting**  In the picture below note two kinds of crosspoints: boundary crosspoints (where a cut intersects the crust) and interior crosspoints (where two cuts intersect).

Now we can define a mapping between pieces and the unique crosspoint at the “top” of the piece (by rotating the pizza to make sure no cut is “horizontal”). Note that every interior crosspoint is mapped to from exactly one piece. The same is true for every “upper” boundary crosspoint (i.e., half of the boundary crosspoints) except the one of the top of the figure which is mapped to from two pieces. Therefore, the number of pieces is

\[
C(n) = 1 + n + \binom{n}{2} = \frac{n^2 + n + 2}{2}
\]