PROOFS

Fibonacci Numbers  Fibonacci (1170?-1250?) was a mathematician from Pisa who studied the following (not very realistic :) ) problem:

A farmer raises rabbits. Each rabbit pair gives birth to a another rabbit pair when it turns one month old, and then to one rabbit pair each month thereafter. Rabbits never die. How many rabbit pairs will the farmer have at the end of the \( n \)th month if he starts with one newborn rabbit pair in the first month?

The number of rabbits in month \( n \) is denoted by \( F_n \). We have the following recurrence:

\[
\begin{align*}
F_0 &= 0 \\
F_1 &= 1 \\
F_n &= F_{n-1} + F_{n-2} \quad \text{for } n \geq 2
\end{align*}
\]

Hence the sequence of Fibonacci numbers:

\[
\begin{array}{cccccccccc}
F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 \\
0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34
\end{array}
\]

Suprisingly, Fibonacci numbers are useful in Computer Science! Their most important property is that they grow (truly!) exponentially. In fact, we have the following.

Proposition 12.1

\[
F_n = (\varphi^n - \psi^n)/\sqrt{5}
\]

where \( \varphi > \psi \) are the two roots of the equation \( x^2 - x - 1 = 0 \).

\( \varphi = (1 + \sqrt{5})/2 \) is known as the Golden Ratio and \( \psi = 1 - \varphi = -1/\varphi = (1 - \sqrt{5})/2 \).

Proof: First of all, what do the roots of a quadratic equation have to do with the solution to a recurrence?

We ask the question, can \( F_n = F_{n-1} + F_{n-2} \) have a solution of the form \( F_n = r^n \)? This is trivially so if \( r = 0 \) but this won’t work later when we also consider the values of \( F_0 \) and \( F_1 \). Assuming \( r \neq 0 \),
plugging in, and dividing both sides by \( r^{n-2} \) gives us \( r^2 = r + 1 \) hence \( r^2 - r - 1 = 0 \) so this works for both solutions to this quadratic equation, denoted \( \varphi \) and \( \psi \) above.

As we shall see (by strong induction, below) the solution to the Fibonacci recurrence (and this means satisfying the conditions (1), (2) and (3) above) is unique. This is intuitively clear, after all we had only one way to compute \( F_2, \ldots, F_9 \) above. We could even prove this by induction. So how we get from \( \varphi^n \) or \( \psi^n \) which satisfy condition (3) but not conditions (1) and (2), to a solution that satisfies all three conditions?

It is not hard to see that (because (3) is homogenous) any linear combination of solutions to (3) is also a solution to (3). So for any \( \lambda_1, \lambda_2 \in \mathbb{R} \) the sequence defined by \( s_n = \lambda_1 \varphi^n + \lambda_2 \psi^n \) satisfies \( s_n = s_{n-1} + s_{n-2} \). Moreover, we can compute \( \lambda_1, \lambda_2 \) to make sure that \( s_0 = 0 \) and \( s_1 = 1 \) as follows.

We get the system of equations in unknowns \( \lambda_1, \lambda_2 \):

\[
\begin{align*}
\lambda_1 + \lambda_2 &= 0 \\
\lambda_1 \varphi + \lambda_2 \psi &= 1
\end{align*}
\]

Solving, we get \( \lambda_1 = 1/(\varphi - \psi) = 1/\sqrt{5} \) and \( \lambda_2 = 1/(\psi - \varphi) = -1/\sqrt{5} \). We conclude that \( s_n = (\varphi^n - \psi^n)/\sqrt{5} \) satisfies simultaneously \( s_0 = 0 \), \( s_1 = 1 \), and \( s_n = s_{n-1} + s_{n-2} \).

**Lemma 12.2** \( \forall n \in \mathbb{N} \) \( F_n = s_n \).

We prove this by *strong* induction. In fact, we use the following version of strong induction, which follows from the version we presented initially.

This is a variant of the **Strong Induction Principle** which is particularly well applicable to proving facts about the Fibonacci numbers explain next,

**Proof Pattern** **STRONG INDUCTION (FIBONACCI EDITION)**

**(BASE CASE 1)** Prove \( P(n_0) \).

**(BASE CASE 2)** Prove \( P(n_0 + 1) \).

**(INDUCTION STEP)** Let \( k \in \mathbb{N} \) s.t. \( k \geq n_0 + 1 \). Prove \( P(k-1) \land P(k) \Rightarrow P(k+1) \).

Details in class.

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Fibonacci Numbers and the Golden Ratio (Section) have many elegant mathematical properties and fascinating connections with naturally occurring phenomena, see [http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci](http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci).

Here are some interesting identities to be shown by induction, sometimes by strong induction (Fibonacci edition).
We may do one of these in class, try to do the others.

\[ F_0 + F_1 + \cdots + F_n = F_{n+2} - 1 \]
\[ F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1} \]
\[ F_1 + F_3 + \cdots + F_{2n-1} = F_{2n} \]
\[ F_0 - F_1 + F_2 - F_3 + \cdots - F_{2n-1} + F_{2n} = F_{2n-1} - 1 \]

\[ \binom{2n}{0} + \binom{2n-1}{1} + \cdots + \binom{n+1}{n-1} + \binom{n}{n} = F_{2n+1} \]

**Summation manipulations** “Doing sums” is a traditional mathematical skill just like integration. In fact, the two have some similar properties, for example linearity:

\[ \sum_{i=1}^{n} (c E_i) = c \sum_{i=1}^{n} E_i \]
\[ \sum_{i=1}^{n} (E_i + H_i) = \left( \sum_{i=1}^{n} E_i \right) + \left( \sum_{i=1}^{n} H_i \right) \]

Here \( c \) does not depend on \( i \), while \( E_i \) and \( H_i \) are expressions that depend on \( i \). Here is an example:

\[ \sum_{i=1}^{n} (2i - 1) = 2 \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 2 \cdot \frac{n(n+1)}{2} - n = n^2 + n - n = n^2 \]

**Example 12.3** Denote

\[ \zeta_n(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \]

Prove that \( \forall n \zeta_n(2) < 2. \]

**Solution:** Try to prove this directly by induction. You will get into trouble because the IH is not saying “enough”.

Here is a proof that invents the idea of examining a different sum, one that can be summed up “telescopically”.

**Lemma 12.4**

\[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1} \]

\[ \text{The notation comes from Riemann’s zeta function (google it). Combining this result with a bit of calculus, it follows that the series that adds the reciprocals of the squares converges. Contrast this with the harmonic series. In fact, we have shown that this series converges to a number between 1 and 2. Using more calculus, Euler has shown that the limit is, in fact, } \pi^2/6. \]
Proof:
\[
\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \left( \sum_{k=1}^{n} \frac{1}{k} \right) - \left( \sum_{k=1}^{n} \frac{1}{k+1} \right) = \\
= \left( \sum_{k=1}^{n} \frac{1}{k} \right) - \left( \sum_{\ell=2}^{n+1} \frac{1}{\ell} \right) = \left( 1 + \sum_{k=2}^{n} \frac{1}{k} \right) - \left( \sum_{\ell=2}^{n} \frac{1}{\ell} + \frac{1}{n+1} \right) \\
= 1 + \sum_{k=2}^{n} \frac{1}{k} - \sum_{\ell=2}^{n} \frac{1}{\ell} - \frac{1}{n+1} = 1 - \frac{1}{n+1}
\]

Now we use the lemma to finish the solution of the example. First we observe that \(1/k^2 < 1/(k-1)k\).

Using also a “change of variable” in summation we get:

\[
\zeta_n(2) = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \frac{1}{(k-1)k} = 1 + \sum_{i=1}^{n-1} \frac{1}{i(i+1)} = 1 + 1 - \frac{1}{n}
\]

In the last step we applied the lemma (for \(n-1\) instead of \(n\)). Therefore \(\zeta_n(2) < 2\).

The proof also suggest that if we insist on using induction, what we should prove is \(\forall n \; \zeta_n(2) < 2 - (1/n)\). Try it.