GRAPH THEORY

We proved in the previous lecture that in a tree the number of edges is completely determined by the number of nodes: $|E| = |V| - 1$.

Example 20.1  Prove that if we erase any edge in a tree then the resulting graph is not connected anymore.

Solution: If we erase an edge then the resulting graph cannot be a tree, by the contrapositive of “tree implies $|E| = |V| - 1$”. But erasing an edge does not create cycles so the resulting graph is still acyclic. The only way it can fail to be a tree is if it is disconnected. ■

Definition 20.2  An edge is cut edge in a graph if erasing it strictly increases the number of connected components.

Example 20.1 shows that in a tree every edge is a cut edge. But how many connected components result from such an erasure?

Example 20.3  Prove that erasing a cut edge in a graph increases the number of connected components by exactly one.

Solution: Let $e \equiv u-v$ be a cut edge in $G = (V, E)$. Erasing $e$ produces the graph $G_e = (V, E \setminus \{e\})$. $u$ and $v$ must belong to the same connected component of $G$ (why?), call it $D$, and this is the only connected component of $G$ affected by the erasure. Suppose toward a contradiction, that in $G_e$ the component $D$ splits into three or more distinct components. Let $D_1, D_2, D_3$ be three of these components and consider three distinct vertices $w_1 \in D_1, w_2 \in D_2, w_3 \in D_3$. In $G$ there existed paths $w_1 \cdots w_2, w_2 \cdots w_3, w_3 \cdots w_1$ but in $G_e$ these paths cannot exist. Thus, $e$ appears in all three of these paths in $G$.

We will show that this situation implies that there is a walk (hence a path) in $G_e$ between at least two of the three vertices $w_1, w_2, w_3$, which contradicts the fact that in $G_e$, these are distinct connected components.
Consider the paths \( w_1 \cdots w_2 \), \( w_2 \cdots w_3 \), \( w_3 \cdots w_1 \) as sequences of vertices and edges which gives them “direction”. All three paths traverse \( e \) but there are only two traversal directions for \( e \) so by PHP two of the paths must traverse \( e \) in the same direction. W.l.o.g. we can assume that these two paths are \( p_1 \equiv w_1 \cdots u \cdots v \cdots w_2 \) and \( p_2 \equiv w_2 \cdots u \cdots v \cdots w_3 \). From \( p_1 \) and \( p_2 \) we construct in \( G \) a walk \( w_1 \cdots u \cdots w_2 \) that does not contain \( e \) and is therefore also a walk in \( G_e \).

**Example 20.4** Prove that cut edges cannot be part of any cycle.

**Solution:** We prove the contrapositive, i.e., we prove that any edge that belongs to a cycle is not a cut edge.

Let \( e \) belong to a cycle \( C \) in \( G \). If two vertices \( u \) to \( v \) are connected by a path in \( G \) that uses \( e \) then erasing \( e \) still leaves \( u \) and \( v \) connected, because we can replace \( e \) with the path that is the part of \( C \) obtained by deleting \( e \) from \( C \) thus obtaining a walk (hence a path) from \( u \) to \( v \).

In Proposition 18.9 we proved that “where there is a walk there is a path”. One might think that something similar is true for cycles, i.e., “where there is a closed walk there is a cycle”. This is false, however. A trivial counterexample consists of a walk of length 0. A more significant counterexample would be any closed walk in a tree with two or more nodes; it cannot contain a cycle because trees are acyclic. However, we do have the following:

**Proposition 20.5** Any closed walk of non-zero length that traverses an edge exactly once contains a cycle (here “contains” means that the edges of the cycle are among the edges of the walk and are traversed in the same direction).

**Proof:** The walk cannot have length 1 because it is closed and it cannot have length 2, i.e., be \( u \cdots v \cdots u \), because it would traverse the edge \( u \cdots v \) twice. Thus, it must have length \( \geq 3 \).

We have three cases.

**Case 1:** the walk is \( u \cdots v \cdots \), has length \( \geq 3 \), and it traverses the edge \( \{u, v\} \) exactly once. Then, the portion \( v \cdots u \) of the walk has length \( \geq 2 \). By Proposition 18.9 there exists a path from \( v \) to \( u \). This path cannot have length 1 because then it would be \( v \cdots u \) and the original walk \( u \cdots v \cdots u \) would traverse the edge \( \{u, v\} \) twice. Hence this path has length \( \geq 2 \) and together with the edge \( u \cdots v \) it forms a cycle. Note that this cycle is not necessarily contained in the original closed walk. For that we would a strengthened version of Proposition 18.9: every walk contains a path with the same endpoints. Although we did not prove this (it’s somewhat messy), it is actually true and you should recall this for further use.

**Case 2:** the walk is \( u \cdots \cdots v \cdots u \), has length \( \geq 3 \), and it traverses the edge \( \{u, v\} \) exactly once. This case is similar to Case 1 and we obtain again a cycle.

**Case 3:** the walk is \( u \cdots \cdots v \cdots w \cdots \cdots \), it has length \( \geq 3 \), it traverses the edge \( \{v, w\} \) exactly once, \( u \neq v \), and \( u \neq w \). Again we invoke Proposition 18.9 (strengthened version) for the portions \( u \cdots \cdots v \), etc.
and $w \cdots u$ of the walk and obtain a path $p$ from $u$ to $v$ and a path $q$ from $w$ to $u$. Importantly, neither $p$ nor $q$ contains the edge $\{v, w\}$. We now have some subcases.

**Case 3a:** $v$ occurs in $q$. Then the portion $w \cdots v$ of $q$ must have length $\geq 2$ otherwise $\{v, w\}$ is traversed twice. It follows that $v \cdots w \cdots v$ is a cycle.

**Case 3b:** $w$ occurs in $p$. Similar to Case 3a, $w \cdots v \cdots w$ is a cycle.

**Case 3c:** $v$ does not occur in $q$ and $w$ does not occur in $p$. In this case, $p$ and $q$ still have vertices in common, certainly $u$ one such vertex, but they are all distinct from $v$ and from $w$. Among all these common vertices let $z$ be the vertex that is closest to $v$ in $p$. Then the portion $z \cdots v$ of $p$ and the portion $w \cdots z$ of $q$ have only $z$ in common. Therefore $z \cdots v \cdots w \cdots z$ is a cycle.

In all the cases the cycle whose existence we proved was formed of edges that already appeared in the original walk.

**Example 20.6** Prove that adding an edge to an acyclic graph creates at most one cycle.

**Solution:** Let $u, v$ be two distinct non-adjacent vertices in an acyclic graph $G = (V, E)$. We add $u-v$ thus producing $G_{uv} = (V, E \cup \{u-v\})$ and we wish to show that $G_{uv}$ has at most one cycle.

Suppose, toward a contradiction, that $G_{uv}$ has at least two distinct cycles $C_1$ and $C_2$. Since $G_{uv}$ was acyclic $u-v$ must belong to both $C_1$ and $C_2$. Since $C_1$ and $C_2$ are distinct one of them must contain an edge that is not in the other one. Let $e$ be that edge.

Deleting $u-v$ from $C_1$ gives us a path from $u$ to $v$ in $G$. Deleting $u-v$ from $C_2$ gives us a path from $v$ to $u$ in $G$. Concatenating these two walks gives us a closed walk from $u$ to $u$ that traverses $e$ exactly once. By the proposition above, such a closed walk must contain a cycle, which contradicts the acyclicity of $G$.

**Proposition 20.7** For any graph $G = (V, E)$ the following statements are equivalent:

(i) $G$ is a tree, i.e., it is connected and acyclic.

(ii) $G$ is connected and $|E| = |V| - 1$.

(iii) $G$ is minimally connected, i.e., it is connected and $G_e = (V, E \setminus \{e\})$, for any edge $e$, is disconnected (equivalently, every edge is a cut edge).

(iv) $G$ is maximally acyclic, i.e., it is acyclic and or any two non-adjacent vertices $u$ and $v$, adding the edge $u-v$ creates a cycle.
(v) Any two vertices of $G$ are connected by a unique path.

Proof:

(i)$\Rightarrow$(ii) We proved this in Lecture 19.

(ii)$\Rightarrow$(iii) We proved this above.

(iii)$\Rightarrow$(iv) Let $G$ be minimally connected, i.e., all its edges are cut edges. Then $G$ is acyclic. Indeed, if $G$ had a cycle then the edges in that cycle cannot be cut edges, as proved above.

Now let $u$ and $v$ be non-adjacent vertices in $G$. We add $u-v$ and obtain $G_{uv} = (V, E \cup \{u-v\})$. Since $G$ is connected there is a path from $u$ to $v$ in $G$. Adding $u-v$ to this path produces a cycle in $G_{uv}$.

(iv)$\Rightarrow$(v) Let $u, v$ be any two vertices of $G$. If $u = v$ then they are connected by a unique path, the path of length 0. If $u$ and $v$ are adjacent, then we have the path $u-v$. Suppose, toward a contradiction, that there is another path from $u$ to $v$. Then this path together with $u-v$ forms a cycle, which contradicts the acyclicity of $G$.

We are left with the case when $u$ and $v$ are non-adjacent. Then $G_{uv} = (V, E \cup \{u-v\})$ has at least one cycle, $C$. This cycle must contain $u-v$. Deleting $u-v$ from $C$ gives us a walk, in fact a path, from $u$ to $v$, which is in $G$.

Now suppose there were two distinct such path in $G$ from $u$ to $v$. Each of them, together with $u-v$ creates a distinct cycle in $G_{uv}$. But we have shown above that adding an edge to an acyclic graph creates at most one cycle so we have a contradiction.

(v)$\Rightarrow$(i) The graph $G$ is connected because there is a path between any two nodes. Suppose, toward a contradiction, that $G$ has a cycle. Let $u$ and $v$ be two distinct nodes in this cycle. The cycle yields two distinct paths from $u$ to $v$ so we have a contradiction. ■