Problem 1: Prove that if every vertex in an undirected graph $G$ has degree greater than or equal to 2, there is at least one cycle in $G$.

Solution:

Maximal Path:

Consider a maximal path $p$ in $G$. Let the end vertices of $p$ be $u$ and $v$. Now, consider $u$. We know that $u$ must have at least two edges coming from it because every vertex in the graph has degree of two or greater. One edge of $u$ is present in the maximal path itself. The other edge must lead to some vertex $w$. If $w$ is not present in $p$, then we could extend $p$ with the edge $(u, w)$, which would make $p$ not maximal. Thus, we know that $w$ must be in $p$. Since there cannot be edges from a vertex to itself and since we cannot have two edges between the same pair of vertices, we are guaranteed that $u \neq w$ and that $(u, w)$ is not present in $p$. Thus, we have two distinct paths between $u$ and $w$: the path between $u$ and $w$ in $p$ and the edge $(u, w)$. These two paths form a cycle in $G$.

Proof by Contradiction:

Assume for contradiction that every vertex has a degree greater than or equal to 2, but the graph does not contain a cycle. Since $G$ does not contain a cycle, it is a forest and there must exist some vertex $v$ such that $v$ is a leaf. Then we reach a contradiction, as every vertex needs to have a degree of at least 2.
**Problem 2:** Krishna goes off into a room and does the following: He flips a fair coin, and if the result is heads, then he rolls one die, and if the result is tails, then he rolls two dice. He comes back and tells you the sum of the dice (potentially just one number) is 2. What is the probability he flipped heads?

**Solution:**

We first note that the sample space $\Omega$ could be represented as the result of the coin toss, followed by the results of the die roll(s).

We then define the following events:

$A :=$ The event Krishna rolled die with a sum of 2.

$B :=$ The event Krishna flipped a heads.

We are asked to determine $\Pr[B|A]$. We can rewrite this as follows:

$$
\Pr[B|A] = \frac{\Pr[A|B] \Pr[B]}{\Pr[A]}
= \frac{\Pr[A|B] \Pr[B]}{\Pr[A|B] \Pr[B] + \Pr[A|\bar{B}] \Pr[\bar{B}]}
$$

We know that, if the coin turned up heads, the probability of rolling a 2 occurs if the only die rolled is a 2:

$$
\Pr[A|B] = \frac{1}{6}
$$

Similarly, if the coin turned up tails, then rolling a sum of 2 can only happen when two 1’s are rolled, so we have:

$$
\Pr[A|\bar{B}] = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}
$$

Thus, because the coin is fair and $\Pr[\bar{B}] = \Pr[B] = \frac{1}{2}$, we arrive at a final answer:

$$
\Pr[B|A] = \frac{\Pr[A|B] \Pr[B]}{\Pr[A|B] \Pr[B] + \Pr[A|\bar{B}] \Pr[\bar{B}]}
= \frac{\frac{1}{6}(\frac{1}{2})}{\frac{1}{6}(\frac{1}{2}) + \frac{1}{36}(\frac{1}{2})}
= \frac{6}{7}
$$
Problem 3: Suppose you have an unlimited number of balls labeled 3 and 5. You are given a target sum \( n \geq 8 \). Show that you can always pick out some combination of balls such that their sum matches the target sum.

Solution:

Proof by strong induction: Let \( P(n) \) be the claim that there is some combination of 3 and 5 balls such that their sum is \( n \).

Induction Hypothesis: For all \( 8 \leq j \leq k \), \( P(j) \) is true.

Base Cases: \( P(8) \). We can select one ball labeled 3, and one ball labeled 5. Thus, we have \( 3 + 5 = 8 \)

\( P(9) \). We can do this using 3 balls labeled with value 3.

\( P(10) \). We can do this using 2 balls labeled with value 5.

Induction Step: We wish to show \( P(k + 1) \) holds.

After handling the base cases, we know that it must be true that \( k + 1 \geq 11 \). By the strong induction hypothesis, we know there is some combination of balls for which the labels sum to \( k - 2 \), since \( 8 \leq k - 2 \leq k \). We can then add one ball labeled 3 to this set, resulting in a total sum of \( k + 1 \), as desired.

Thus, by strong induction, we can always choose balls such that they sum to \( n \) for any \( n \geq 8 \).
Problem 4: For any natural number $n$, $\exists$ a number $m$ composed of digits 5 and 0 only, and $m$ is divisible by $n$.

Solution:

Given $n + 1$ numbers, at least 2 of them have the same remainder when divided by $n$ using the pigeonhole principle.
Consider the numbers 5, 55, 555... and take $n + 1$ of them. In these $n + 1$ numbers, let $x$ and $y$ be the two that have the same remainder when divided by $n$. Without loss of generality, let $x$ be the larger of the two numbers. The difference between $x$ and $y$ will consist of just 5’s and 0’s

\[
\begin{align*}
  x &= k \times n + r, k \in \mathbb{Z}, r \in \mathbb{N} \\
  y &= l \times n + r, l \in \mathbb{Z}, r \in \mathbb{N} \\
  x - y &= (k - l) \times n
\end{align*}
\]

We therefore have shown that $x - y$ is divisible by $n$, and since $x - y$ consists only of 5’s and 0’s, we have found an $m$ that satisfies the constraints of the question.