CIS 160
Recitation Guide - Week 10

Topics Covered: Digraphs, Maximum Path, k-colorable

Problem 1
Prove that every directed acyclic graph (DAG) has a vertex with in-degree of 0.

Solution:
Consider directed acyclic graph \( G = (V, E) \). We prove using contradiction that this graph has a vertex with in-degree 0. For the sake of contradiction, assume there are no vertices in \( G \) with in-degree 0.

Let \( P \) be a directed path of maximum length in \( G \). Consider the vertices on this path to be \( v_1, v_2, \ldots, v_k \), where \( 1 \leq k \leq |V| \). We thus know that \( v_2, \ldots, v_k \) do not have in-degree of 0. As every vertex on this path has to have in-degree of greater than 0, there exists an edge going from some vertex \( u \) to \( v_1 \). We now have two cases:

Case 1: \( u \) is not a vertex on \( P \), meaning there exists a path \( P' = u, v_1, v_2, \ldots, v_k \). \( P' \) is a longer path, contradiction.

Case 2: \( u \) is on \( P \). Assume \( u = v_i \) (\( 1 < i \leq k \)). Then, \( G \) contains a cycle: \( v_1, v_2, \ldots, v_i, v_1 \), which contradicts with the fact that \( G \) is a directed acyclic graph.

Therefore, by contradiction, we have shown that every directed acyclic graph must have at least one vertex with in-degree of 0.
Problem 2

Let $P_1$ and $P_2$ denote two paths in a connected graph $G$ with maximum length. Prove that $P_1$ and $P_2$ have a common vertex.

Solution:

Assume towards a contradiction that $P_1$ and $P_2$ do not share a common vertex. Since the graph is connected, there exists a shortest path connecting $P_1$ to $P_2$ with endpoints at vertices $u$ in $P_1$ and $v$ in $P_2$. Call this shortest path connecting $u$ to $v$ $P_3$. $P_3$ contains no vertices in $P_1$ or $P_2$ other than $u$ and $v$ (If it did, then we could find a shorter path connecting vertices in $P_1$ and $P_2$ by cutting out the extra vertices in $P_3$.)

Call the endpoints of $P_1$ $a$ and $b$ and the endpoints of $P_2$ $c$ and $d$. Since $u$ is in $P_1$, there exists paths from $a$ to $u$ and from $b$ to $u$. Call the maximum of the two paths $P_4$. (if $u$ is equal to $a$ (or $b$), let $P_4$ be the path from $b$ (or $a$) to $u$).

Since $v$ is in $P_2$, there exists paths from $v$ to $c$ and from $v$ to $d$. Call the maximum of the two paths $P_5$. (If $v$ is equal to $c$ (or $d$), let $P_5$ be the path from $v$ to $d$ (or $c$)).

By combining paths $P_4$, $P_3$, and $P_5$ to get the path $P_4 P_3 P_5$, we obtain a path that is longer than $P_1$ and $P_2$, thus contradicting the assumption that $P_1$ and $P_2$ were paths of maximum length.
Problem 3
Prove that, in any graph, there must exist a path between any vertex with odd degree and some other vertex with odd degree.

Solution:
If there are no vertices with odd degree then the statement is vacuously true. From here, we prove the statement for graphs with at least one vertex with odd degree.

First, we take care of the case where there is only one vertex with odd degree. This case is impossible since there must be an even number of vertices with odd degree.

Thus, we now prove the statement for graphs with at least two vertices with odd degree. Let $u$ be an arbitrary vertex with odd degree.

Case 1: $u$ is the only vertex of odd degree in its connected component. This case is impossible because we know that every graph (and therefore connected component) must have an even number of vertices with odd degree.

Case 2: there is some other vertex $w$ of odd degree in the connected component that contains $u$. Then there exists a path between $u$ and $w$, since it is a connected component.

In all (possible) cases we have shown that $u$ is connected by some path to some other vertex of odd degree. Since our original choice of $u$ was arbitrary, every vertex of odd degree must be connected by some path to some other vertex of odd degree.