Problem 1: Let’s say we are playing a game. I first roll a fair 6-sided die. If the number that shows is divisible by 3, I roll again and I pay you the dollar amount that shows up on the second roll. If not, then I flip a fair coin. If it’s tails, I take 10 dollars from you, and if it’s heads, I pay you 5 dollars. What is your expected payoff?

Solution:

We will denote the payoff using a random variable $X$. We use the formula for the expected value of a random variable for each value $i$ that $X$ can take on.

$$E[X] = \sum_i i \Pr[X = i]$$

For $X$ to take on the value of 1, the number on the first die must have been divisible by 3 and the number of the second die must have been 1.

Let $A$ be the event that the number on the first die is divisible by 3. We can then write:

$$\Pr[X = 1] = \Pr[X = 1 \cap A] + \Pr[X = 1 \cap \bar{A}]$$

$$= \Pr[X = 1|A] \times \Pr[A]$$

$$= \frac{2}{6} \times \frac{1}{6}$$

Note that the second die roll is dependent on the first, so although we use multiplication, we can’t justify it with “independence.”

Similarly, $\Pr[X = 2], \Pr[X = 3], \Pr[X = 4], \text{ and } \Pr[X = 6]$ all equal $\frac{2}{6} \times \frac{1}{6}$.

For $X$ to take on the value of 5, either the number on the first die is divisible by 3 and the number on the second die is 5 or the number on the first die is not divisible by 3 and the coin toss lands as heads. Thus, using $A$ as defined above,

$$\Pr[X = 5] = \Pr[X = 5|A] \times \Pr[A] + \Pr[X = 5|\bar{A}] \times \Pr[\bar{A}]$$

$$= \frac{2}{6} \times \frac{1}{6} + \frac{4}{6} \times \frac{1}{2}$$

Finally, for $X$ to take on the value of -10, the number on the first die must not be divisible by 3 and the coin must land as tails. Thus, $Pr[X = -10] = \frac{4}{6} \times \frac{1}{2}$. Thus, the expected value is:

$$E[X] = \left(\frac{2}{6} \times \frac{1}{6}\right) (1 + 2 + 3 + 4 + 6)$$

$$+ \left(\frac{2}{6} \times \frac{1}{6} + \frac{4}{6} \times \frac{1}{2}\right) (5)$$

$$+ \left(\frac{4}{6} \times \frac{1}{2}\right) (-10) = -\frac{1}{2}$$
Alternatively, we can sum over the outcomes of the sample space (rather than summing over all possible values of \( X \)):

If the first roll is divisible by 3 (which happens with probability \( \frac{2}{6} = \frac{1}{3} \)), then the payoff for the outcome is just equal to the second roll. The probability that the second roll is equal to any value between 1 and 6 is \( \frac{1}{6} \) since the die is fair. Thus, the probability that the first roll is divisible by 3 and the second roll is equal to any particular value is \( \frac{1}{3} \cdot \frac{1}{6} \) by independence of the two rolls. Thus for all outcomes in which the first die’s value was divisible by 3 and the second die had a value \( k \) (for integer \( k \) where \( 1 \leq k \leq 6 \)), the value of the outcome is \( k \) with probability \( \frac{1}{3} \cdot \frac{1}{6} \).

If the roll is not divisible by 3 (which happens with probability \( \frac{4}{6} = \frac{2}{3} \)), then the payoffs are equal to \(-10\) or 5, depending on the flip (since the coin is fair, each has probability \( \frac{1}{2} \)). Thus, each outcome in which the first roll was not divisible by 3 occurs with probability \( \frac{2}{3} \cdot \frac{1}{2} \) (since the roll and the flip are independent). Thus, the expected value is:

\[
E[X] = \frac{1}{3} \times \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) + \frac{2}{3} \times \frac{1}{2}(-10 + 5)
\]

\[
= \frac{1}{18}(21) + \frac{1}{3}(-5) = -\frac{1}{2}
\]
Problem 2: There are $n$ people in a room. Each pair of people has probability $p$ of being friends (uniform probability across all pairs of people). What is the expected number of friend groups of size $m$ in the room (in terms of $n$, $p$, and $m$)? Friend groups are groups of $m$ people in which everyone in the group is friends with everyone else in the group.

Solution:

Let the random variable $X$ denote the number of groups of size $m$ where everyone is friends with everyone else within that group. We are asked to find $\mathbb{E}[X]$.

Note that there are $\binom{n}{m}$ distinct groups of $m$ people among $n$ people. We can then label the groups of $m$ from 1 up to $\binom{n}{m}$.

We use an indicator variable $X_i$ where $X_i = 1$ if group $i$ is a friend group and 0 otherwise. $\Pr[X_i = 1]$ for all $i$ is $p^{\binom{m}{2}}$ because every possible friendship between any two people in the group must exist, and these friendships are independent of one another.

We can then compute the expectation:

\[
\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{\binom{n}{m}} X_i\right] \\
= \sum_{i=1}^{\binom{n}{m}} \mathbb{E}[X_i] \quad \text{(Linearity of Expectation)} \\
= \sum_{i=1}^{\binom{n}{m}} 1 \times \Pr[X_i = 1] + 0 \times \Pr[X_i = 0] \\
= \sum_{i=1}^{\binom{n}{m}} \Pr[X_i = 1] \\
= \sum_{i=1}^{\binom{n}{m}} p^{\binom{m}{2}} \\
= \binom{n}{m} \cdot p^{\binom{m}{2}}
\]