Recitation Guide - Week 13

Topics Covered: Geometric and Binomial Random Variables, Matchings

Problem 0:

Prove the Memoryless Property for geometric random variables: For a geometric random variable $X$ with parameter $p$ and for $n > 0$,

$$\Pr[X = n + k \mid X > k] = \Pr[X = n]$$

Solution:

$$\Pr[X = n + k \mid X > k] = \frac{\Pr[X = n + k \cap X > k]}{\Pr[X > k]}$$

$$= \frac{\Pr[X = n + k]}{\Pr[X > k]} \quad (\text{since } n > 0)$$

$$= \frac{p(1 - p)^{n+k-1}}{(1 - p)^k}$$

$$= p(1 - p)^{n-1} = \Pr[X = n]$$
Problem 1:
Calculate the expectation of a geometric random variable $X$ with parameter $p$.

Solution:
First, we need some more mechanics:

**Conditional Expectation.** The following is the definition of conditional expectation.

$$E[Y \mid Z = z] = \sum_y y \cdot \Pr[Y = y \mid Z = z],$$

where the summation is over all possible values $y$ that the random variable $Y$ can assume.

**Law of Total Expectation** For any random variables $X$ and $Y$,

$$E[X] = \sum_y E[X \mid Y = y] \Pr[Y = y]$$

*Proof:*

$$E[X] = \sum_x x \cdot \Pr[X = x]$$

$$= \sum_x x \cdot \sum_y \Pr[X = x \mid Y = y] \Pr[Y = y]$$

$$= \sum_y \Pr[Y = y] \cdot \sum_x x \cdot \Pr[X = x \mid Y = y]$$

$$= \sum_y \Pr[Y = y] \cdot E[X \mid Y = y]$$

Now let us calculate the expectation of a geometric random variable $X$ using the memoryless property of the geometric random variable. Let $Y$ be a random variable that is 0, if the first flip results in tails and that is 1, if the first flip is a heads.

Using total expectation we have

$$E[X] = E[X \mid Y = 0] \Pr[Y = 0] + E[X \mid Y = 1] \Pr[Y = 1]$$

Let us try to determine $E[X \mid Y = 0]$.

$$E[X \mid Y = 0] = \sum_{x=1}^{\infty} x \cdot \Pr[X = x \mid Y = 0]$$

$$= 1 \cdot \Pr[X = 1 \mid Y = 0]$$

$$+ \sum_{x=2}^{\infty} x \cdot \Pr[X = x \mid Y = 0]$$
Note that $\Pr[X = 1|Y = 0] = 0$ (consider what these events mean):

$$= \sum_{x=2}^{\infty} x \cdot \Pr[X = x|Y = 0]$$

Note that $\Pr[X = x|Y = 0] = \Pr[X = x|X > 1] = \Pr[X = x - 1]$ (by the memoryless property):

$$= \sum_{x=2}^{\infty} x \cdot \Pr[X = x - 1]$$
$$= \sum_{x=1}^{\infty} (x + 1) \cdot \Pr[X = x]$$
$$= \sum_{x=1}^{\infty} x \cdot \Pr[X = x] + \sum_{x=1}^{\infty} \Pr[X = x]$$

$\therefore \ E[X|Y = 0] = E[X] + 1$

Returning to $E[X]$:

$$E[X] = E[X|Y = 0] \Pr[Y = 0] + E[X|Y = 1] \Pr[Y = 1]$$
$$= (E[X] + 1)(1 - p) + 1 \cdot p$$

$\therefore \ pE[X] = 1$
$$E[X] = \frac{1}{p}$$
**Problem 2:** Prove that any tree has at most one perfect matching.

**Solution:**

We prove the claim by strong induction on vertices.

Let $P(n)$ be the claim that a tree with $n$ vertices has at most one perfect matching.

**Base Cases:** $P(1)$ is true, as there is no perfect matching on a single vertex. $P(2)$ is also true, as a tree with two vertices is itself a perfect matching.

**Induction Hypothesis:** For some integer $k \geq 1$, assume $P(j)$ for all $1 \leq j \leq k$.

**Induction Step:** Consider a tree $T = (V, E)$ with $k+1$ vertices. We know there exists some leaf $\ell$ in this graph. Let the only neighbor of $\ell$ in the graph be $v$. Because $\ell$ has one neighbor, any perfect matching of $T$ must have the edge $v - \ell$.

Consider the forest induced by $V \setminus \{v, \ell\}$. Each connected component of this forest is a tree with $\leq k - 1$ vertices. Therefore, by the induction hypothesis, each of these trees has at most one perfect matching. There are now two cases:

**Case 1:** There is a tree in the forest that does not have a perfect matching.

Because $v - \ell$ must be contained in any matching of $T$, it must be the case that all other vertices do not share an edge with $v$ in a perfect matching of $T$. Therefore, all other vertices in $T$ must have matchings in the forest induced by $V \setminus \{v, \ell\}$. If one such tree in the forest does not have a perfect matching, then $T$ does not have a perfect matching, and therefore $T$ still has at most one perfect matching.

**Case 2:** All trees in the forest have exactly one perfect matching.

If all trees in the forest have exactly one perfect matching, then those unique perfect matchings, along with $v - \ell$ form the unique perfect matching for $T$. Note that because $v - \ell$ is contained in any perfect matching of $T$, there can not exist a matching of $T$ where $v$ is connected to another one of its neighbors.

Since $T$ has exactly one perfect matching, it has at most one perfect matching, proving the claim.

**Alternate Solution:**

Consider an arbitrary tree $T = (V, E)$. Suppose towards contradiction that there exist two distinct matchings, $M$ and $M'$ of $T$. Consider the subgraph $G$ containing all vertices of $T$ and the edges from both $M$ and $M'$.

By assumption, there must exist some vertex $v$ in $G$ with degree 2. Consider the connected component that contains $v$. We claim that no vertex in this connected component can have degree 1. Suppose towards contradiction that some vertex $u$ in this connected component had degree 1. There are two cases for the neighbor of $u$, say $w$: either $u - w$ is in both $M$ and $M'$, or $u - v$ is in exactly one of the two. If $u - w$ is in both $M$ and $M'$, then the connected component is just $u - w$, and has no vertex of degree 2, a contradiction. If $u - w$ is in exactly one of $M$ or $M'$, then $u - w$ have an edge in one matching, but $u$ does not have a neighbor in the other matching, since it has degree 1, which is again a contradiction.
Therefore, there exists a connected component in a subgraph of $T$ in which every vertex has degree of 2. From recitation week 10, we know this connected component has a cycle, contradicting that $T$ was a tree. Thus, any tree has at most one perfect matching.